64th International Mathematical Olympiad
Chiba, Japan, 2nd–13th July 2023

IMO 2023 Problems, Marking Schemes and Solutions

Note:
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Saturday, 8. July 2023

Problem 1. Determine all composite integers $n > 1$ that satisfy the following property: if $d_1, d_2, \ldots, d_k$ are all the positive divisors of n with $1 = d_1 < d_2 < \cdots < d_k = n$, then d_i divides $d_{i+1} + d_{i+2}$ for every $1 \leqslant i \leqslant k - 2$.

Problem 2. Let ABC be an acute-angled triangle with $AB < AC$. Let Ω be the circumcircle of ABC. Let S be the midpoint of the arc CB of Ω containing A. The perpendicular from A to BC meets BS at D and meets Ω again at $E \neq A$. The line through D parallel to BC meets line BE at L. Denote the circumcircle of triangle BDL by ω. Let ω meet Ω again at $P \neq B$.
Prove that the line tangent to ω at P meets line BS on the internal angle bisector of $\angle BAC$.

Problem 3. For each integer $k \geqslant 2$, determine all infinite sequences of positive integers $a_1, a_2, \ldots$ for which there exists a polynomial P of the form $P(x) = x^k + c_{k-1}x^{k-1} + \cdots + c_1 x + c_0$, where $c_0, c_1, \ldots, c_{k-1}$ are non-negative integers, such that

$$P(a_n) = a_{n+1}a_{n+2}\cdots a_{n+k}$$

for every integer $n \geqslant 1$.

Language: English

Time: 4 hours and 30 minutes.
Each problem is worth 7 points.

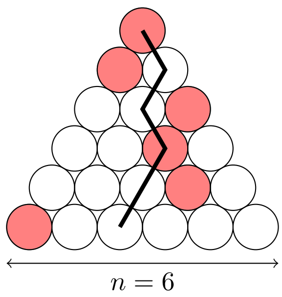
Sunday, 9. July 2023

Problem 4. Let $x_1, x_2, \ldots, x_{2023}$ be pairwise different positive real numbers such that

$$a_n = \sqrt{(x_1 + x_2 + \cdots + x_n)\left(\frac{1}{x_1} + \frac{1}{x_2} + \cdots + \frac{1}{x_n}\right)}$$

is an integer for every $n = 1, 2, \ldots, 2023$. Prove that $a_{2023} \geqslant 3034$.

Problem 5. Let n be a positive integer. A *Japanese triangle* consists of $1 + 2 + \cdots + n$ circles arranged in an equilateral triangular shape such that for each $i = 1, 2, \ldots, n$, the i^{th} row contains exactly i circles, exactly one of which is coloured red. A *ninja path* in a Japanese triangle is a sequence of n circles obtained by starting in the top row, then repeatedly going from a circle to one of the two circles immediately below it and finishing in the bottom row. Here is an example of a Japanese triangle with $n = 6$, along with a ninja path in that triangle containing two red circles.



In terms of n, find the greatest k such that in each Japanese triangle there is a ninja path containing at least k red circles.

Problem 6. Let ABC be an equilateral triangle. Let A_1, B_1, C_1 be interior points of ABC such that $BA_1 = A_1C$, $CB_1 = B_1A$, $AC_1 = C_1B$, and

$$\angle BA_1C + \angle CB_1A + \angle AC_1B = 480°.$$

Let BC_1 and CB_1 meet at A_2, let CA_1 and AC_1 meet at B_2, and let AB_1 and BA_1 meet at C_2. Prove that if triangle $A_1B_1C_1$ is scalene, then the three circumcircles of triangles AA_1A_2, BB_1B_2 and CC_1C_2 all pass through two common points.

(Note: a scalene triangle is one where no two sides have equal length.)

Language: English

Time: 4 hours and 30 minutes.
Each problem is worth 7 points.

Marking scheme for Problem 1

Problem 1.
Determine all composite integers $n > 1$ that satisfy the following property: if $d_1, d_2, \ldots, d_k$ are all the positive divisors of n with $1 = d_1 < d_2 < \cdots < d_k = n$, then d_i divides $d_{i+1} + d_{i+2}$ for every $1 \leqslant i \leqslant k-2$.

Marking scheme

The solution is subdivided into two parts.
(A) Proving that $n = p^r$ $(r \geq 2)$ satisfies the condition .... **2 points**
(B) Proving that $n \neq p^r$ $(r \geq 2)$ does not satisfy the condition .... **6 points**

A total of 2 *points for Part (A) can be given only if the mark for Part (B) is* 0.

Example: 2 points are given to the following if the mark in Part (B) is 0 "The condition is obviously true when $n = p^r$ because the divisors of p^r are p^i".

The following general items are rewarding:
- **(C1)** Giving concrete (counter-)examples .... **0 point**
- **(C2)** Considering any factorization $n = p_1^{e_1} \cdots p_r^{e_r}$ .... **0 point**
- **(C3)** Writing $d_i d_{k+1-i} = n$ .... **0 point**
- **(C4)** Considering the two smallest prime divisors $p < q$ of n .... **0 point**
- **(C5)** Just considering $j \geq 1$ such that $1 < p < \cdots < p^j < q$ are the first $j+2$ divisors . **0 point**
- **(C6)** Just considering the divisor $p^r || n$ for the smallest prime divisor p of n .... **0 point**

In the 2-points regime of **part A**, partial marks, up to 1 point, are given as follows:
- **(A1)** Just claiming p^r $(r \geq 2)$ works (for infinitely many p, e.g. for odd p) .... **1 point**
- **(A2)** Proving that $n = p^r$ satisfies the condition for a particular p (e.g. $p = 2$) .... **1 point**
- **(A3)** Proving that $n = p^2$ satisfies the condition .... **0 point**
- **(A4)** Proving that $n = p^e$(for a fixed $e \geq 3$) satisfies the condition .... **1 point**
- **(A5)** Noticing that the divisors of p^r are p^i .... **1 point**

The point in Part (A1) can also be awarded in the 1-point regime for Part (A).

In **part B**, marks are given as follows:

(B1) Getting a *relevant expression* such as $\frac{n}{q} \mid \frac{n}{p^j} + \frac{n}{p^{j-1}}$, or $d_i \mid d_{i+1}^2$ (for all $1 \leq i \leq k-1$), or $d_i \mid d_{i+1}$ (for all $1 \leq i \leq k-1$), or $p \mid d_{\ell-1}$ and $p \mid d_{\ell+1}$ where $d_\ell = \frac{n}{p^r}$ $(r = \mathrm{val}_p(n))$ ...... **4 points**

For the relevant expression $p \mid d_{\ell-1}$ and $p \mid d_{\ell+1}$, the first part is 3 points and the second part is 1 point (they are additive).

(B2) Using a relevant expression to prove that there exists only one prime divisor and concluding .... **2 points**

To get 2 *points for (B2), it is necessary to get full score (or* -1 *point of deduction) for (B1).*

In (B1), partial marks, up to 3 points, are given as follows:

(B1-1) (C4) + (C5) + Noticing $p^{j-1} \mid p^j + q$ .... **1 point**
(B1-2) Obtaining $j = 1$, or $d_2 = p$ and $d_3 = q$ .... **2 points**
(B1-3) Reverting the list of divisors of n, and obtaining $\frac{n}{q} < \frac{n}{p^j} < \cdots < n$ .... **2 points**

(B1-4) Just rewriting the condition $d_i \mid d_{i+1} + d_{i+2} \iff \frac{n}{d_{i+2}} \mid \frac{n}{d_i} + \frac{n}{d_{i+1}}$ .... **0 point**
(B1-5) Clearing up denominators to obtain $d_i d_{i+1} \mid d_{i+1} d_{i+2} + d_i d_{i+2}$ .... **1 point**
(B1-6) Obtaining $d_i \mid d_{i+1} d_{i+2}$ .... **2 points**

(B1-7) Claiming $d_1 \mid d_2$ because $d_1 = 1$ .... **0 point**
(B1-8) Using the condition in the problem statement to obtain $d_{k-2} \mid d_{k-1}$ .... **0 point**
(B1-9) Obtaining $d_{k-2} \mid d_{k-1}$ and $d_{k-3} \mid d_{k-2}$ .... **2 points**
(B1-10) Obtaining $d_2 \mid d_3$ .... **3 points**

(B1-11) Considering the divisor $d_\ell = \frac{n}{p^r}$ ($r = \mathrm{val}_p(n)$) for the smallest prime p dividing n .... **1 point**

Partial marks in (B1) are not additive.

In (B2), partial marks, up to 1 point, are given as follows:

(B2-1) Proving that $\gcd(d_2, d_3) > 1$ implies $p \mid d_i$ (for all $2 \le i \le k$) for a prime p .... **1 point**
(B2-2) Proving that $p \mid d_i$ and $p \mid d_{i+2}$ imply $p \mid d_{i+1}$ for a prime p .... **1 point**

Deductions

(D1) Not verifying the base case of the induction .... -1 **point**
(D2) Any minor mistake which can be fixed in a single-line sentence .... -1 **point**

Solutions

Answer: $n = p^r$ is a prime power for some $r \geq 2$.

Solution 1. It is easy to see that such an $n = p^r$ with $r \geq 2$ satisfies the condition as $d_i = p^{i-1}$ with $1 \geq i \geq k = r + 1$ and clearly
$$p^{i-1} \mid p^i + p^{i+1}.$$

Now, let us suppose that there is a positive integer n that satisfies the divisibility condition of the problem and that has two different prime divisors p and q. Without lost of generality, we assume $p < q$ and that they are the two smallest prime divisors of n. Then there is a positive integer j such that
$$d_1 = 1,\, d_2 = p, \ldots, d_j = p^{j-1}, d_{j+1} = p^j, d_{j+2} = q,$$
and it follows that
$$d_{k-j-1} = \frac{n}{q}, d_{k-j} = \frac{n}{p^j},\, d_{k-j+1} = \frac{n}{p^{j-1}}, \ldots, d_{k-1} = \frac{n}{p},\, d_k = n.$$
Thus
$$d_{k-j-1} = \frac{n}{q} \mid d_{k-j} + d_{k-j+1} = \frac{n}{p^j} + \frac{n}{p^{j-1}} = \frac{n}{p^j}(p+1). \tag{1}$$
This gives $p^j \mid q(p+1)$, which is a contradiction since $\gcd(p, p+1) = 1$ and $p \neq q$.

Solution 1'. We present here a more technical way of finishing Solution 1 after obtaining (1). We let $v_p(m)$ denote the p-adic valuation of m. Notice that $v_p(n/q) = v_p(n)$ as $\gcd(p, q) = 1$ and that
$$v_p\left(\frac{n}{p^j}(p+1)\right) = v_p(n) - j$$
as $\gcd(p, p+1) = 1$. But (1) implies
$$v_p(n) = v_p(n/q) \leq v_p\left(\frac{n}{p^j}(p+1)\right) = v_p(n) - j$$
which is a contradiction. Thus n has only one prime divisor as desired.

Solution 2. We start by proving the following claim:

Claim. $d_i \mid d_{i+1}^2$ for all $1 \leq i \leq k-2$.

Proof. Since $d_i d_{k+1-i} = n$, we have the equivalence:
$$d_{k-i-1} \mid d_{k-i} + d_{k-i+1} \iff \frac{n}{d_{i+2}} \mid \frac{n}{d_{i+1}} + \frac{n}{d_i}.$$
We multiply both sides by $d_i d_{i+1} d_{i+2}$ and cancel the n's to get
$$d_i d_{i+1} \mid d_i d_{i+2} + d_{i+1} d_{i+2}.$$
Hence,
$$d_i \mid d_{i+1} d_{i+2}. \tag{2}$$
Moreover, by the condition of the problem,
$$d_i \mid d_{i+1}(d_{i+1} + d_{i+2}) = d_{i+1}^2 + d_{i+1} d_{i+2}.$$
Combining this with (2) we get that $d_i \mid d_{i+1}^2$ for all $1 \leq i \leq k-2$. □

Let $d_2 = p$ be the smallest prime divisor of n. By induction on i we prove that $p \mid d_i$ for all $2 \le i \le k-1$. The base case $d_2 = p$ is obvious. Let us suppose that $p \mid d_j$ for some $2 \le j \le k-2$. Then we have that

$$p \mid d_j \mid d_{j+1}^2 \implies p \mid d_{j+1}$$

as p is prime, which completes the induction. This implies that n has to be a prime power, as otherwise there would be another prime q that divides n and we would get that $p \mid q$ which is obviously false.

We finally check that the powers of p satisfy the condition in the statement of the problem as in Solution 1.

Solution 3. We start by proving the following claim:

Claim. $d_i \mid d_{i+1}$ for every $1 \le i \le k-1$.

Proof. We prove the Claim by induction on i; it is trivial for $i = 1$ because $d_1 = 1$. Suppose that $2 \le i \le k-1$ and the Claim is true for $i-1$, i.e. $d_{i-1} \mid d_i$. By the induction hypothesis and the problem condition, $d_{i-1} \mid d_i$ and $d_{i-1} \mid d_i + d_{i+1}$, so $d_{i-1} \mid d_{i+1}$.

Now consider the divisors $d_{k-i} = \dfrac{n}{d_{i+1}}$, $d_{k-i+1} = \dfrac{n}{d_i}$, $d_{k-i+2} = \dfrac{n}{d_{i-1}}$. By the problem condition,

$$\frac{d_{k-i+1} + d_{k-i+2}}{d_{k-i}} = \frac{\dfrac{n}{d_i} + \dfrac{n}{d_{i-1}}}{\dfrac{n}{d_{i+1}}} = \frac{d_{i+1}}{d_i} + \frac{d_{i+1}}{d_{i-1}}$$

is an integer. We conclude that $\dfrac{d_{i+1}}{d_i}$ is an integer, so $d_i \mid d_{i+1}$. □

By the Claim, n cannot have two different prime divisors because the smallest one would divide the other one. Hence, n must be a power of a prime, and powers of primes satisfy the condition of the problem as we saw in Solution 1.

Solution 3'. The following Claim is the same as Solution 3; only the order of the induction is different.

Claim. $d_i \mid d_{i+1}$ for every $1 \le i \le k-1$.

Proof. We prove the Claim by induction on i from the case $i = k-1$ to $i = 1$, Since $d_{k-2} \mid d_{k-1} + d_k$ and $d_{k-2} \mid d_k = n$, we have $d_{k-2} \mid d_{k-1}$, hence the base case $i = k-1$ of the Claim. Suppose that $d_i \mid d_{i+1}$ holds for some $2 \le i \le k-1$. Since $d_{k-i} \mid d_{k+1-i} + d_{k+2-i}$, $d_{k-i} = \dfrac{n}{d_{i+1}}$, $d_{k+1-i} = \dfrac{n}{d_i}$, and $d_{k+2-i} = \dfrac{n}{d_{i-1}}$, we have

$$\frac{n}{d_{i+1}} \mid \frac{n}{d_i} + \frac{n}{d_{i-1}} \iff d_{i-1}d_i \mid d_{i-1}d_{i+1} + d_i d_{i+1}.$$

Since $\dfrac{d_{i+1}}{d_i}$ is an integer, we have $d_{i-1} \mid d_{i-1}\dfrac{d_{i+1}}{d_i} + d_{i+1}$, so $d_{i-1} \mid d_{i+1}$. Combining this with $d_{i-1} \mid d_i + d_{i+1}$, we conclude $d_{i-1} \mid d_i$, so the Claim is proved by induction. □

The rest of the proof is exactly the same as Solution 3.

Solution 4. The following is another solution for Part (B).

Assume n is not a prime power and satisfies the condition of the problem statement.

Consider the smallest prime p dividing n and the divisor $d_\ell = \dfrac{n}{p^r}$ $(r = \mathrm{val}_p(n))$.

Then $d_{\ell-1}$ divides $d_\ell + d_{\ell+1}$. Notice that p does not divide d_ℓ.

But we will prove p divides $d_{\ell-1}$ and $d_{\ell+1}$, and we will get a contradiction.

- If p does not divide $d_{\ell+1}$, then $d_{\ell+1} = \dfrac{n}{p^r q} < d_\ell$, which is a contradiction.
- If p does not divide $d_{\ell-1}$, then $d_{\ell-1} = \dfrac{n}{p^r q}$, but then

$$d_{\ell-1} = \frac{n}{p^r q} < \frac{n}{p^{r-1} q} < \frac{n}{p^r} = d_\ell,$$

  which is a contradiction.

Marking scheme for Problem 2

Problem 2.
Let ABC be an acute-angled triangle with $AB < AC$. Let Ω be the circumcircle of ABC. Let S be the midpoint of the arc CB of Ω containing A. The perpendicular from A to BC meets BS at D and meets Ω again at $E \neq A$. The line through D parallel to BC meets line BE at L. Denote the circumcircle of triangle BDL by ω. Let ω meet Ω again at $P \neq B$.
Prove that the line tangent to ω at P meets line BS on the internal angle bisector of $\angle BAC$.

Marking scheme

Applying the Markscheme

In the following sections, unless otherwise stated, the relevant item must have a **complete proof** (not just a statement or conjecture) to score the mark.

Marks within a section are **not additive**. The students mark will be the maximum of:

- **Full Solution**
- **Significant Progress**
- **General Steps + Towards a Solution**
- **Other Paths**

For example, if a student were to complete **(G1)**, **(G2)**, **(T5)** and **(T3)**, they would score 1 in **General Steps** and 2 in **Towards a Solution**. These can add to each other (point 3 above) meaning the student scores $\boxed{3}$ overall. If the student also did **(S1)**, they would score $\boxed{4}$.

Additional Clarifications

- For points requiring proofs of parallel lines, these must be **explicitly** stated or clearly marked on the diagram. Proving equivalent angle equalities and not stating that this means the relevant lines are parallel is **NOT** sufficient to score the points.
- *Reducing the problem to proving* $\mathcal{S}$ (where $\mathcal{S}$ is a statement e.g. $XA = XP$).
  - To score these points, the student must **explicitly** state that, with the results they have proved so far, $\mathcal{S}$ is sufficient to finish the problem **AND** prove why this is the case.
  - Simply having proved results that, combined with $\mathcal{S}$, would be sufficient to solve the problem but not **explicitly** stating and proving this is **NOT** sufficient to score the points.

Full Solution

- Complete solution .... 7 **points**
- Complete solution for the incorrect† configuration which translates to the correct one 7 **points**
- Essentially complete solution with minor error .... 6 **points**

Significant Progress (not additive to anything)

(S1) Proving at least 2 of $\{PD \parallel QS, AP \parallel S'Q, \angle DPA = 90°\}$ .... 4 **points**

(S2) **Explicitly** stating that $\triangle SS'Q \sim \triangle DAP$ **AND** mentioning the parallel sides of these triangles 5 **points**

- Simply marking/proving equal angles in the triangles is **NOT** sufficient.

(S3) $XA = XP$ where X is defined as **either** $X := BD \cap AS'$ or $X := BD \cap PQ$ ...... 5 **points**

(S4) $XA = XP$ where $X := PQ \cap AS'$ .... 3 **points**

(S5) Reducing the problem to proving $XA = XP$ (with any of the three definitions of X given in **(S3)** and **(S4)**) .... 3 **points**

(S6) Under a $\sqrt{ac}$ inversion at B, prove that $D^*P^* = D^*C$ where D^*, P^* are the images of D, P (as in *Solution 12*) .... 5 **points**

General Steps (not additive within the section but additive to *Towards a Solution*)

(G1) L, P, S collinear .... 1 **point**

(G2) P, D, A' collinear **OR** $\angle DPA = 90°$ .... 1 **point**

(G3) AD, PS', ω concur (at T) .... 1 **point**

Towards a Solution (not additive within the section but additive to *General Steps*)

(T1) $PD \parallel QS$ **OR** $AP \parallel S'Q$ .... 2 **points**

(T2) [**Requires (G3)**] $(P, D; T, B) = -1$ .... 2 **points**

(T3) [**Requires (G2)**] $APTR$ cyclic ($T := AD \cap \omega$) where $R := AS' \cap PD$ .... 2 **points**

(T4) $\gamma := \odot APT$ ($T := AD \cap \omega$) is orthogonal to ω **OR** tangesnt to ω at P passes through centre of γ .... 2 **points**

(T5) Reducing the problem to proving BS passes through the midpoint of AR where either $R := AS' \cap PA'$ or $R := AS' \cap \odot APT$ ($T := AD \cap \omega$) .... 2 **points**

(T6) [**Requires (G3)**] $AXTB$ cyclic .... 2 **points**

†The problem statement is phrased to preclude solutions having to treat multiple configurations, but a contestant might still e.g. misread the order and work with the wrong configuration. Such solutions are not deducted points.

Other Paths (not additive to anything)

(O1) Reducing the problem to proving **one** of the following: .... 2 **points**

- $(Y, D; P, A') = -1$ (where $Y = BS' \cap PA'$).
- $\angle RBS = \angle SBP$ (where $R := PA' \cap AS'$).

Not Worth Points (on their own)

(N1) AS' tangent to $\odot ABD$ .... 0 **points**

(N2) AP, BS', ω concur (at M) .... 0 **points**

(N3) BE', AE, ω concur (where E' is the point diametrically opposite E on Ω) .... 0 **points**

(N4) PS', BE', AE concur (at T) .... 0 **points**

(N5) $PXBR$ cyclic (with any definition of X or R) .... 0 **points**

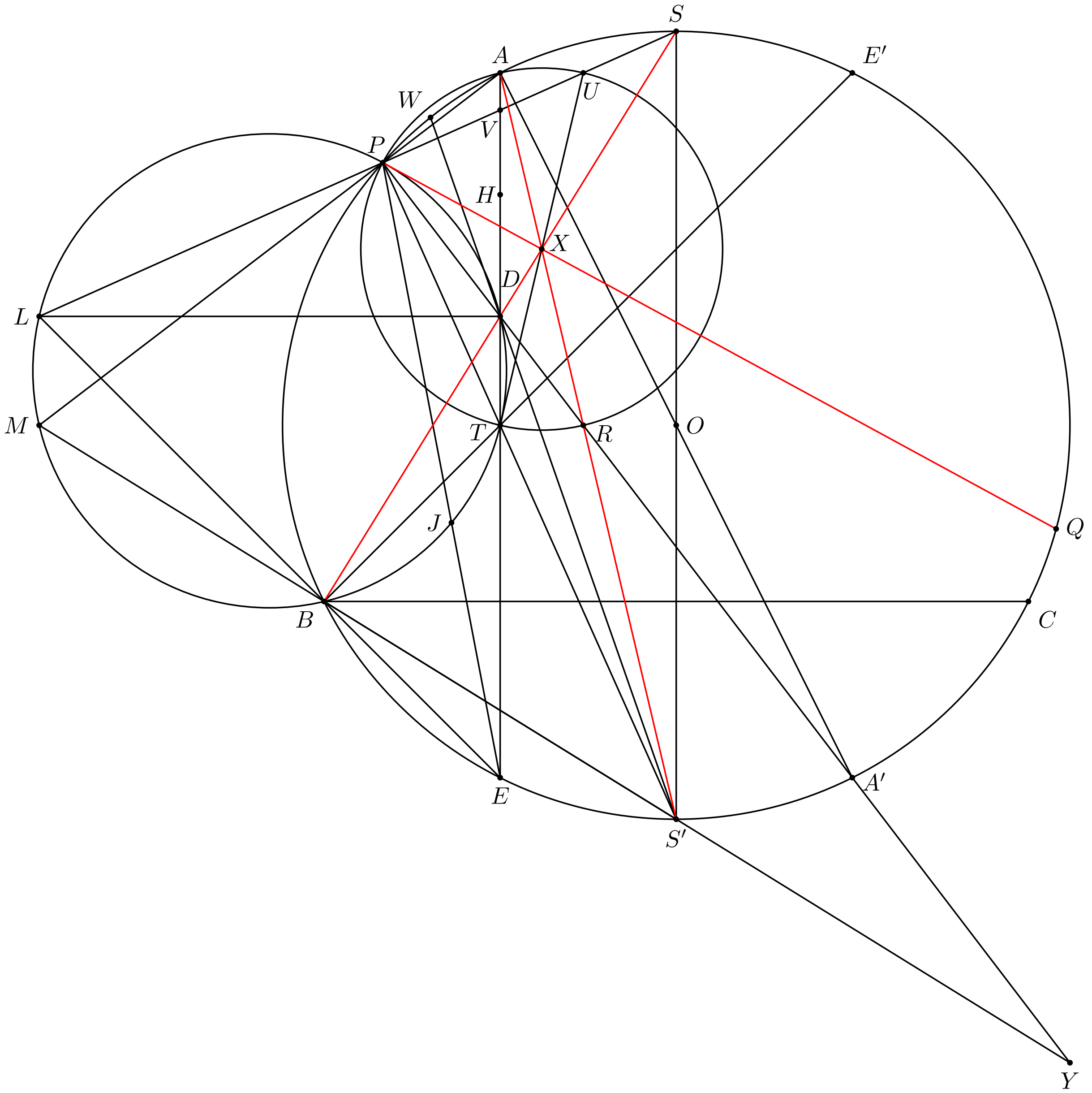
(N6) $PE = SQ$ **OR** $PS \parallel EQ$ .... 0 **points**

(N7) Equal angles or parallel lines marked on a diagram with no proof .... 0 **points**

Remark about computational solutions

In general, computational approaches, unless substantially complete, will not be awarded more than a few points. A mere translation of geometry into trigonometry, complex numbers, Cartesian or barycentric coordinates etc. will be awarded **0 points**. Any essentially incomplete computational attempt will get **0 points** unless the results are interpreted in geometrical terms, in which case this constitutes a valid proof of those results, giving points according to the above scheme.

Diagram

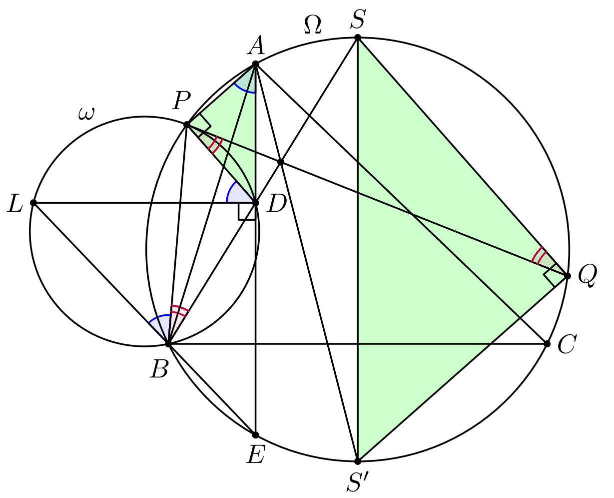


Solutions

Many of the solutions given below are **sketches** and do not necessarily present sufficient detail to score full marks.

Solution 1.

Let S' be the midpoint of arc BC of Ω, diametrically opposite to S so SS' is a diameter in Ω and AS' is the angle bisector of $\angle BAC$. Let the tangent of ω at P meet Ω again at $Q \neq P$, then we have $\angle SQS' = 90°$.



We will show that triangles APD and $S'QS$ are similar and their corresponding sides are parallel. Then it will follow that the lines connecting the corresponding vertices, namely line AS', that is the angle bisector of $\angle BAC$, line PQ, that is the tangent to ω at P, and DS are concurrent. Note that the sides AD and $S'S$ have opposite directions, so the three lines cannot be parallel. First we show that $AP \perp DP$. Indeed, from cyclic quadrilaterals $APBE$ and $DPLB$ we can see that

$$\angle PAD = \angle PAE = 180° - \angle EBP = \angle PBL = \angle PDL = 90° - \angle ADP.$$

Then, in triangle APD we have $\angle DPA = 180° - \angle PAD - \angle ADP = 90°$.

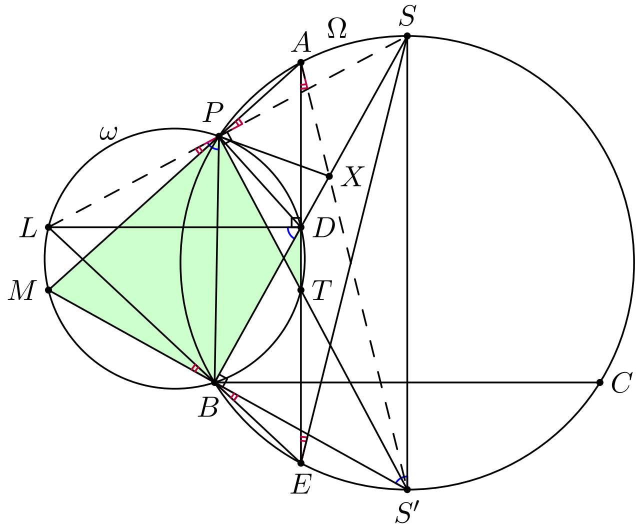
Now we can see that:

- Both lines ADE and SS' are perpendicular to BC, so $AD \parallel S'S$.
- Line PQ is tangent to circle ω at P so $\angle DPQ = \angle DBP = \angle SBP = \angle SQP$; it follows that $PD \parallel QS$.
- Finally, since $AP \perp PD \parallel QS \perp S'Q$, we have $AP \parallel S'Q$ as well.

Hence the corresponding sides of triangles APD and $S'QS$ are parallel completing the solution.

Solution 2. Again, let S' be the midpoint of arc BC, diametrically opposite to S, so $AES'S$ is an isosceles trapezoid, and $\angle S'BS = \angle S'PS = 90°$. Let lines AE and PS' meet at T and let AP and $S'B$ meet at point M.

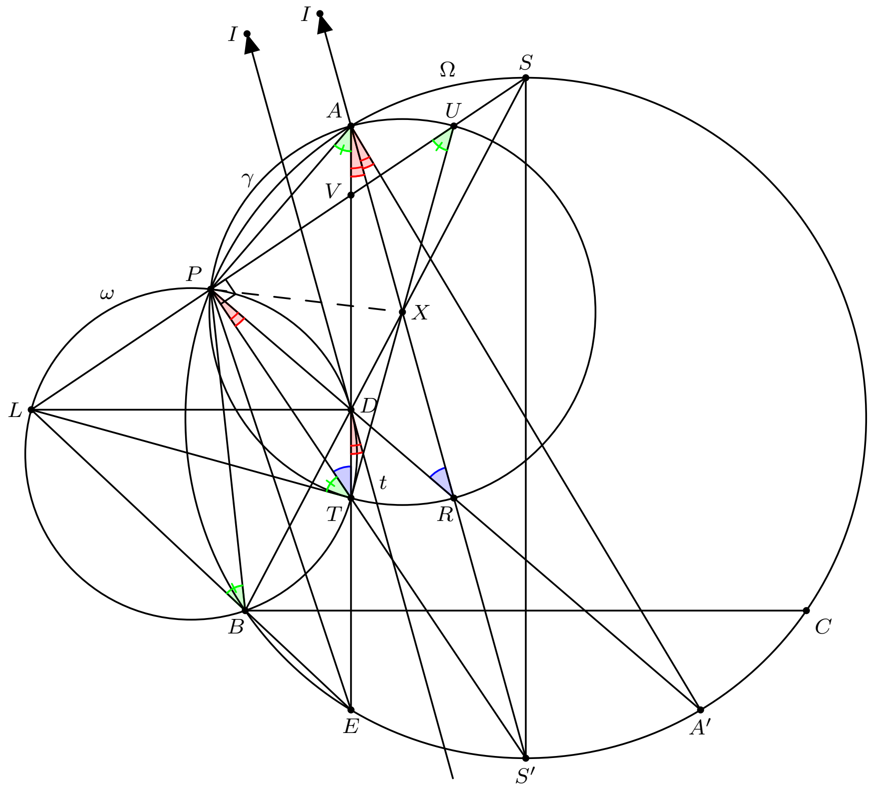
We will need that points L, P, S are collinear, and points T and M lie on circle ω.



- From $\angle LPB = \angle LDB = 90° - \angle BDE = 90° - \angle BSS' = \angle SS'B = 180° - \angle BPS$ we get $\angle LPB + \angle BPS = 180°$, so L, P and S are indeed collinear.
- Since SS' is a diameter in Ω, lines LPS and PTS' are perpendicular. We also have $LD \parallel BC \perp AE$ hence $\angle LDT = \angle LPT = 90°$ and therefore $T \in \omega$.
- By $\angle LPM = \angle SPA = \angle SEA = \angle EAS' = \angle EBS' = \angle LBM$, point M is concyclic with B, P, L so $M \in \omega$.

Now let X be the intersection of line BDS with the tangent of ω at P and apply Pascal's theorem to the degenerate cyclic hexagon $PPMBDT$. This gives points $PP \cap BD = X$, $PM \cap DT = A$ and $MB \cap TP = S'$ are collinear so X lies on line AS', that is the bisector of $\angle BAC$.

Solution 3. Let A' and S' be the points of Ω diametrically opposite to A and S respectively. It is well-known that E and A' are reflections with respect to SS' so AS' is the angle bisector of $\angle EAA'$. Define point T to be the intersection of AE and PS'. As in the previous two solutions, we have: $\angle DPA = 90°$ so PD passes through A'; points L, P, S are collinear; and $T \in \omega$.



Let lines AS' and PDA' meet at R. From the angles of triangles PRS' and PTE we get

$$\angle ARP = \angle AS'P + \angle S'PA' = \angle AEP + \angle EPS' = \angle ATP$$

so points A, P, T, R are concyclic. Denote their circle by γ. Due to $\angle RPA = \angle DPA = 90°$, segment AR is a diameter in γ.

We claim that circles ω and γ are perpendicular. Let line LPS meet γ again at $U \neq P$, and consider triangles PLT and PTU. By $\angle LPT = \angle TPU = 90°$ and

$$\angle PTL = \angle PBL = 180° - \angle EBP = \angle PAE = \angle PAT = \angle PUT,$$

triangles PLT and PTU are similar. It follows that the spiral similarity that takes PLT to PTU, maps ω to γ and the angle of this similarity is $90°$, so circles ω and γ are indeed perpendicular.

Finally, let lines BDS and ARS' meet at X. We claim that X bisects AR, so point X is the centre of γ and, as ω and γ are perpendicular, PX is tangent to ω.

Let t be the tangent of ω at D. From $\angle(DT, t) = \angle TPD = \angle S'PA' = \angle EAS'$ it can be seen that $t \parallel AS'$. Let I be the common point at infinity of t and AS'. Moreover, let lines LPS and $ADTE$ meet at V. By projecting line AS' to circle ω through D, then projecting ω to line AE through L, finally projecting AE to Ω through P, we find

$$\frac{AX}{RX} = (A, R; X, I) \overset{D}{=} (T, P; B, D) \overset{L}{=} (T, V; E, D) \overset{P}{=} (S', S; E, A') = -1,$$

so X is the midpoint of AR.

Solution 4. Let $X \coloneqq DS \cap AS'$. Our goal is to prove that PX is tangent to ω. As in other solutions show $\angle APD = 90°$.

Claim 1. Line AX is tangent to the circumcircle of triangle ABD.

Proof. Note that line AX intersects Ω again at S' and $AE \parallel SS'$ so:

$$\angle DAX = \angle EAS' = \angle SBA = \angle DBA$$

which proves the claim. □

Claim 2. $XA = XP$.

Proof. Let O be the centre of Ω, $F = AX \cap DO$, $G = DX \cap OA$ and $H = OX \cap AD$.

By Ceva's Theorem on triangle AOD, we have

$$\frac{AH}{HD} \cdot \frac{DF}{FO} \cdot \frac{OG}{GA} = 1 \Longrightarrow \frac{AH}{HD} = \frac{FO}{DF} \cdot \frac{GA}{OG}.$$

Since lines AD and AO are isogonal we know that AX bisects $\angle BAC$, and therefore, by the angle bisector theorem,

$$\frac{FO}{DF} = \frac{AO}{AD}.$$

On the other hand, lines AD and OS are parallel and thus $\triangle AGD \sim \triangle OGS$, which gives us

$$\frac{AG}{GO} = \frac{AD}{SO} = \frac{AD}{AO}.$$

Hence,

$$\frac{AH}{HD} = \frac{AO}{AD} \cdot \frac{AD}{AO} = 1$$

and thus H is the midpoint of AD.

Therefore, since $\angle APD = 90°$ this gives us $HA = HP$, and because $OA = OP$ we may conclude that $XA = XP$. □

Finally, combining Claims 1 and 2 gives us

$$XP^2 = XA^2 = XD \cdot XB$$

which finishes the solution by power of X with respect to circle ω.

Solution 5. Let $X \coloneqq DS \cap AS'$ and T be the second intersection of AD with ω. As in other solutions, $\angle APD = 90°$, L, P, S are collinear and P, T, S' are collinear. It is enough to prove that PX is tangent to ω.

Now let $R \coloneqq PD \cap AS'$ and let J denote the second intersection of PE with ω.

Claim 1. The cyclic quadrilateral $PDTB$ is harmonic.

Proof. Since $\angle APD = 90°$, line PR passes through A' and therefore PS' bisects $\angle EPR$. Thus, we may conclude that $TD = TJ$, and since LT is a diameter of ω, we have

$$-1 = (L, T; D, J) \stackrel{E}{=} (B, D; T, P)$$

as required. □

Claim 2. Point X is the centre of a circle passing through points A, P, T, R.

Proof. Since PS' bisects $\angle EPR$ we have

$$\angle TPR = \angle JPT = \angle EPS' = \angle EAS' = \angle TAR$$

which implies that $APTR$ is a cyclic quadrilateral.

Furthermore, note that

$$\angle TPR = \angle TAR \implies \angle TPD = \angle TAS'$$

and thus the tangent to ω at D is parallel to line AS'.

Hence, if we let ∞ denote the point at infinity along line AS', we obtain:

$$-1 = (B, D; T, P) \stackrel{D}{=} (X, \infty; A, R)$$

which implies that X is the midpoint of AR.

Since $\angle APR = 90°$ this gives us that X is the circumcentre of triangle APR, and we have already seen that $APTR$ is cyclic. □

Finally, since $PDTB$ is harmonic, $XP = XT$ and $X \in BD$, point X must be the intersection of the tangents to ω at P and T, which finishes the solution.

Solution 6. Let $T := BE' \cap AE$, R lie on PD such that $TR \parallel BC$ and finally let the tangents to ω at P, T intersect at X.

Claim 1. $T \in \omega$.

Proof.

$$\angle LDT = 90° = \angle EBE' = \angle EBT = \angle TBL$$

□

As in other solutions, we have seen that L, P, S and P, D, A' and P, T, S' are collinear.

Claim 2. $(P, T; D, B) = -1$

Proof.

$$(P, T; D, B) \stackrel{L}{=} (LP \cap AE, T; D, E) \stackrel{P}{=} (S, S'; A', E) = -1$$

where the last result follows from SS' being the perpendicular bisector of $A'E$. □

Claim 3. $APTR$ cyclic on a circle with centre X and diameter AR.

Proof. The cyclic and diameter part follow from:

$$\angle ATR = 90° = \angle A'PA = \angle RPA$$

To show X is the centre observe $XP = XT$ and:

$$\angle RTD = 90° \implies \angle PDT = 90° + \angle PRT \implies \angle PXT = 2\angle PDT - 180° = 2\angle PRT$$

which is enough to show X is the centre of $APTR$. □

From Claim 3, X lies on the tangent at P and the line DB (which passes through S). Therefore it's enough to show that X lies on AS', the internal bisector of $\angle BAC$. This follows from:

$$\angle PAX \stackrel{Claim\ 4}{=} 90° - \angle DTP = 90° - \angle SBP = 90° - \angle SS'P = \angle PSS' = \angle PAS'$$

which completes the proof.

Solution 7.

As in other solutions, P, D, A' are collinear. Define $X := AS' \cap SB$ and $R := AS' \cap PA'$. Let W be the second intersection of $\odot ABC$ and $S'D$ and $Y := BS' \cap PD$. It is enough to show $\angle BPD = \angle DPX$. Angle chasing:

$$\angle XRP = \angle ARP = 90^\circ - \angle PAS' = SBP = \angle XBP$$

so $PXBR$ is cyclic and hence $\angle DPX = \angle RPX = \angle RBX$ so it is enough to show $\angle DBP = \angle RBD$. Since $\angle YBD = 90^\circ$, it is enough to show $(P, R; D, Y) = -1$.

$$(P, R, D, Y) \overset{S'}{=} (P, A, W, B) \overset{D}{=} (A', E; S', S) = -1$$

Solution 8. Let $X = BS \cap AS'$. As $AS'A'S$ is a rectangle and $AD \parallel SS'$ we have:

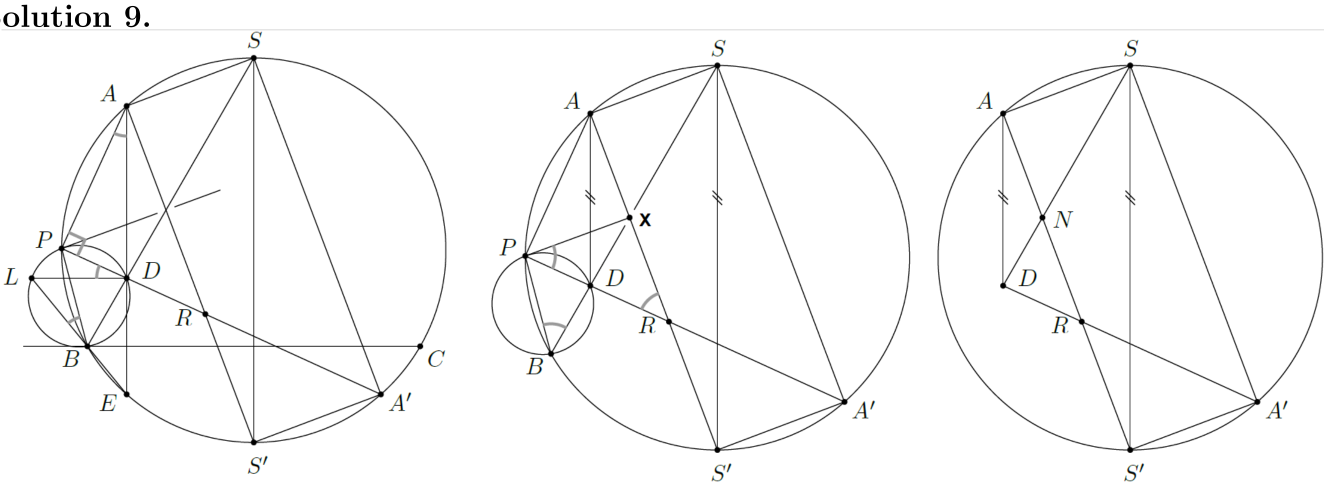
$$\frac{XR}{SA'} = \frac{DX}{DS} = \frac{AX}{AS'} = \frac{AX}{SA'} \implies XR = AX$$

As in other solutions, $\angle DPA = 90^\circ$ so X is the centre of $\odot APR$. This gives:

$$\angle RPX = \angle ARP \overset{AS' \parallel SA'}{=} \angle SA'P = \angle SBP = \angle DBP$$

so PX is tangent to $\odot BPD$.

Solution 9.



Left Figure: Introduce A', S' as in other solutions and prove that P, D, A' are collinear.

Middle figure: We can now reconstruct the diagram from rectangle $ASA'S'$ (and no longer require points L, C, E):

- First, take point D on a line through A parallel to the diagonal SS'.
- Then take $B = SD \cap \Omega$ and $P = AD \cap \Omega$.

Let the tangent to ω at P meet the angle bisector AS' at X. Angle-chasing we have:

$$\angle RPX = \angle DBP = \angle SBP = \angle SS'P = 90^\circ - \angle PAS' = \angle XRP$$

so $XR = XP$ and, as $\angle RPA = 90^\circ$, this means X is the midpoint of AR.

Right figure: Now we can also remove points B, P and it suffices to prove that the intersection $N = DS \cap AS'$ is also the midpoint of AR. We have two pairs of similar triangles namely $NRD \sim SA'D$ and $AND \sim S'NS$ giving:

$$\frac{NR}{SA'} = \frac{DN}{DS} = \frac{AN}{AS'}.$$

Since $SA' = AS'$ we get $NR = AN$.

Solution 10. Define $X \coloneqq BS \cap AS'$. Angle chasing gives AS' tangent to $\odot ABD$ so $XD \cdot XB = XA^2$. Thus it is sufficient to prove $XA = XP$.

Let ℓ be the perpendicular bisector of AP. As in other solutions, $\angle DPA = 90^\circ$ so the midpoint of AD lies on ℓ. Also, the midpoint of SS' is the centre of Ω so also lies on ℓ. As $AD \parallel SS'$, we can consider the dilation at X taking $AD \to S'S$ which also takes the midpoint of AD to the midpoint of SS'. Hence $X \in \ell$ so $XA = XP$ as required.

Solution 11. As in other solutions define $T \coloneqq PS' \cap AE$ then we have $T \in \omega$. We also have P, D, A' collinear. Define $X \coloneqq AS' \cap BS$ then we have:

$$\angle TBX = \angle TBD = \angle TPD = \angle S'PA' \overset{S'A'=S'E}{=} \angle EAS' = \angle TAX$$

so $AXTB$ is cyclic. From this:

$$\angle TAX = \angle EAS' \overset{EA \parallel SS'}{=} \angle SBA = \angle XBA = \angle XTA \implies XA = XT$$

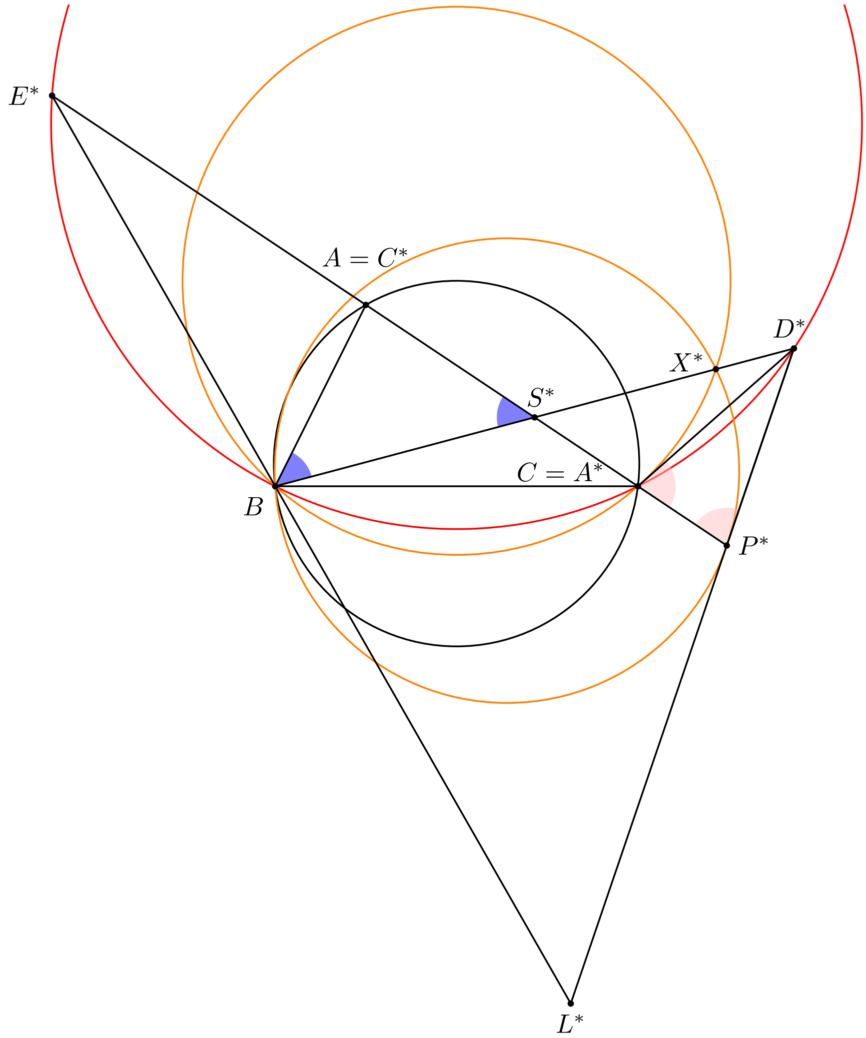
and also:

$$\angle TPA = \angle S'PA = 90^\circ - \angle EAS' = 90^\circ - \angle TAX$$

Combining these is enough to show that X is the circumcentre of $\triangle APT$ and so $XA = XP$. From here we can finish as in other solutions.

Solution 12. As in other solutions, AS' is tangent to $\odot ABD$. Let $X := AS' \cap BD$.

Perform a $\sqrt{ac}$ inversion (i.e. invert about B with radius $\sqrt{BA \cdot BC}$ and then reflect over the bisector of $\angle CBA$) and denote the image of a point Z by Z^*.



From properties of angles under this inversion we have:

$$\angle S^*BA = \angle CBS = 90^\circ - \frac{\angle BAC}{2} = 90^\circ - \frac{\angle BAS^*}{2} \implies \measuredangle D^*BA = \measuredangle S^*BA = \measuredangle AS^*B \qquad (\dagger)$$

Angle-chasing using directed angles we then get

$$\begin{aligned}
\measuredangle D^*CP^* &= \measuredangle D^*CE^* \\
&= \measuredangle D^*BE^* && (A, E, D \text{ collinear} \implies D^*CBE^* \text{ cyclic}) \\
&= \measuredangle D^*BL^* \\
&= \measuredangle ABL^* + \measuredangle D^*BA \\
&= \measuredangle ABL^* + \measuredangle AS^*B && (\text{Using } (\dagger)) \\
&= \measuredangle BD^*L^* + \measuredangle AS^*B && (BC \parallel LD \implies AB \text{ tangent to } \odot BL^*D^*) \\
&= \measuredangle BD^*L^* + \measuredangle P^*S^*D^* \\
&= \measuredangle S^*D^*P^* + \measuredangle P^*S^*D^* && (BLPD \text{ cyclic} \implies D^*, P^*, L^* \text{ collinear}) \\
&= \measuredangle CP^*D^* && (\text{Angles in } \triangle P^*S^*D^*)
\end{aligned}$$

so $D^*C = D^*P^*$.

As XA tangent to $\odot ABD$ we get $\odot BCX^*$ is tangent to line CD^* which, combined with the above, gives

$$D^*B \cdot D^*X^* = D^*C^2 = (D^*P^*)^2$$

which means $\odot BP^*X^*$ is tangent to line D^*P^*. Inverting back we get line XP is tangent to $\odot BPD \equiv \omega$ as required.

Solution 13. Let X be the intersection of AS' and the tangent at P to ω. As in other solutions, L, P, S are collinear. Then we have

$$\angle XPA = \angle QPA = \angle QPS + \angle SPA = \angle PBL + \angle SPA = \angle PAE + \angle SPA.$$

From $SS' \parallel AE$ we have $SAES'$ is an isosceles trapezium so $\angle SPA = \angle EAS'$. Combining this with the above gives

$$\angle XPA = \angle PAE + \angle EAS' = \angle PAS' = \angle PAX.$$

Hence $XA = XP$.

Now, as in other solutions, show that $\odot ADB$ is tangent to line $\overline{AXS'}$ from which we get:

$$\text{Pow}_{\omega}(X) = XD \cdot XB = XP^2 = XA^2 = \text{Pow}_{\odot ABD}(X)$$

Thus X lies on the radical axis of ω and $\odot ABD$, but this is exactly line BD.

Marking scheme for Problem 3

Problem 3.
For each integer $k \geq 2$, determine all infinite sequences of positive integers $a_1, a_2, \ldots$ for which there exists a polynomial $P(x) = x^k + c_{k-1}x^{k-1} + \cdots + c_1 x + c_0$, where $c_0, c_1, \ldots, c_{k-1}$ are non-negative integers, such that

$$P(a_n) = a_{n+1}a_{n+2}\cdots a_{n+k}$$

for every integer $n \geq 1$.

Marking scheme

We expect all solutions to follow the same structure in two steps.

1. First, one should prove some properties about the sequence (a_n) (e.g. increasing or unbounded non-decreasing). This part typically only requires that P is injective (or increasing) and satisfies $P(a_n) = a_{n+1}a_{n+2}\cdots a_{n+k}$.
2. Then, control the differences $a_{n+1} - a_n, \ldots, a_{n+k} - a_n$ using the fact that $P(x)$ is a polynomial, and conclude that (a_n) must be an arithmetic progression. This usually involves arguments such as finding estimates using the second-to-largest term of the polynomial, using some "infinitary" pigeonhole principle, etc.

The marking scheme is divided into three regimes. The final score is the maximum among the scores obtained in each regime.

Regime A (≤ 1 point)

The following points are non-additive.

List of observations worth 0 points:

- Proving that $P(x) \geq x^k$ .... 0 **points**
- Proving that $P(x) > P(y)$ for $x > y$ .... 0 **points**
- Proving that $P(x) \neq P(y)$ for $x \neq y$ .... 0 **points**
- Proving that if (a_n) is eventually constant, then it is constant .... 0 **points**
- Proving that if (a_n) is eventually an arithmetic progression, then it is an arithmetic progression from the start .... 0 **points**
- Proving that the sequence $a_n = cn$ satisfies the condition of the problem 0 **points**
- Proving that the sequence $a_n = n+c$ satisfies the condition of the problem 0 **points**
- Proving that the sequence $a_n = c$ satisfies the condition of the problem 0 **points**
- Claiming that non-decreasing arithmetic progressions (a_n) satisfy the condition of the problem without justification .... 0 **points**
- Proving that (a_n) is constant when $P(x) = x^k$ .... 0 **points**

Making partial progress with specific values of k (typically in the case $k = 2$) will be awarded at most one point:

(A.1) For a specific value of k, the sequence (a_n) is either constant or increasing 1 **point**

(A.2) For a specific value of k, the sequence (a_n) is either constant or unbounded non-decreasing . . . . 1 **point**

Identifying *all* the solutions with proof is also worth a point (if a student forgets about constant sequences, the mark is awarded anyway). When the student mistakenly identifies a larger class of sequences (e.g., all arithmetic progressions) as solutions, the point is awarded regardless.

(A.3) Proving that every non-decreasing arithmetic progression (a_n) satisfies the condition of the problem (to get credit, the polynomial $P(x)$ corresponding to the sequence must be written out explicitly) . . . . 1 **point**

Regime B (≤ 6 points)

Regime B is divided into two subsections corresponding to the two main steps of the solution. The items in each subsection are *not* additive, but the two subsections are additive with each other for a maximum of **6 points**.

Specific values of k. If a statement is proved for a particular value of k where, when replacing this specific value of k by a general parameter, the argument immediately generalizes to a complete argument for general value of k, then one point will be deducted from the corresponding item. If the students also claims that the argument immediately generalizes to a complete argument for general value of k, no points will be deducted.

First step (≤ 3 points)

(B.1) The sequence (a_n) is either constant or increasing . . . . 3 **points**

(B.2) The sequence (a_n) is either constant or unbounded non-decreasing . . . . . 3 **points**

(B.3) The sequence (a_n) is non-decreasing . . . . 2 **points**

(B.4) The sequence (a_n) is either (eventually) constant or goes to infinity . . . 2 **points**

(B.5) The sequence (a_n) is either (eventually) constant or unbounded . . . . 1 **point**

(B.6) The sequence (a_n) is either (eventually) constant or contains an increasing subsequence . . . . 1 **point**

(B.7) The sequence (a_n) is either bounded or goes to infinity . . . . 1 **point**

(B.8) There exists infinitely many n with $a_n \leq \min\{a_{n+1}, \ldots, a_{n+k}\}$ . . . . 1 **point**

(B.9) The sequence (a_n) has a non-decreasing subsequence . . . . 0 **points**

(B.10) For every n we have $a_n \leq \max\{a_{n+1}, \ldots, a_{n+k}\}$ . . . . 0 **points**

(B.11) For every n we have $a_n \leq (a_{n+1} + \cdots + a_{n+k})/k$ . . . . 0 **points**

(B.12) For every n we have $a_n \leq \sqrt[k]{a_{n+1} \cdots a_{n+k}}$ . . . . 0 **points**

(B.13) If the sequence (a_n) is eventually constant, then it is constant . . . . 0 **points**

(B.14) If the sequence (a_n) is non-decreasing and not (eventually) constant, then it is strictly increasing . . . . 0 **points**

Second step (≤ 4 points)

For this part of the marking scheme, students may assume any combination of the facts mentioned in the first step, in particular, that (a_n) is either constant or increasing, or any weaker statement, for a maximum of **4 points**. To gain full points, students must *explicitly* state the assumption, given that they do not provide a proof.

(B.15) The sequence (a_n) is an arithmetic progression .... 4 **points**

(B.16) The sequence (a_n) is eventually an arithmetic progression .... 3 **points**

(B.17) There exists a constant b for which $(a_{n+1} - a_n) + \cdots + (a_{n+k} - a_n) = b$ for all sufficiently large n .... 2 **points**

(B.18) We have $a_{n+k+1} - a_{n+1} = k(a_{n+1} - a_n)$ for infinitely many n .... 2 **points**

(B.19) There exist integers $b_1, \ldots, b_k$ with $P(x) = (x+b_1)\cdots(x+b_k)$ such that $a_{n+i} = a_n+b_i$ for all $1 \leq i \leq k$ for infinitely many n .... 2 **points**

(B.20) There exist integers $b_1, \ldots, b_k$ such that $a_{n+i} = a_n + b_i$ for all $1 \leq i \leq k$ for infinitely many n .... 1 **point**

(B.21) There exists a constant b for which $(a_{n+1} - a_n) + \cdots + (a_{n+k} - a_n) = b$ for infinitely many n .... 1 **point**

(B.22) The differences $a_{n+1} - a_n$ are bounded for infinitely many n .... 1 **point**

(B.23) The sums of differences $(a_{n+1} - a_n) + \cdots + (a_{n+k} - a_n)$ are bounded for infinitely many n .... 1 **point**

(B.24) If (a_n) is unbounded and $P(a_n) = (a_n + b_1)\cdots(a_n + b_k)$ for infinitely many n, then $P(x) = (x + b_1)\cdots(x + b_k)$ .... 0 **points**

(B.25) Other general facts about polynomials .... 0 **points**

Regime C (≥ 6 points)

At most one point will be deducted from a full solution for a minor flaw such as

(C.1) Only proving that (a_n) is *eventually* a non-decreasing arithmetic progression -1 **point**

(C.2) Absence of verification that all non-decreasing arithmetic progressions are indeed solutions .... -1 **point**

(C.3) Algebraic mistake in verifying that all non-decreasing arithmetic progressions are indeed solutions, resulting in a wrong answer .... -1 **point**

(C.4) Claiming that only increasing arithmetic progressions are solutions (i.e., forgetting about constant sequences) .... -1 **point**

(C.5) Solving the problem completely for a particular value of k where, when replacing this specific value of k by a general parameter, the argument immediately generalizes to a complete solution for general value of k .... -1 **point**

(C.6) Solving the problem completely for a particular value of k where, when replacing this specific value of k by a general parameter, the argument immediately generalizes to a complete solution for general value of k *and claiming* so .... -0 **points**

(C.7) Making a typo (e.g., writing down the polynomial $P(x) = x(x+d)\cdots(x+(k-1)d)$), when it is clear that the student understands the mathematics ...... -0 **points**

(C.8) Other deductions might be added to this list as imperfections are discovered in students' scripts .... -1 **point**

Solutions

Remarks. Solution 1 and its variants first establish that (a_n) is increasing and then finish using polynomial considerations. Solutions 2 and 3 only use a weaker version of (a_n) being increasing.

The following arguments and observations are implicit in the solutions given below.

Suppose the sequence (a_n) is an arithmetic progression with common difference $d \geq 0$. Then it satisfies the condition with

$$P(x) = (x+d)\cdots\big(x+kd\big).$$

This settles one direction. Now suppose (a_n) is a sequence satisfying the condition. We will show that it is a non-decreasing arithmetic progression.

Since $P(x)$ has non-negative integer coefficients, it is strictly increasing on the positive real line. In particular, it holds that, for any positive integer x, y,

$$P(x) < P(y) \quad \Longleftrightarrow \quad x < y.$$

Furthermore, if the sequence (a_n) is eventually constant, then $P(x) = x^k$ and the sequence (a_n) is actually constant. Indeed, if $P(x)$ were not the polynomial x^k, then $P(a_n) = a_{n+1}\cdots a_{n+k}$ cannot be satisfied for n such that $a_n = \cdots = a_{n+k}$. By a descending induction, we conclude that (a_n) is constant. Thus we can restrict to the case (a_n) is not eventually constant.

Solution 1. We assume that (a_n) is not eventually constant.

Step 1. The first goal is to show that the sequence must be *increasing*, i.e. $a_n < a_{n+1}$ for all $n \geq 1$.

First, by comparing the two equalities

$$P(a_n) = a_{n+1}a_{n+2}\cdots a_{n+k},$$
$$P(a_{n+1}) = a_{n+2}\cdots a_{n+k}a_{n+k+1},$$

we observe that

$$a_n < a_{n+1} \quad \Longleftrightarrow \quad P(a_n) < P(a_{n+1}) \quad \Longleftrightarrow \quad a_{n+1} < a_{n+k+1}, \tag{1}$$
$$a_n > a_{n+1} \quad \Longleftrightarrow \quad P(a_n) > P(a_{n+1}) \quad \Longleftrightarrow \quad a_{n+1} > a_{n+k+1}, \tag{2}$$
$$a_n = a_{n+1} \quad \Longleftrightarrow \quad P(a_n) = P(a_{n+1}) \quad \Longleftrightarrow \quad a_{n+1} = a_{n+k+1}. \tag{3}$$

Claim 1. $a_n \leq a_{n+1}$ for all $n \geq 1$.

Proof. Suppose, to the contrary, that $a_{n(0)-1} > a_{n(0)}$ for some $n(0) \geq 2$. We will give an infinite sequence of positive integers $n(0) < n(1) < \cdots$ satisfying

$$a_{n(i)-1} > a_{n(i)} \text{ and } a_{n(i)} > a_{n(i+1)}.$$

Then $a_{n(0)}, a_{n(1)}, a_{n(2)}, \ldots$ is an infinite decreasing sequence of positive integers, which is absurd.

We construct such a sequence inductively. If we have chosen $n(i)$, then we let $n(i+1)$ be the smallest index larger than $n(i)$ such that $a_{n(i)} > a_{n(i+1)}$. Note that such an index always exists and satisfies $n(i)+1 \leq n(i+1) \leq n(i)+k$ because $a_{n(i)} > a_{n(i)+k}$ by (2). We need to check that $a_{n(i+1)-1} > a_{n(i+1)}$. This is immediate if $n(i+1) = n(i)+1$ by construction. If $n(i+1) \geq n(i)+2$, then $a_{n(i+1)-1} \geq a_{n(i)}$ by minimality of $n(i+1)$, and so $a_{n(i+1)-1} \geq a_{n(i)} > a_{n(i+1)}$. □

We are now ready to prove that the sequence a_n is increasing. Suppose $a_n = a_{n+1}$ for some $n \geq 1$. Then we also have $a_{n+1} = a_{n+k+1}$ by (3), and since the sequence is non-decreasing we have $a_n = a_{n+1} = a_{n+2} = \cdots = a_{n+k+1}$. We repeat the argument for $a_{n+k} = a_{n+k+1}$ and get that the sequence is eventually constant, which contradicts our assumption. Hence

$$a_n < a_{n+1} \text{ for all } n \geq 1.$$

Step 2. The next and final goal is to prove that the sequence a_n is an *arithmetic progression*. Observe that we can make differences of terms appear as follows

$$\begin{aligned} P(a_n) &= a_{n+1}a_{n+2}\cdots a_{n+k} \\ &= \big(a_n + (a_{n+1} - a_n)\big)\big(a_n + (a_{n+2} - a_n)\big)\cdots\big(a_n + (a_{n+k} - a_n)\big). \end{aligned}$$

We will prove that, for n large enough, the sum

$$(a_{n+1} - a_n) + (a_{n+2} - a_n) + \cdots + (a_{n+k} - a_n)$$

is equal to the coefficient c_{k-1} of the term x^{k-1} in P. The argument is based on the following claim.

Claim 2. There exists a bound A with the following properties:

1. If $(e_1, \ldots, e_k)$ is a k-tuple of positive integers with $e_1 + \cdots + e_k > c_{k-1}$, then for every $x \geq A$ we have $P(x) < (x + e_1)(x + e_2)\cdots(x + e_k)$.

2. If $(e_1, \ldots, e_k)$ is a k-tuple of positive integers with $e_1 + \cdots + e_k < c_{k-1}$, then for every $x \geq A$ we have $P(x) > (x + e_1)(x + e_2)\cdots(x + e_k)$.

Proof. It suffices to show parts 1 and 2 separately, because then we can take the maximum of two bounds.

We first show part 1. For each single $(e_1, \ldots, e_k)$ such a bound A exists since

$$P(x) - (x + e_1)(x + e_2)\cdots(x + e_k) = (c_{k-1} - (e_1 + \cdots + e_k))x^{k-1} + (\text{terms of degree} \leq k - 2)$$

has negative leading coefficient and hence takes negative values for x large enough.

Suppose A is a common bound for all tuples $e = (e_1, \ldots, e_k)$ satisfying $e_1 + \cdots + e_k = c_{k-1} + 1$ (note that there are only finitely many such tuples). Then, for any tuple $e' = (e'_1, \ldots, e'_k)$ with $e'_1 + \cdots + e'_k > c_{k-1}$, there exists a tuple $e = (e_1, \ldots, e_k)$ with $e_1 + \cdots + e_k = c_{k-1} + 1$ and $e'_i \geq e_i$, and then the inequality for e' follows from the inequality for e.

We can show part 2 either in a similar way, or by using that there are only finitely many such tuples. □

Take A satisfying the assertion of Claim 2, and take N such that $n \geq N$ implies $a_n \geq A$. Then for each $n \geq N$, we have

$$(a_{n+1} - a_n) + \cdots + (a_{n+k} - a_n) = c_{k-1}.$$

By taking the difference of this equality and the equality for $n + 1$, we obtain

$$a_{n+k+1} - a_{n+1} = k(a_{n+1} - a_n)$$

for every $n \geq N$.

We conclude using an extremal principle. Let $d = \min\{a_{n+1} - a_n \mid n \geq N\}$, and suppose it is attained at some index $n \geq N$. Since

$$kd = k(a_{n+1} - a_n) = a_{n+k+1} - a_{n+1} = \sum_{i=1}^{k}(a_{n+i+1} - a_{n+i})$$

and each summand is at least d, we conclude that d is also attained at $n+1, \ldots, n+k$, and inductively at all $n' \geq n$. We see that the equation $P(x) = (x + d)(x + 2d)\cdots(x + kd)$ is true for infinitely many values of x (all $a_{n'}$ for $n' \geq n$), hence this is an equality of polynomials. Finally we use (backward) induction to show that $a_{n+1} - a_n = d$ for every $n \geq 1$.

Solution 1.a. We provide an alternative proof of Claim1.

Proof (Alternative proof of Claim 1). If $a_{n_0-1} \geq a_{n_0}$, then

$$a_{n_0}a_{n_0+1}\cdots a_{n_0+k-1} = P(a_{n_0-1}) \geq P(a_{n_0}) = a_{n_0+1}a_{n_0+2}\cdots a_{n_0+k}.$$

So $a_{n_0} \geq a_{n_0+k}$. Let a_{n_1} be the first term after a_{n_0} that is not larger than a_{n_0}; so $n_1 \leq n_0 + k$ and $a_{n_1-1} \geq a_{n_1}$.

By the same argument as above, we find $n_2 \in (n_1, n_1 + k]$ such that $a_{n_2-1} \leq a_{n_2}$. Continue this way, we see that $a_{n_0} \geq a_{n_1} \geq a_{n_2} \geq \cdots$. Thus all a_{n_i}'s and $P(a_{n_i})$'s are bounded. But moreover, as $n_i - n_{i-1} \leq k$; so all a_j's are bounded. (Indeed, for each a_j, one can find n_i such that $j \in [n_i + 1, n_i + k]$ and thus $a_j \leq P(a_{n_i})$.) Let a_M be the maximal element in the sequence. So $P(a_M) = a_{M+1}a_{M+2}\cdots a_{M+k} \leq a_M^k$. So we must have $P(x) = x^k$ and $a_{M+1} = a_{M+2} = \cdots = a_{M+k}$. Reverse induction shows that a_n is a constant sequence. (Obviously, constant sequence satisfies the condition of the problem.)

Solution 1.b. Step 2. We also provide an alternative way to finish the solution. Now, we assume that a_n is strictly increasing.

Suppose that $P(x) = x^k+c_{k-1}x^{k-1}+\cdots+c_0$. Put $C = c_0+\cdots+c_{k-1}+1$ so that $P(m) \leq m^k+Cm^{k-1}$ for any $m \in \mathbb{Z}_{>0}$.

Observer that

$$a_{n+1}a_{n+2}\cdots a_{n+k} = P(a_n) \leq a_n^k + Ca_n^{k-1}.$$

So each $a_{n+i} \leq a_n + C$ for every $i = 1, \ldots, k$ because every $a_{n+j} > a_n$.

Now if $a_n > C^k \cdot 2^k$ (and obviously $a_n > d_i$ for each i), we write $a_{n+i} = a_n + \delta_i$ with every $\delta_i \in (0, C)$ and see that

$$\begin{aligned} a_n^k + c_{k-1}a_n^{k-1} + \cdots + c_0 &= P(a_n) = a_{n+1}a_{n+2}\cdots a_{n+k} \\ &= (a_n + \delta_1)(a_n + \delta_2)\cdots(a_n + \delta_k). \\ &= a_n^k + (\delta_1 + \cdots + \delta_k)a_n^{k-1} + \cdots + \delta_1\cdots\delta_k. \end{aligned}$$

From this, we deduce that c_i is equal to the ith elementary symmetric polynomial in $\delta_1, \ldots, \delta_k$. (A easy way to see this is to view both sides of the above equality as N-based numbers.)

It follows that

$$P(x) = (x + \delta_1)\cdots(x + \delta_k)$$

(In particular, these δ_i's are *independent* of n.) So, when $a_m > C^k \cdot 2^k$, we always have $a_{m+i} = a_m + \delta_i$ for each $i = 1, \ldots, k$. Considering the same for a_{m+1}, we deduce that $a_{m+i} = a_{m+1} + \delta_{i-1} + \delta_1$. Thus $\delta_i = i\delta_1$, and thus $a_n, a_{n+1}, \ldots$ form an arithmetic progression (when n is sufficiently large).

By an easy induction backwards, we see that the entire sequence a_n is an arithmetic progression.

Solution 1.c. Step 2. We provide yet another alternative way to finish the solution. Define $d_i(n) := a_{n+i} - a_n$ for $i = 1, \ldots, k$. Because (a_n) is increasing, it holds that $0 < d_1 < \cdots < d_k$. We write

$$P(a_n) = (a_n + d_1)(a_n + d_2)\cdots(a_n + d_k) \geq (a_n)^k + d_k(a_n)^{k-1}.$$

If d_k were greater than the coefficient c_{k-1} of x_{k-1} in P, we would get a contradiction for large enough a_n. Thus $d_k(n)$ is bounded, and so are all other d_i.

Thus, some tuple $(d_1(n), \ldots, d_k(n))$ occurs infinitely often. Denote it by $(D_1, \ldots, D_k)$ and hence

$$P(x) = (x + C_1)\cdots(x + C_k)$$

as polynomials. In particular, no other tuple can occur infinitely many times. Hence the difference $a_{n+1} - a_n$ is eventually constant to some number $D_1 = d$, and $P(x) = (x + d)\cdots(x + kd)$. We conclude a in the other solutions.

Solution 2. We assume that (a_n) is not eventually constant. In this solution, we first prove an alternative version of Claim 1.

Claim 3. There exist infinitely many $n \geq 1$ with

$$a_n \leq \min\{a_{n+1}, \ldots, a_{n+k}\}.$$

Proof. Suppose not, then for all but finitely many $n \geq 1$, it holds that $a_n > \min\{a_{n+1}, \ldots, a_{n+k}\}$. Hence for all large enough n, there always exist some $1 \leq l \leq k$ such that $a_n > a_{n+l}$. This induces an infinite decreasing sequence $a_n > a_{n+l_1} > a_{n+l_2} > \cdots$ of positive integers, which is absurd. □

We use Claim 3 to quickly settle the case $P(x) = x^k$. In that case, for every n with $a_n \leq \min\{a_{n+1}, \ldots, a_{n+k}\}$, since $a_{n+1} \cdots a_{n+k} = a_n^k$, it implies $a_n = a_{n+1} = \cdots = a_{n+k}$. This shows that the sequence is eventually constant, which contradicts our assumption.

From now on, assume

$$P(x) > x^k \text{ for all } x > 0.$$

Claim 4. For every $M > 0$, there exists some $N > 0$ such that $a_n > M$ for all $n > N$.

Proof. Suppose there exists some $M > 0$, such that $a_n \leq M$ for infinitely many n. For each i with $a_i \leq M$, we consider the k-tuple $(a_{i+1}, \ldots, a_{i+k})$. Then each of the terms in the k-tuple is bounded from above by $P(a_i)$, and hence by $P(M)$ too. Since the number of such k-tuples is bounded by $P(M)^k$, we deduce by the Pigeonhole Principle that there exist some indices $i < j$ such that $(a_{i+1}, \ldots, a_{i+k}) = (a_{j+1}, \ldots, a_{j+k})$. Since a_n is uniquely determined by the k terms before it, we conclude that $a_{i+k+1} = a_{j+k+1}$ must hold, and similarly $a_{i+l} = a_{j+l}$ for all $l \geq 0$, so the sequence is eventually periodic, for some period $p = j - i$.

Take K such that $a_n = a_{n+p}$ for every $n \geq K$. Then, by taking the products of the inequalities

$$a_n^k < P(a_n) = a_{n+1} \cdots a_{n+k}$$

for $K \leq n \leq K + p - 1$, we obtain

$$\begin{aligned}
\prod_{n=K}^{K+p-1} a_n^k &< \prod_{n=K}^{K+p-1} a_{n+1} \cdots a_{n+k} \\
&= a_{K+1} a_{K+2}^2 \cdots a_{K+k-1}^{k-1} \left(\prod_{n=K+k}^{K+p} a_n \right)^k a_{K+p+1}^{k-1} \cdots a_{K+p+k-2}^2 a_{K+p+k-1} \\
&= \left(\prod_{n=K}^{K+p-1} a_n \right)^k \qquad \text{(by periodicity)},
\end{aligned}$$

which is a contradiction. □

Write $P(x) = x^k + c_{k-1}x^{k-1} + Q(x)$, where $Q(x)$ is of degree at most $k - 2$. Take M such that $x > M$ implies $x^{k-1} > Q(x)$.

Claim 5. There exist non-negative integers $b_1, \cdots, b_k$ such that $P(x) = (x + b_1) \cdots (x + b_k)$, and such that, for infinitely many $n \geq 1$, we have $a_{n+i} = a_n + b_i$ for every $1 \leq i \leq k$.

Proof. By Claims 3 and 4, there are infinitely many n such that

$$a_n > M \text{ and } a_n \leq \min\{a_{n+1}, \ldots, a_{n+k}\}.$$

Call such indices n to be *good*. We claim that if n is a good index then

$$\max\{a_{n+1}, \ldots, a_{n+k}\} \leq a_n + c_{k-1}.$$

Indeed, if $a_{n+i} \geq a_n + c_{k-1} + 1$, then together with $a_n \leq \min\{a_{n+1}, \dots, a_{n+k}\}$ and $a_n^{k-1} > Q(a_n)$, we have

$$a_n^k + (c_{k-1}+1)a_n^{k-1} > a_n^k + c_{k-1}a_n^{k-1} + Q(a_n) = P(a_n) \geq (a_n + c_{k-1} + 1)a_n^{k-1},$$

a contradiction.

Hence for each good index n, we may write $a_{n+i} = a_n + b_i$ for all $1 \leq i \leq k$ for some choices of $(b_1, \dots, b_k)$ (which may depend on n) and $0 \leq b_i \leq c_{k-1}$. Again by Pigeonhole Principle, some k-tuple $(b_1, \dots, b_k)$ must be chosen for infinitely such good indices n. This means that the equation $P(a_n) = (a_n + b_1) \cdots (a_n + b_k)$ is satisfied by infinitely many good indices n. By Claim 4, a_n is unbounded among these a_n's, hence $P(x) = (x + b_1) \cdots (x + b_k)$ must hold identically. ☐

Claim 6. We have $b_i = ib_1$ for all $1 \leq i \leq k$.

Proof. Call an index n *excellent* if $a_{n+i} = a_n + b_i$ for every $1 \leq i \leq k$. From Claim 5 we know there are infinitely many excellent n.

We first show that for any pair $1 \leq i < j \leq k$ there is $1 \leq l \leq k$ such that $b_j = b_i + b_l$. Indeed, for such i and j and for excellent n, $a_n + b_j$ (which is equal to a_{n+j}) divides $P(a_{n+i}) = \prod_{l=1}^{k}(a_n + b_i + b_l)$, and hence divides $\prod_{l=1}^{k}(b_i + b_l - b_j)$. Since $a_n + b_j$ is unbounded among excellent n, we have $\prod_{l=1}^{k}(b_i + b_l - b_j) = 0$, hence there is l such that $b_j = b_i + b_l$.

In particular, $b_j = b_i + b_l \geq b_i$, i.e. $(b_1, \dots, b_k)$ is non-decreasing.

Suppose $b_1 = 0$ and n is an excellent number. In particular, it holds that $a_n = a_{n+1}$. Moreover, since

$$a_{n+k+1}P(a_n) = a_{n+1} \cdots a_{n+k+1} = a_{n+1}P(a_{n+1}),$$

we have $a_n = a_{n+1} = a_{n+k+1}$, which divides $P(a_{n+i}) = \prod_{l=1}^{k}(a_n + b_i + b_l)$ for each $1 \leq i \leq k$. Hence a_n divides $\prod_{l=1}^{k}(b_i + b_l)$. By the same reasoning, we have $b_i + b_l = 0$ for some l, but since $b_i, b_l \geq 0$ we obtain $b_i = 0$ for each $1 \leq i \leq k$.

Now suppose $b_1 \geq 1$. Then, for each $1 \leq i < j \leq k$, we have $b_j - b_i = b_l \geq b_1 \geq 1$, hence $(b_1, \dots, b_k)$ is strictly increasing. Therefore, the $k-1$ elements $b_2 < b_3 < \cdots < b_k$ are exactly equal to $b_1 + b_1 < \cdots < b_1 + b_{k-1}$, since they cannot be equal to $b_1 + b_k$. This gives $b_i = ib_1$ for all $1 \leq i \leq k$ as desired. ☐

Claim 6 implies $P(x) = (x+d)(x+2d)\cdots(x+kd)$ for some $d \geq 1$, and there are infinitely many indices n with $a_{n+i} = a_n + id$ for $1 \leq i \leq k$. By backwards induction, $P(a_{n-1}) = a_n \cdots a_{n+k-1}$ implies $a_{n-1} = a_n - d$, and so on. Thus $a_1, \dots, a_n$ forms an arithmetic progression with common difference d. Since n can be arbitrarily large, the whole sequence is an arithmetic progression too, as desired.

Solution 3.

We assume that a_n is not eventually constant. We start with some general observations on sequences of integers.

Claim 7.

(1) If a sequence $x_1, x_2, x_3, \dots$ consists of integers and satisfies $x_n = \frac{1}{k}(x_{n-1} + \cdots + x_{n-k})$ for all $n > k$, then it is a constant sequence.

(2) If a sequence $x_1, x_2, x_3, \dots$ consists of integers and satisfies $x_n = \frac{1}{k}(x_{n-1} + \cdots + x_{n-k}) + c$ for all $n > k$, where c is a constant, then it is an arithmetic progression.

Proof.

(1) We define the *variance* of $k+1$ variables $y_1, \dots, y_{k+1}$ to be

$$V(y_1, \dots, y_{k+1}) := \sum_{1 \le i < j \le k+1} (y_i - y_j)^2.$$

We claim that $V(x_{n-k-1}, \dots, x_{n-1}) \ge V(x_{n-k}, \dots, x_n)$ and that equality holds if and only if $x_n = x_{n-k-1}$. Indeed, V is symmetric and

$$V(t, x_{n-k}, \dots, x_{n-1}) = kt^2 - 2\left(\sum_{i=1}^{k} x_{n-i}\right) t + c',$$

where c' is a constant independent of t. The right-hand side attains its minimum at $t = \frac{1}{k}\sum_{i=1}^{k} x_{n-i} = x_n$. If all inequalities in the chain are actually equalities

$$V(x_{n-k-1}, \dots, x_{n-1}) \ge V(x_{n-k}, \dots, x_n) \ge \cdots \ge V(x_n, \dots, x_{n+k}),$$

then $x_{n-k-1} = x_n, x_{n-k} = x_{n+1}, \dots, x_{n-1} = x_{n+k}$ and all the terms are actually equal. So, if $x_{n-k-1}, x_{n-k}, \dots, x_{n+k}$ are not all equal, then at least one inequality is strict.

Since the variance of integers is a positive integer, the sequence $V(x_n, \dots, x_{n+k})$ is eventually constant, which, by above, implies that the sequence x_n itself is eventually constant. Backtracking shows that x_n is constant from the beginning.

(2) Apply part 1 to the sequence $x_n - x_{n+1}$. □

Claim 8. Either the sequence a_n is eventually constant, or $\lim_{n\to\infty} a_n = \infty$.

Proof. For every n, we have

$$\max\{a_{n+1}, \dots, a_{n+k}\} \ge (a_{n+1} \cdots a_{n+k})^{1/k} = P(a_n)^{1/k} \ge a_n.$$

Consider the following subsequence. Let $n(0)$ be any index. Inductively, let $n(i+1)$ be one of the indices $n(i)+1, \dots, n(i)+k$ for which $a_{n(i+1)} = \max\{a_{n(i)+1}, \dots, a_{n(i)+k}\}$. We obtain a non-decreasing subsequence $a_{n(0)}, a_{n(1)}, a_{n(2)}, \dots$ where all adjacent indices are at most k apart, i.e. $n(i+1) - n(i) \le k$. Then either $a_{n(i)}$ is eventually constant or $\lim_{i\to\infty} a_{n(i)} = \infty$.

First suppose $a_{n(i)}$ is eventually constant. Take f and i_0 such that $a_{n(i)} = f$ for every $i \ge i_0$. By the construction of $n(i)$, we have $a_{n(i)+1}, \dots, a_{n(i)+k} \le a_{n(i+1)} = f$ for every $i \ge i_0$. We have, for every $i \ge i_0$,

$$f^k = a_{n(i)}^k \le P(a_{n(i)}) = a_{n(i)+1} \cdots a_{n(i)+k} \le a_{n(i+1)}^k = f^k,$$

which implies $a_{n(i)+1} = \cdots = a_{n(i)+k} = f$. Therefore a_n is eventually constant.

Next suppose $\lim_{i\to\infty} a_{n(i)}$. Pick c such that $P(x) \le (x+c)^k$. For every n, let i be the number such that $n(i-1) \le n < n(i)$; notice that $n+1 \le n(i) \le n+k$. Then

$$(a_n + c)^k \ge P(a_n) = a_{n+1} \cdots a_{n+k} \ge 1^{k-1} a_{n(i)} = a_{n(i)},$$

hence $a_n \ge a_{n(i)}^{1/k} - c$. Since $a_{n(i)}^{1/k} \to \infty$ we also have $a_n \to \infty$. □

Since we assumed that a_n is not eventually constant, we will thereafter assume that

$$\lim_{n\to\infty} a_n = \infty.$$

Call n to be *nice* if $a_n < \min\{a_{n+1}, \dots, a_{n+k}\}$. Since $\lim_{n\to\infty} a_n = \infty$, there are infinitely many nice integers. Indeed, for large enough M, $\max\{n \in \mathbb{N} \mid a_n < M\}$ is nice .

Claim 9. There exists $A > 0$ such that $a_{n+1}, \dots, a_{n+k} \in (a_n, a_n + A]$ for all nice n.

Proof. For the sake of contradiction, assume that for all positive integers A there exists nice n such that $\max\{a_{n+1}, \dots, a_{n+k}\} > a_n + A$, hence $P(a_n) = a_{n+1} \cdots a_{n+k} > a_n^{k-1}(a_n + A)$, which means $P(x) > x^{k-1}(x + A)$ for some $x > 0$. This cannot be true for all positive integers A as the coefficient of x^{k-1} in $P(x)$ is finite. □

Therefore for all nice n,

$$a_{n+1} - a_n, a_{n+2} - a_n, \dots, a_{n+k} - a_n \in [1, A].$$

By Pigeonhole Principle, there exists $(b_1, \dots, b_k) \in [1, A]^k$ such that, for infinitely many n (now called *extra nice*), it holds that $(a_{n+1} - a_n, \dots, a_{n+k} - a_n) = (b_1, \dots, b_k)$, i.e.

$$P(a_n) = a_{n+1} \cdots a_{n+k} = (a_n + b_1) \cdots (a_n + b_k),$$

but since there are infinitely many extra nice n and $\lim_{n\to\infty} a_n = \infty$, we must have

$$P(x) = (x + b_1) \cdots (x + b_k).$$

Let c_{k-1} be the coefficient of the term x^{k-1} in P. It holds that $c_{k-1} = b_1 + \cdots + b_k$.

Claim 10. Define $b_0, b_{-1}, b_{-2}, \dots$ by the (backward) recurrence formula

$$b_n := \frac{b_{n+1} + \cdots + b_{n+k} - c_{k-1}}{k},$$

for $n \le 0$. Then, for every integer $N \le 0$, there are infinitely many $n > -N$ satisfying

$$a_{n+i} = a_n + b_i \quad (N \le i \le k).$$

Proof. We use (descending) induction on N. For $N = 0$, we take extra nice indices n. Suppose the claim is true for $N + 1$. Write $a_{n+N} = a_n + z$, where z is an integer that might depend on n. Then, comparing the "coefficient of a_n^{k-1}" in the equality

$$\begin{aligned}(a_n + z)^k + c_{k-1}(a_n + z)^{k-1} + (\text{terms of degree} \le k-2) &= P(a_{n+N}) \\ &= a_{n+N+1} \cdots a_{n+N+k} = (a_n + b_{N+1}) \cdots (a_n + b_{N+k}),\end{aligned}$$

we obtain $kz + c_{k-1} = b_{N+1} + \cdots + b_{N+k}$ (because a_n can be large enough). By definition of the sequence b_n, we conclude that $z = b_N$. □

By Claim 10, we have $b_N = a_{n+N} - a_n$ for infinitely many n, in particular b_N is an integer. Applying part 2 Claim 7 to the sequence b_{k-n}, we conclude that the sequence b_{k-n} $(n \ge 0)$ is an arithmetic progression. In particular, $(b_0 = 0, b_1, \dots, b_k)$ is an arithmetic progression.

Now take an extra nice m, then $a_m, \dots, a_{m+k}$ is an arithmetic progression, which, together with $P(a_n) = a_{n+1} \cdots a_{n+k}$, recursively defines all terms before and after it. Therefore a_n is an arithmetic progression.

Problem 4 marking scheme

Problem 4.
Let $x_1, x_2, \ldots, x_{2023}$ be pairwise different real positive numbers such that

$$a_n = \sqrt{(x_1 + x_2 + \cdots + x_n)\left(\frac{1}{x_1} + \frac{1}{x_2} + \cdots + \frac{1}{x_n}\right)}$$

is an integer for every $n = 1, 2, \ldots, 2023$. Prove that $a_{2023} \geq 3034$.

Marking scheme

The marking scheme consists of two parts: Part A, which is worth 1 point, and Part B, which is worth 6 points. The points for the two parts are additive, hence adds up to a complete mark of 7 points. However, within each part, the items are not additive.

- Part A: if an increasing sequence of positive integers $a_1, a_2, \ldots$ has the property that $(a_{n+1} - a_n, a_{n+2} - a_{n+1}) \neq (1, 1)$ for all n, then $a_{2023} \geq 3034$.
- Part B: if $a_{n+1} - a_n = 1$ and $a_{n+2} - a_{n+1} = 1$, then $x_{n+2} = x_{n+1}$ (which contradicts the assumption that the real numbers x_i are pairwise different).

If a student gets 0 points for both parts, then they may obtain at most 1 point as the final score by making certain observations. This is described in the "Zero or one regime" section.

For a minor algebraic mistake that does not impact the validity of the argument, at most 1 point is deducted.

Part A (≤ 1 point)

A student will obtain a maximum of 1 point according to the progress the student makes.

(A.1) Observing that if $a_{n+2} \geq a_n + 3$ for all n, then $a_{2023} \geq 3034$ .... 1 **point**

(A.2) Observing that if $a_{n+1} = a_n + 1$ implies $a_{n+2} \geq a_{n+1} + 2$, then $a_{2023} \geq 3034$ .... 1 **point**

(A.3) Observing that if $a_{n+1} = a_n + 1$ and $a_{n+2} = a_{n+1} + 1$ together lead to a contradiction, then $a_{2023} \geq 3034$ .... 1 **point**

(A.4) Writing down the expression $a_{n+2} \geq a_n + 3$ .... 0 **points**

Part B (≤ 6 points)

A student will obtain the following points for items in the following list. Partial points are not additive. For all items, the variable n for the index can always be shifted to give an equivalent item; for example, item (B.9) is equivalent to giving an algebraic characterization of $a_n = a_{n-1} + 1$.

(B.1) Proving that $a_{n+2} \geq a_n + 3$ for all n .... 6 **points**

(B.2) Proving that $a_{n+1} = a_n + 1$ and $a_{n+2} = a_{n+1} + 1$ together lead to a contradiction . 6 **points**

(B.3) Proving that $a_{n+1} = a_n + 1$ and $a_{n+2} = a_{n+1} + 1$ together imply $x_{n+2} = x_{n+1}$ ..... 6 **points**

(B.4) Proving that $a_{n+2} = a_n + 2$ implies $x_{n+2} = x_{n+1}$ .... 6 **points**

(B.5) Obtaining one of the items (B.1)–(B.4) for a specific value of $n \geq 2$ ...... $\min(n, 5)$ **points**

(B.6) Obtaining one of the items (B.1)–(B.4) for a specific value of n and a specific value of a_n (e.g., proving that $(a_2, a_3, a_4) = (3, 4, 5)$ leads to a contradiction) .... 0 **points**

(B.7) Obtaining both (B.9) and (B.13) .... 3 **points**

(B.8) Obtaining both (B.10) and (B.13) .... 2 **points**

(B.9) Giving an algebraic characterization of $a_{n+1} = a_n + 1$, such as

$$\frac{1}{x_{n+1}}(x_1 + \cdots + x_n) = x_{n+1}\left(\frac{1}{x_1} + \cdots + \frac{1}{x_n}\right),$$

or

$$x_{n+1} = \sqrt{\frac{x_1 + \cdots + x_n}{\frac{1}{x_1} + \cdots + \frac{1}{x_n}}},$$

or an equivalent expression obtained by moving terms around .... 2 **points**

- The characterizations

$$\frac{1}{x_{n+1}}(x_1 + \cdots + x_n) + x_{n+1}\left(\frac{1}{x_1} + \cdots + \frac{1}{x_n}\right) = 2\sqrt{(x_1 + \cdots + x_n)\left(\frac{1}{x_1} + \cdots + \frac{1}{x_n}\right)}$$

or

$$(a_n + 1)^2 = a_n^2 + 1 + \frac{1}{x_{n+1}}(x_1 + \cdots + x_n) + x_{n+1}\left(\frac{1}{x_1} + \cdots + \frac{1}{x_n}\right)$$

do *not* count.

(B.10) Obtaining item (B.9) for a specific value of $n \geq 2$ .... 1 **point**

(B.11) Obtaining item (B.9) for a specific value of n and a specific value of a_n .... 0 **points**

(B.12) Attempting to algebraically analyze the difference $a_{n+1} - a_n$ .... 0 **points**

- An example of an attempt at an algebraic analysis is showing that

$$a_{n+1}^2 = a_n^2 + 1 + \frac{1}{x_{n+1}}(x_1 + \cdots + x_n) + x_{n+1}\left(\frac{1}{x_1} + \cdots + \frac{1}{x_n}\right)$$

(B.13) Considering the two conditions $a_{n+2} - a_{n+1} = 1$ and $a_{n+1} - a_n = 1$ *at the same time*, or combining the two conditions .... 1 **point**

- Obtaining either (A.2) or (A.3) automatically results in obtaining this item.

(B.14) Attempting to algebraically analyze the two differences $a_{n+1} - a_n$ and $a_{n+2} - a_{n+1}$ at the same time .... 0 **points**

- An example at such an attempt is proving both

$$a_{n+1}^2 = a_n^2 + 1 + \frac{1}{x_{n+1}}(x_1 + \cdots + x_n) + x_{n+1}\left(\frac{1}{x_1} + \cdots + \frac{1}{x_n}\right)$$

and

$$a_{n+2}^2 = a_{n+1}^2 + 1 + \frac{1}{x_{n+1}}(x_1 + \cdots + x_{n+1}) + x_{n+2}\left(\frac{1}{x_1} + \cdots + \frac{1}{x_{n+1}}\right).$$

(B.15) Obtaining both items (B.16) and (B.17) . . . . 2 **points**

(B.16) Attempting to algebraically analyze the difference $a_{n+2} - a_n$ in a meaningful way . 1 **point**

- An example of an attempt at an algebraic analysis is showing that

$$
\begin{aligned}
a_{n+2}^2 = a_n^2 &+ (x_{n+1} + x_{n+2})\left(\frac{1}{x_1} + \cdots + \frac{1}{x_n}\right) \\
&+ (x_1 + \cdots + x_n)\left(\frac{1}{x_{n+1}} + \frac{1}{x_{n+2}}\right) + (x_{n+1} + x_{n+2})\left(\frac{1}{x_{n+1}} + \frac{1}{x_{n+2}}\right).
\end{aligned}
$$

- Obtaining (A.1) does *not* automatically result in obtaining this item.

(B.17) Observing that

$$(x_n + x_{n+1})\left(\frac{1}{x_n} + \frac{1}{x_{n+1}}\right) > 4$$

(or observing $(x_n + x_{n+1})(\frac{1}{x_n} + \frac{1}{x_{n+1}}) \geq 4$ with the equality condition $x_n = x_{n+1}$) for a general index n, or observing an equivalent statement . . . . 1 **point**

(B.18) Writing down the expression $a_{n+2} - a_n = 2$ . . . . 0 **points**

(B.19) Other random algebraic manipulations . . . . 0 **points**

Zero or one regime

If a student gets 0 points so far, they may receive a final score of at most 1 point by making certain observations. Examples and non-examples are listed below.

(O.1) Proving that $a_1 = 1$ . . . . 0 **points**

(O.2) Proving that $a_n \geq n$ for all n . . . . 0 **points**

(O.3) Proving that $a_{n+1} \geq a_n + 1$ for all n . . . . 0 **points**

(O.4) Proving that $a_n > n$ for some $n \geq 2$ . . . . 1 **point**

(O.5) Proving that $a_n \geq n + 1$ for some $n \geq 2$ . . . . 1 **point**

Solutions

Solution 1. We start with some basic observations. First note that the sequence $a_1, a_2, \ldots, a_{2023}$ is increasing and thus, since all elements are integers, $a_{n+1} - a_n \geq 1$.

Claim 1. If $a_{n+1} - a_n = 1$ and $a_{n+2} - a_{n+1} = 1$, then $x_{n+2} = x_{n+1}$.

Proof 1.1. We start by observing that

$$\begin{aligned} a_{n+1}^2 &= (x_1 + \cdots + x_{n+1}) \left(\frac{1}{x_1} + \cdots + \frac{1}{x_{n+1}} \right) \\ &= (x_1 + \cdots + x_n) \left(\frac{1}{x_1} + \cdots + \frac{1}{x_n} \right) + 1 \\ &\quad + \frac{1}{x_{n+1}}(x_1 + \cdots + x_n) + x_{n+1} \left(\frac{1}{x_1} + \cdots + \frac{1}{x_n} \right) \\ &\geq a_n^2 + 1 + 2\sqrt{\frac{1}{x_{n+1}}(x_1 + \cdots + x_n) \cdot x_{n+1} \left(\frac{1}{x_1} + \cdots + \frac{1}{x_n} \right)} \\ &= a_n^2 + 1 + 2a_n \\ &= (a_n + 1)^2, \end{aligned}$$

where we used AM-GM to obtain the inequality. In particular, if $a_{n+1} = a_n + 1$, then

$$\frac{1}{x_{n+1}}(x_1 + \cdots + x_n) = x_{n+1} \left(\frac{1}{x_1} + \cdots + \frac{1}{x_n} \right). \tag{1}$$

Similarly, from $a_{n+2} = a_{n+1} + 1$ we obtain

$$\frac{1}{x_{n+2}}(x_1 + \cdots + x_{n+1}) = x_{n+2} \left(\frac{1}{x_1} + \cdots + \frac{1}{x_{n+1}} \right).$$

We can rewrite this relation as

$$\frac{x_{n+1}}{x_{n+2}} \left(\frac{1}{x_{n+1}} (x_1 + \cdots + x_n) + 1 \right) = \frac{x_{n+2}}{x_{n+1}} \left(x_{n+1} \left(\frac{1}{x_1} + \cdots + \frac{1}{x_n} \right) + 1 \right).$$

From (1), we conclude that $x_{n+1} = x_{n+2}$. □

Proof 1.2. Write $b_i = x_1 + x_2 + \cdots + x_i$, $c_i = \frac{1}{x_1} + \frac{1}{x_2} + \cdots + \frac{1}{x_i}$, and $d_i = a_{i+1} - a_i$. By definition,

$$\sqrt{b_n c_n} + d_n = \sqrt{(b_n + x_{n+1})(c_n + 1/x_{n+1})}$$

Squaring both sides,

$$b_n c_n + 2d\sqrt{b_n c_n} + d_n^2 = (b_n + x_{n+1})(c_n + 1/x_{n+1}),$$

and then by completing squares, we obtain

$$(\sqrt{c_n x_{n+1}} - \sqrt{b_n / x_{n+1}})^2 = (d_n - 1)(2\sqrt{b_n c_n} + d_n + 1).$$

If $a_{n+2} = a_{n+1} + 1$ and $a_{n+1} = a_n + 1$, then

$$x_{n+1} = \sqrt{\frac{b_n}{c_n}}$$

and

$$x_{n+2} = \sqrt{\frac{b_{n+1}}{c_{n+1}}} = \sqrt{\frac{b_n + \sqrt{\frac{b_n}{c_n}}}{c_n + \sqrt{\frac{c_n}{b_n}}}} = \sqrt{\frac{b_n}{c_n}} = x_{n+1}.$$ □

It follows immediately that $a_{n+2} \geq a_n + 3$. Since $a_1 = 1$, we get

$$a_{2023} = (a_{2023} - a_{2021}) + (a_{2021} - a_{2019}) + \cdots + (a_3 - a_1) + a_1 \geq 3 \cdot 1011 + 1 = 3034.$$

Solution 2. The trick is to compare a_{n+2} and a_n directly.
Claim 2. We have $a_{n+2} > a_n + 2$ for all n.

Proof. Observe that

$$\begin{aligned}
a_{n+2}^2 &= (x_1 + \cdots + x_{n+2}) \left(\frac{1}{x_1} + \cdots + \frac{1}{x_{n+2}} \right) \\
&= (x_1 + \cdots + x_n) \left(\frac{1}{x_1} + \cdots + \frac{1}{x_n} \right) + (x_{n+1} + x_{n+2}) \left(\frac{1}{x_{n+1}} + \frac{1}{x_{n+2}} \right) \\
&\quad + (x_1 + \cdots + x_n) \left(\frac{1}{x_{n+1}} + \frac{1}{x_{n+2}} \right) + (x_{n+1} + x_{n+2}) \left(\frac{1}{x_1} + \cdots + \frac{1}{x_n} \right) \\
&\geq a_n^2 + (x_{n+1} + x_{n+2}) \left(\frac{1}{x_{n+1}} + \frac{1}{x_{n+2}} \right) \\
&\quad + 2\sqrt{(x_{n+1} + x_{n+2}) \left(\frac{1}{x_{n+1}} + \frac{1}{x_{n+2}} \right) (x_1 + \cdots + x_n) \left(\frac{1}{x_1} + \cdots + \frac{1}{x_n} \right)} \\
&= a_n^2 + (x_{n+1} + x_{n+2}) \left(\frac{1}{x_{n+1}} + \frac{1}{x_{n+2}} \right) + 2a_n \sqrt{(x_{n+1} + x_{n+2}) \left(\frac{1}{x_{n+1}} + \frac{1}{x_{n+2}} \right)},
\end{aligned}$$

where we used AM-GM to obtain the inequality. Furthermore, we have

$$(x_{n+1} + x_{n+2}) \left(\frac{1}{x_{n+1}} + \frac{1}{x_{n+2}} \right) > 4$$

because $x_{n+1} \neq x_{n+2}$ by assumption. Therefore, it follows that

$$a_{n+2}^2 > a_n^2 + 4 + 4a_n = (a_n + 2)^2.$$

□

Because a_{n+2} and a_n are both positive integers, we conclude that

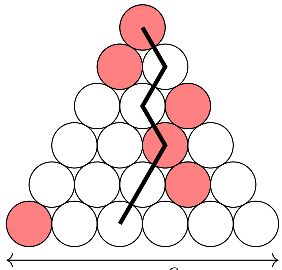
$$a_{n+2} \geq a_n + 3.$$

Then we finish as in Solution 1.

Marking scheme for Problem 5

Problem 5.

Let n be a positive integer. A *Japanese triangle* consists of $1 + 2 + \cdots + n$ circles arranged in an equilateral triangular shape such that for each $i = 1, 2, \ldots, n$, the i^{th} row contains exactly i circles, exactly one of which is coloured red. A *ninja path* in a Japanese triangle is a sequence of n circles obtained by starting in the top row, then repeatedly going from a circle to one of the two circles immediately below it and finishing in the bottom row. Here is an example of a Japanese triangle with $n = 6$, along with a ninja path in that triangle containing two red circles.



In terms of n, find the greatest k such that in each Japanese triangle there is a ninja path containing at least k red circles.

Marking scheme

Solution is divided into building blocks the marks, and the points in each block is given by the maximum of the marks they get. The total score is defined by

$$\max(U + \max(C + \max(A, B), D, E, P), S).$$

Upper Bound (2 Points)

(U1) Complete construction and the proof of the upper bound $\lfloor \log_2 n \rfloor + 1$ $(= \lceil \log_2(n+1) \rceil)$ 2 **points**

(U2) Guessing the upper bound $\lfloor \log_2 n \rfloor + 1$ without a construction .... 0 **points**

(U3) Construction (with some *explanation* of the structure, see Comments) that gives the correct bound for all n without proof .... 1 **point**

(U4) Proof of a weaker upper bound up to additive constant $(\log_2 n + c)$ .... 1 **point**

Lower Bound

Common part for solutions 1, 2, 3 (1 Point)

(C1) Explicitly define labels: the maximum number of red circles on ninja-paths starting (or ending) at that circle .... 1 **point**

(C2) Other attempts with random labels .... 0 **points**

Solution 1 (4 Points)

(A1) Observing the local relation that the label is determined by the labels of the circles directly above (or below) and whether the circle is red. .... 0 **points**

(A2-1) Observing that if the labels of i-th row circles are $v_1, v_2, \cdots, v_i$ and v_m is the maximum, then the labels of $(i+1)$-th row are at least $v_1, v_2, \cdots, v_m, v_m, v_{m+1}, \cdots, v_i$. .... 1 **point**

(A2-2) Prove $\sigma_{i+1} \geq \sigma_i + v_m$ .... 1 **point**

(A3) Prove $\sigma_{i+1} \geq \sigma_i + v_m + 1$ .... 2 **points**

(A4) Prove $\sigma_{2^j} \geq j\, 2^j + 1$ .... 3 **points**

(A5) Complete proof .... 4 **points**

Solution 2, 3 (4 Points)

(B1) Observation : no ninja-path between two red circles with same label .... 0 **points**

(B2) Claim 2 (for Sol 2): $e_i \leq l$, where l is top row of i label
Claim 4 (for Sol 3): $f_i \leq l$, where l is top row of $1, 2, \cdots, i$ label .... 2 **points**

(B3) Claim 3 (for Sol 2): $e_i \leq 2^i$
Claim 5 (for Sol 3): $f_1 + \cdots + f_j \leq n - \lfloor \frac{n}{2^j} \rfloor$ .... 3 **points**

(B4) Complete proof .... 4 **points**

Solution 4 (5 Points)

(D1) Defining partial order between red circles .... 0 **points**

(D2) Relating Mirsky's theorem to the problem .... 1 **point**

(D3) Proving Claim 6 .... 2 **points**

(D4) Relating Mirsky's theorem to the problem and proving Claim 6 .... 3 **points**

(D5) Complete proof .... 5 **points**

Solution 5 (5 Points)

(E1) Claim that the lower bound can be proved probabilistic method (without any detail). 0 **points**

(E2) Reduce the problem to the lemma 1. .... 2 **points**

(E3) Complete proof .... 5 **points**

Partial Credits for Lower Bound

Not additive to A, B, C, D schemes.

(P1) Prove weaker lower bound $c \log_2 n$ .... 1 **point**

Minor flaws

(M1) Extending the definition of ninja path without mentioning (so that ninja path can start and end in the middle row) . . . . -0 **points**

(M2) Off-by-one error or minor computational mistakes . . . . -0 **points**

Small Cases

(S1) Fully solving (proving both upper and lower bound) the problem for the cases $n \leq 6$. 0 **points**

(S2) Fully solving the cases $n = 7$ (or any $n = 2^k - 1 > 3$) . . . . 1 **point**

Comments

(U3) There could be several constructions, for example, top-down or bottom-up (Figure 1). Top-down construction consists of blocks of 2^i rows with a possible part of final block. Bottom-up construction's bottom block consists of $\lceil \frac{n}{2} \rceil$ rows, and other blocks will be inductively constructed. To get point with (U3), there should be an evidence of considering what would like to be for general n by writing down or drawing the blocks with indicating heights.

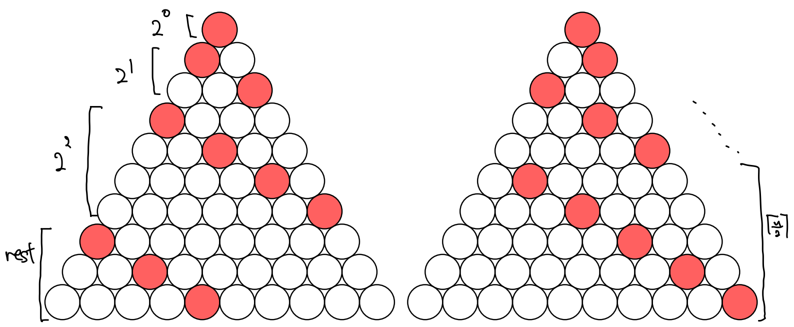


Figure 1: Examples of "explanation". Left: "top-down" construction, right: "bottom-up".

(U4) Example of a construction that gives weaker upper bound up to additive constant ($\log_2 n + c$) is shown in Figure 2.

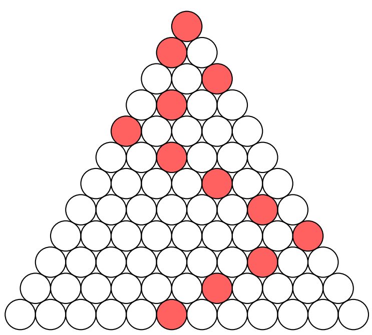


Figure 2: "wrong" zigzag (a weaker bound).

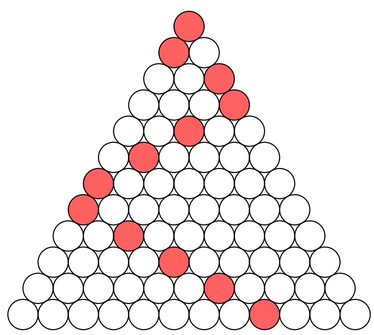


Figure 3: "correct" zigzag.

(U4) Example of a construction that gives weaker upper bound up to additive constant ($\log_2 n + c$) is shown in Figure 2.

(A) Same as solution 3, We can label each circle with the maximum number of red circles on a ninja-path *starting* at that circle. Similar argument to solution 1 holds true for this labelling, and corresponding marks in section (A) will be awarded to the (partial) solution.

(A3) To get credits in A3, students must evaluate the sum of each row. In other words, if they give observation in (A2-1) and they claim that the label increases by 1 when coloured red, but they do not give any evaluation on the total sum of each row, they will *not* be awarded.

(D2) Relating Mirsky's theorem to the problem is equivalent to claim that, to show $k \geq 1 + \lfloor \log_2(n) \rfloor$, it is sufficient to show that the red circles cannot be partitioned into $\leq \log_2(n)$ antichains. In particular, it's not the name of Mirsky's theorem that counts.

(D3) The student can get points from (D3) by proving equivalent statement without defining partial order between red circles.

(S2) The student gets the credit if they show the upper bound for $n = 7$ and the lower bound for $n = 4$. Here are some tricky examples of the construction for $n = 7$ (Fig 4).

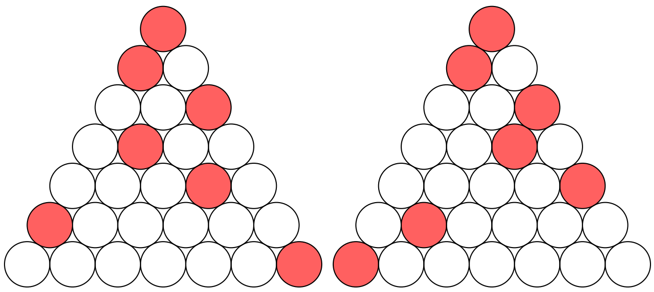


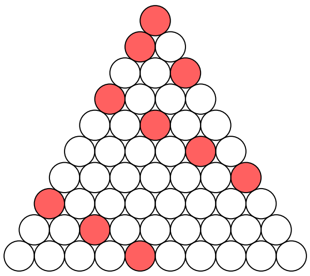
Figure 4: $n = 7$.

Solutions

Answer: The maximum value is $k = 1 + \lfloor \log_2 n \rfloor$.

Solution 1. Write $N = \lfloor \log_2 n \rfloor$ so that we have $2^N \leq n \leq 2^{N+1} - 1$.

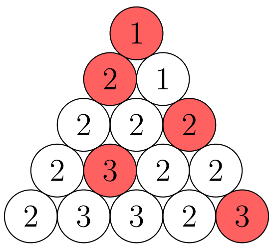
We first provide a construction where every ninja-path passes through at most $N + 1$ red circles. For the row $i = 2^a + b$ for $0 \leq a \leq N$ and $0 \leq b < 2^a$, we colour the $(2b + 1)$-th circle.



Then every ninja-path passes through at most one red circle in each of the rows $2^a, 2^a+1, \ldots, 2^{a+1}-1$ for each $0 \leq a \leq N$. It follows that every ninja-path passes through at most $N + 1$ red circles.

We now prove that for every Japanese triangle, there exists a ninja-path going through at least $N+1$ red circles. In the following section, we extend the definition of ninja-path so that a consecutive subsequence of ninja-path (which may start and end in the middle rows) is also called ninja-path. Note that a subsequence of ninja-path contains less or equal red circles than the original, hence it suffices to prove in this extended setting.

For each circle C, we assign the maximum number of red circles in a ninja-path that starts at the top of the triangle and ends at C.



Note that

- if C is not red, then the number assigned to C is the maximum of the number assigned to the one or two circles above C, and
- if C is red, then the number assigned to C is one plus the above maximum.

Write $v_1, \ldots, v_i$ for the numbers in row i, and let v_m be the maximum among these numbers. Then the numbers in row $i + 1$ will be at least

$$v_1, \ldots, v_{m-1}, v_m, v_m, v_{m+1}, \ldots, v_i,$$

not taking into account the fact that one of the circles in row $i + 1$ is red. On the other hand, for the red circle in row $i+1$, the lower bound on the assigned number can be increased by 1. Therefore the sum of the numbers in row $i + 1$ is at least

$$(v_1 + \cdots + v_i) + v_m + 1.$$

Using this observation, we prove the following claim.

Claim 1. Let σ_k be the sum of the numbers assigned to circles in row k. Then for $0 \le j \le N$, we have $\sigma_{2^j} \ge j \cdot 2^j + 1$.

Proof. We use induction on j. This is clear for $j = 0$, since the number in the first row is always 1. For the induction step, suppose that $\sigma_{2^j} \ge j \cdot 2^j + 1$. Then the maximum value assigned to a circle in row 2^j is at least $j+1$. As a consequence, for every $k \ge 2^j$, there is a circle on row k with number at least $j+1$. Then by our observation above, we have

$$\sigma_{k+1} \ge \sigma_k + (j+1) + 1 = \sigma_k + (j+2).$$

Then we get

$$\sigma_{2^{j+1}} \ge \sigma_{2^j} + 2^j(j+2) \ge j \cdot 2^j + 1 + 2^j(j+2) = (j+j+2)2^j + 1 = (j+1)2^{j+1} + 1.$$

This completes the inductive step. □

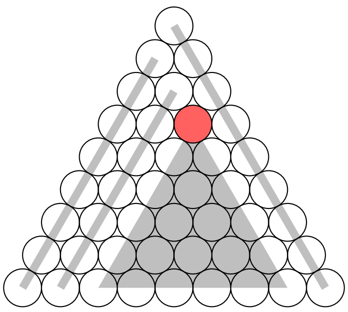
For $j = N$, this immediately implies that some circle in row 2^N has number at least $N+1$. This shows that there is a ninja-path passing through at least $N+1$ red circles.

Solution 2. As in solution 1, a consecutive subsequence of ninja-path is also called ninja-path.

We give an alternative proof that there exists a ninja-path passing through at least $N+1$ red circles. Assign numbers to circles as in the previous solution, but we only focus on the numbers assigned to red circles. For each positive integer i, denote by e_i the number of red circles with number i.

Claim 2. If the red circle on row l has number i, then $e_i \le l$.

Proof. Note that if two circles C and C' are both assigned the same number i, then there cannot be a ninja-path joining the two circles. We partition the triangle into a smaller triangle with the red circle in row l at its top along with $l-1$ lines that together cover all other circles.



In each set, there can be at most one red circle with number i, and therefore $e_i \le l$. □

We observe that if there exists a red circle C with number $i \ge 2$, then there also exists a red circle with number $i-1$ in some row that is above the row containing C. This is because the second last red circle in the ninja-path ending at C has number $i-1$.

Claim 3. We have $e_i \le 2^{i-1}$ for every positive integer i.

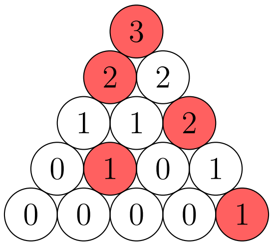
Proof. We prove by induction on i. The base case $i = 1$ is clear, since the only red circle with number 1 is the one at the top of the triangle . We now assume that the statement is true for $1 \le i \le j-1$ and prove the statement for $i = j$. If $e_j = 0$, there is nothing to prove. Otherwise, let l be minimal such that the red circle on row l has number j. Then all the red circles on row $1, \dots, l-1$ must have number less than j. This shows that

$$l - 1 \le e_1 + e_2 + \cdots + e_{j-1} \le 1 + 2 + \cdots + 2^{j-2} = 2^{j-1} - 1.$$

This proves that $l \le 2^{j-1}$, and by Claim 2, we also have $e_j \le l$. Therefore $e_j \le 2^{j-1}$. □

We now see that
$$e_1 + e_2 + \cdots + e_N \le 1 + \cdots + 2^{N-1} = 2^N - 1 < n.$$
Therefore there exists a red circle with number at least $N+1$, which means that there exists a ninja-path passing through at least $N+1$ red circles.

Solution 3. We provide yet another proof that there exists a ninja-path passing through at least $N+1$ red circles. As in solution 1 and 2, a consecutive subsequence of ninja-path is also called ninja-path. We assign to a circle C the maximum number of red circles on a ninja-path *starting* at C (including C itself).



Denote by f_i the number of red circles with number i. Note that if a red circle C has number i, and there is a ninja-path from C to another red circle C', then the number assigned to C' must be less than i.

Claim 4. If the red circle on row l has number less than or equal to i, then $f_i \le l$.

Proof. This proof is same as the proof of Claim 2. The additional input is that if the red circle on row l has number strictly less than i, then the smaller triangle cannot have a red circle with number i. □

Claim 5. We have
$$f_1 + f_2 + \cdots + f_i \le n - \left\lfloor \frac{n}{2^i} \right\rfloor$$
for all $0 \le i \le N$.

Proof. We use induction on i. The base case $i = 0$ is clear as the left hand side is the empty sum and the right hand side is zero. For the induction step, we assume that $i \ge 1$ and that the statement is true for $i-1$. Let l be minimal such that the red circle on row l has number less than or equal to i. Then all the red circles with number less than or equal to i lie on rows $l, l+1, \ldots, n$, and therefore
$$f_1 + f_2 + \cdots + f_i \le n - l + 1.$$
On the other hand, the induction hypothesis together with the fact that $f_i \le l$ shows that
$$f_1 + \cdots + f_{i-1} + f_i \le n - \left\lfloor \frac{n}{2^{i-1}} \right\rfloor + l.$$

Averaging the two inequalities gives
$$f_1 + \cdots + f_i \le n - \frac{1}{2}\left\lfloor \frac{n}{2^{i-1}} \right\rfloor + \frac{1}{2}.$$
Since the left hand side is an integer, we conclude that
$$f_1 + \cdots + f_i \le n - \left\lfloor \frac{1}{2}\left\lfloor \frac{n}{2^{i-1}} \right\rfloor \right\rfloor = n - \left\lfloor \frac{n}{2^i} \right\rfloor.$$
This completes the induction step. □

Taking $i = N$, we obtain

$$f_1 + f_2 + \cdots + f_N \leq n - \left\lfloor \frac{n}{2^N} \right\rfloor < n.$$

This implies that there exists a ninja-path passing through at least $N+1$ red circles.

Comment. Using essentially the same argument, one may inductively prove

$$e_a + e_{a+1} + \cdots + e_{a+i-1} \leq n - \left\lfloor \frac{n}{2^i} \right\rfloor.$$

instead. Taking $a = 1$ and $i = N$ gives the desired statement.

Solution 4 (Mirsky's theorem.). Given a Japanese triangle, we define a relation on its red circles: We say one *precedes* another if it is in a higher row and there is a ninja path passing through both. Since ninja paths can be concatenated, this is clearly a partial order on the set of red circles, and the question asks us to find a lower bound on the length of the longest chain in this poset. Mirsky's theorem states that this length is equal to the smallest number of antichains the poset can be partitioned into. Thus, to show $k \geq 1 + \lfloor \log_2(n) \rfloor$, it is enough to show that the red circles cannot be partitioned into $\leq \log_2(n)$ antichains. This will follow from the following claim:

Claim 6. If the topmost circle of an antichain of red circles is on row l, then the length of the antichain is at most l.

Proof. If the red circle on row l is the i-th from the left, then the $i-1$ down-left diagonals to its left and the $l-i$ down-right diagonals to its right cover all red circles in lower rows which are not preceded by our top circle. Since such diagonals form chains, our antichain can contain at most 1 red circle from each of these $l-1$ diagonals, which gives at most l red circles together with the top circle, as claimed. □

We now prove that any partition into any k antichains must have $k > \log_2(n)$: Suppose $A_1, \ldots, A_k$ is such a partition, where the top circle in part A_i is in row l_i, and they are sorted so that the l_i are in increasing order. Then by the claim, for each i, $|A_i| \leq l_i$. It follows that $|A_1 \cup \cdots \cup A_i| \leq l_1 + \cdots + l_i$. Thus, the first $l_1 + \cdots + l_i + 1$ rows are not fully covered by $A_1 \ldots A_i$, and since the A_j contain all red circles, the first row not covered by them must be l_{i+1}, i.e. $l_{i+1} \leq l_1 + \cdots + l_i + 1$. Clearly $l_1 = 1$ and an immediate induction now shows $l_i \leq 2^{i-1}$. It follows that $n = |A_1 \cup \cdots \cup A_k| \leq l_1 + \cdots + l_k < 2^k$, i.e. $k > \log_2(n)$, as claimed.

Remark. This solution is very similar to solution 2; in fact claims 2 and 3 from solution 2 are just special cases of the claims in this proof: the sets of red circles assigned i are level sets of the poset, which are special antichains that form an optimal partitioning. In other words, solution 2 can be viewed as packaging this solution together with an explicit proof of Mirsky's theorem for this poset instead of using it as a black box. Which solution would be more accessible and natural for a student depends on their familiarity with Mirsky's theorem; the main advantage of considering the theorem is that it immediately sets you on the path of investigating properties of antichains, which are also important for the construction of the example (or vice versa, realizing the construction uses very long antichains could point to using Mirsky's theorem for the bound).

Comment (weaker lower bound). We give a proof for weaker lower bound based on probabilistic method. Select a real number $x \in (0, 1)$. Consider the path P_x in which the $\lceil ix \rceil$-th circle in the i-th row is selected. This is a ninja path, since $\lceil ix \rceil \leq \lceil (i+1)x \rceil \leq \lceil ix \rceil + 1$. Observing this, on the i -th row, the red circle is selected if x lies on a certain interval of length $1/i$. Therefore, The average number of red circles on a ninja path of this kind is $1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} > \ln n$. Thus there exists a ninja path P_x containing at least $\ln n$ red circles.

Solution 5 (a probabilistic proof of the lower bound). This solution was found by by Jeck Lim, inspired by the above probabilistic proof for weaker lower bound (the script was written by Yuya Matsumoto).

The idea was to choose a ninja path randomly from a certain distribution, which consists of well-chosen 2^N ninja paths with equal probability. (The choice of paths depends on the configuration of red circles, unlike in "Comment (weaker lower bound)" of the present marking scheme.) However, for the purpose of decreasing the use of fractions, we avoid the use of the notion of randomness and just consider collections of 2^N ninja paths.

To show the lower bound, it suffices to show that if $n = 2^N$ then there exists a ninja path containing $N+1$ red circles. The case $N = 0$ is obvious. Hereafter we let $n = 2^N$ with $N \geq 1$.

It suffices to give a collection of 2^N ninja paths such that the average number of red circles contained in the paths is $> N$, since then at least one path contains more than N red circles. (Here, a collection may contain the same path more than once. Mathematically, it is a multiset.)

To obtain such a collection, it suffices to prove the following lemma.

Lemma 1. There exists a collection of 2^N ninja paths, *from row* 2^N *to row* 2^{N-1}, satisfying the following properties.

- Each circle in row 2^N is contained in exactly 1 path.
- Each circle in row 2^{N-1} is contained in exactly 2 paths.
- Each red circle in row j, where $2^{N-1} \leq j < 2^N$, is contained in exactly 2 paths.

(The meaning of such (backward, and stopping at the middle) ninja paths would be clear.)

We first show that this lemma indeed implies the lower bound. Combining the collection from row 2^N to row 2^{N-1} and the collection from row 2^{N-1} to 2^{N-2}, with each path in the latter collection counted twice, gives a collection of 2^N ninja paths from row 2^N to row 2^{N-2}. Continuing this, we obtain a collection of 2^N ninja paths from row 2^N to row 1 with the property that the red circle in row j, with $2^{N-m-1} \leq j < 2^{N-m}$, is contained in exactly 2^{m+1} paths. Then the average number of red circles contained in the paths is

$$\frac{1}{2^N}\left(1 + \sum_{m=0}^{N-1} \sum_{j=2^{N-m-1}}^{2^{N-m}-1} 2^{m+1}\right) = \frac{1}{2^N}\left(1 + \sum_{m=0}^{N-1} 2^{N-m-1} \cdot 2^{m+1}\right)$$

$$= \frac{1}{2^N} + \sum_{m=0}^{N-1} 1 = \frac{1}{2^N} + N > N,$$

as desired, where "1+" comes from the red circle in row 2^N.

It remains to prove Lemma 1. (The proof can be visualized by writing the numbers n_h in the circle C_h in the pyramid; from the circle in the lower row with number 1 or 2 we draw 1 or 2 arrows to circle(s) in the upper row.)

We give an inductive construction. We consider, for each $0 \leq i \leq 2^{N-1}$, a collection of 2^N ninja paths from row 2^N to row $2^N - i$ satisfying the following properties.

- Each circle in row j, where $2^N - i \leq j \leq 2^N$, is contained in exactly 1 or 2 paths.
- Each red circle in row j, where $2^N - i \leq j < 2^N$, is contained in exactly 2 paths.

For $i = 0$, the collection of all 2^N circles, considered as "trivial" ninja paths, satisfies the properties. Suppose we have a collection for i with $0 \le i < 2^{N-1}$, and we will construct a collection for $i + 1$.

Let $C_1, \dots, C_{2^N - i}$ be the circles in row $2^N - i$ (numbered from left to right), and let C_a, C_{a+1} be the two circles immediately below the red circle in row $2^N - i - 1$. Let n_h $(1 \le h \le 2^N - i)$ be the number of ninja paths in the collection (for i) containing C_h. The first condition is equivalent to every n_h being equal to 1 or 2.

Case 1. There exist $b_1 \le a$ and $b_2 \ge a + 1$ with $n_{b_1} = 1$ and $n_{b_2} = 1$.

Take b_1 to be the largest among such indices and take b_2 to be the smallest among such indices. We extend each path in the collection (for i) according to the following rule.

- We extend the paths containing C_h with $h \le b_1$ to the circle on the upper right.
- We extend the paths containing C_h with $h \ge b_2$ to the circle on the upper left.
- For each fixed h with $b_1 < h < b_2$, there are exactly two paths containing C_h (because of the definitions of b_1 and b_2), and we extend one of them to the circle on the upper right and one to the upper left.

This construction preserves the first condition. Indeed, the numbers

$$n_1, \dots, n_{b_1 - 1}, 1, \underbrace{2, \dots, 2}_{b_2 - b_1 - 1}, 1, n_{b_2 + 1}, \dots, n_{2^N - i}$$

become

$$n_1, \dots, n_{b_1 - 1}, \underbrace{2, \dots, 2}_{b_2 - b_1}, n_{b_2 + 1}, \dots, n_{2^N - i}.$$

Moreover, the red circle belongs to the $2, \dots, 2$ zone at the middle, hence the second condition is also satisfied.

Case 2. $n_h = 2$ for every $h \le a$, and there exists $h \ge a + 1$ with $n_h = 1$.

Since the sum of n_h is even (because the sum is 2^N), there exist at least two h for which $n_h = 1$. Let $b_1 < b_2 < \dots$ be the indices with $n_h = 1$. Using these b_1 and b_2, apply the same construction as in Case 1. Then, again the first condition is preserved, and since the red circle belongs to the $n_1, \dots, n_{b_1 - 1}$ zone at the left (which are all equal to 2), the second condition is also satisfied.

Case 3. $n_h = 2$ for every $h \ge a + 1$, and there exists $h \le a$ with $n_h = 1$.

Parallel to Case 2.

Case 4. $n_h = 2$ for every $1 \le h \le 2^N - i$.

Since the sum of n_h is equal to 2^N, this implies that $2^N - i = 2^{N-1}$, which contradicts our assumption that $i < 2^{N-1}$.

By the inductive construction, we obtain a collection of 2^N ninja paths from row 2^N to row 2^{N-1}. Comparing the number of paths and the number of circles in row 2^{N-1}, we conclude that each circle in row 2^{N-1} is contained in exactly 2 paths, and hence this collection satisfies the properties stated in Lemma 1.

Marking scheme for Problem 6

Problem 6.
Let ABC be an equilateral triangle. Let A_1, B_1, C_1 be interior points of ABC such that $BA_1 = A_1C$, $CB_1 = B_1A$, $AC_1 = C_1B$, and

$$\angle BA_1C + \angle CB_1A + \angle AC_1B = 480^\circ.$$

Let BC_1 and CB_1 meet at A_2, let CA_1 and AC_1 meet at B_2, and let AB_1 and BA_1 meet at C_2. Prove that if triangle $A_1B_1C_1$ is scalene, then the three circumcircles of triangles AA_1A_2, BB_1B_2 and CC_1C_2 all pass through two common points.

(Note: a scalene triangle is one where no two sides have equal length.)

Marking scheme — general comments

- Partial points from Solution 1 are not additive with partial points from Solution 2.
- Partial points from the same solution are additive within that solution.
- The student's final score will be the **sum of all partial points rounded to the nearest integer**. For example:

$$2 - 0.4 = 2, \quad 4 - 0.8 = 3, \quad 5 - 1.2 = 4, \quad 7 - 1.6 = 5, \quad \text{and so on.}$$

- In general, **computational approaches**, unless substantially complete, will not be awarded more than a few points. A mere translation of geometry into trigonometry, complex numbers, Cartesian or barycentric coordinates etc. will be awarded **0 points**. Any essentially incomplete computational attempt will get **0 points** unless the results are interpreted in geometrical terms, in which case this constitutes a valid proof of those results, giving points according to the marking schemes below.

Zero points

The following items will give no partial points.

(Z1) Characterising the 480° condition in terms of A_1, etc., being a circumcentre ...... 0 **points**

(Z2) Proving that A_1 lies inside $\triangle OBC$ where O is the centre of $\triangle ABC$ .... 0 **points**

(Z3) Proving that $A_1B_2C_1A_2B_1C_2$ is convex .... 0 **points**

(Z4) Proving concurrencies amongst lines defined by points from the following list: A, B, C, A_1, B_1, C_1, A_2, B_2, C_2, A_3, B_3, C_3, X, Y, O .... 0 **points**

(Z5) Constructing point X or Y (e.g., as an isogonal conjugate) .... 0 **points**

(Z6) Constructing points A_3, B_3, C_3 .... 0 **points**

(Z7) Claiming that a quadrilateral is cyclic without proof .... 0 **points**

(Z8) General characterisation of coaxial systems (e.g., via the coaxial lemma, orthogonal circles, etc.) .... 0 **points**

For brevity, we use $\mathcal{S}(ABCD)$ to denote the statement "$ABCD$ is cyclic".

Marking scheme — Solution 1

Solution 1 is divided into three parts.

(P) Proving that point X has equal powers .... 2 **points**

(Q) Proving that point Y has equal powers .... 4 **points**

(R) Completing the problem .... 1 **point**

We detail the partial points and their dependencies (if any) below.

Part (P)

(P1) Proving $\mathcal{S}(A_1B_1A_2B_2)$ or $\mathcal{S}(B_1C_1B_2C_2)$ or $\mathcal{S}(C_1A_1C_2A_2)$ .... 1 **point**

(P2) Proving that ($\mathcal{S}(A_1B_1A_2B_2)$ and $\mathcal{S}(B_1C_1B_2C_2)$ and $\mathcal{S}(C_1A_1C_2A_2)$) implies that X has equal powers in $\delta_A, \delta_B, \delta_C$ .... 1 **point**

(P2x) [**Requires (P2)**] Not proving that $A_1B_2C_1A_2B_1C_2$ is not cyclic .... -0.4 **points**

Part (Q)

(Q1) Proving $\mathcal{S}(ABA_3B_3)$ or $\mathcal{S}(BCB_3C_3)$ or $\mathcal{S}(CAC_3A_3)$ .... 2 **points**

(Q2) Proving that ($\mathcal{S}(ABA_3B_3)$ and $\mathcal{S}(BCB_3C_3)$ and $\mathcal{S}(CAC_3A_3)$) implies that Y has equal powers in $\delta_A, \delta_B, \delta_C$ .... 1 **point**

(Q2x) [**Requires (Q2)**] Not proving that $AC_3BA_3CB_3$ is not cyclic .... -0.4 **points**

(Q3) Completing both **(Q1)** and **(Q2)** .... 1 **point**

Part (R)

(R1) [**Requires (P2) and (Q2)**] Prove that ((**P2**) and (**Q2**)) implies that the three circles have two common points .... 1 **point**

(R1x) [**Requires (R1)**] Not proving that the circles intersect at least twice .... -0.4 **points**

(R1y) [**Requires (R1)**] Not proving $X \neq Y$ .... -0.4 **points**

Marking scheme — Solution 2

Partial points

(K1) Proving $\mathcal{S}(A_1B_1A_2B_2)$ or $\mathcal{S}(B_1C_1B_2C_2)$ or $\mathcal{S}(C_1A_1C_2A_2)$ .... 1 **point**

(K2) Proving that δ_A is the locus of $\operatorname{Pow}_{\omega_B}(Z) = \Gamma_A \operatorname{Pow}_{\omega_C}(Z)$ .... 1 **point**

(K3) Proving the claim $\Gamma_A\Gamma_B\Gamma_C = 1$ .... 3 **points**

(K3x) [**Requires (K2) and (K3)**] Not proving that $\omega_A, \omega_B, \omega_C$ are distinct .... -1 **points**

(K4) Proving the claim implies (if Y lies on two circles, then it lies on all three) .... 1 **point**

(K5) [**Requires (K1), (K2), (K3) and (K4)**] Completing the solution .... 1 **point**

(K5x) [**Requires (K5)**] Not proving that the circles intersect at least twice .... -1 **points**

Solutions

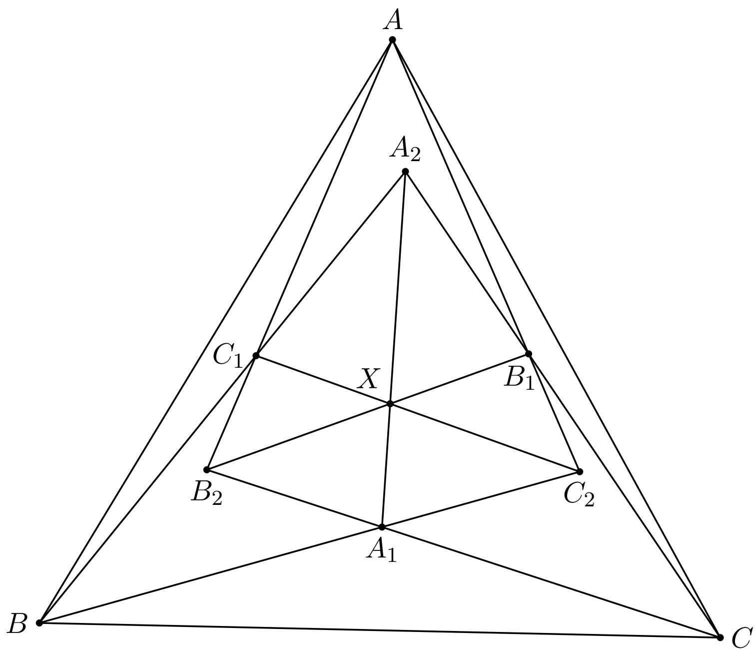
Solution 1. Let $\delta_A, \delta_B, \delta_C$ be the circumcircles of $\triangle AA_1A_2$, $\triangle BB_1B_2$, $\triangle CC_1C_2$. The general strategy of the solution is to find two different points having equal power with respect to $\delta_A, \delta_B, \delta_C$.

Claim. A_1 is the circumcentre of A_2BC and cyclic variations.

Proof. Since A_1 lies on the perpendicular bisector of BC and inside $\triangle BA_2C$, it suffices to prove $\angle BA_1C = 2\angle BA_2C$. This follows from

$$\begin{aligned}\angle BA_2C &= \angle A_2BA + \angle BAC + \angle ACA_2\\ &= \frac{1}{2}\left((180^\circ - \angle AC_1B) + (180^\circ - \angle CB_1A)\right) + 60^\circ\\ &= 240^\circ - \frac{1}{2}\left(480^\circ - \angle BA_1C\right)\\ &= \frac{1}{2}\angle BA_1C\end{aligned}$$

□



The circumcentres above give

$$\angle B_1B_2C_1 = \angle B_1B_2A = \angle B_2AB_1 = \angle C_1AC_2 = \angle AC_2C_1 = \angle B_1C_2C_1$$

and so $B_1C_1B_2C_2$ is cyclic. Likewise $C_1A_1C_2A_2$ and $A_1B_1A_2B_2$ are cyclic. Note that hexagon $A_1B_2C_1A_2B_1C_2$ is not cyclic since

$$\angle C_2A_1B_2 + \angle B_2C_1A_2 + \angle A_2B_1C_2 = 480^\circ \neq 360^\circ.$$

Thus we can apply radical axis theorem to the three circles to show that A_1A_2, B_1B_2, C_1, C_2 concur at a point X and this point has equal power with respect to $\delta_A, \delta_B, \delta_C$.

Let the circumcircle of $\triangle A_2BC$ meet δ_A at $A_3 \neq A_2$. Define B_3 and C_3 similarly.

Claim. BCB_3C_3 cyclic.

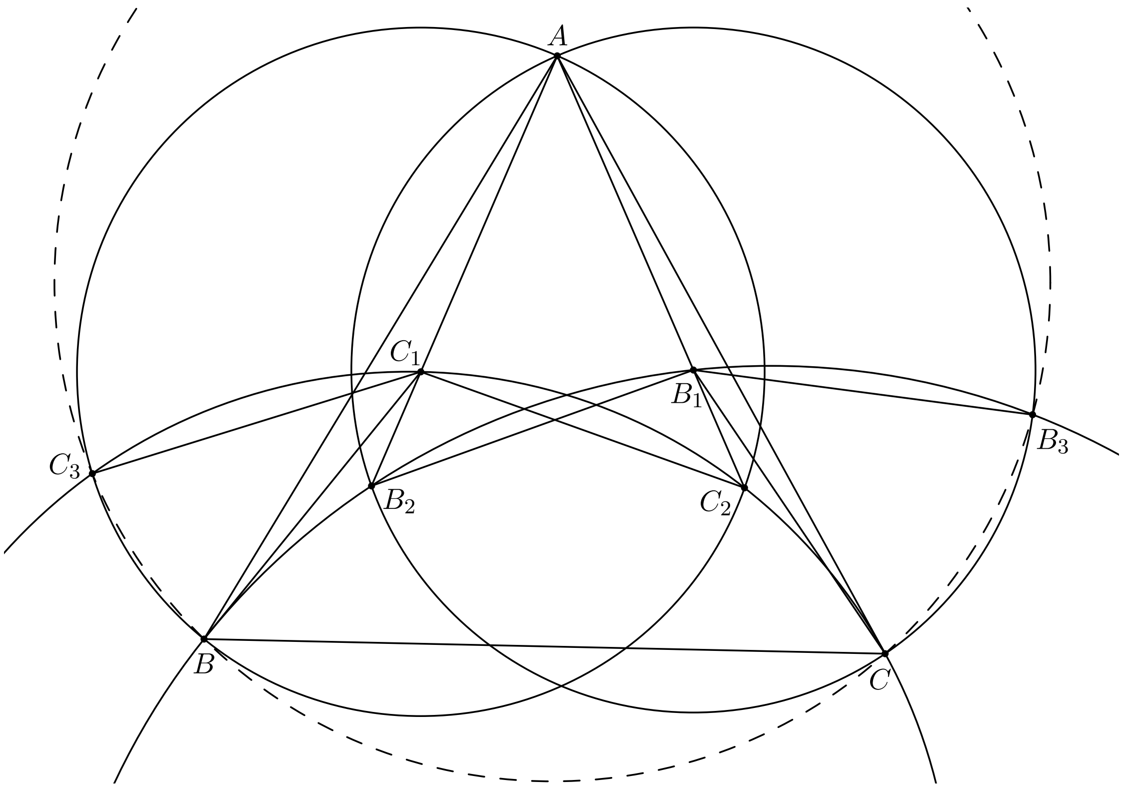
Proof. Using directed angles

$$\begin{aligned}\measuredangle BC_3C &= \measuredangle BC_3C_2 + \measuredangle C_2C_3C \\ &= \measuredangle BAC_2 + \measuredangle C_2C_1C \\ &= 90^\circ + \measuredangle(C_1C, AC_2) + \measuredangle C_2C_1C && (CC_1 \perp AB) \\ &= 90^\circ + \measuredangle C_1C_2B_1.\end{aligned}$$

Similarly $\measuredangle CB_3B = 90^\circ + \measuredangle B_1B_2C_1$. Hence, using $B_1C_1B_2C_2$ cyclic

$$\measuredangle BB_3C = 90^\circ + \measuredangle C_1B_2B_1 = 90^\circ + \measuredangle C_1C_2B_1 = \measuredangle BC_3C$$

as required. □



Similarly CAC_3A_3 and ABA_3B_3 are cyclic. $AC_3BA_3CB_3$ is not cyclic because then AB_2CB_3 cyclic would mean B_2 lies on $\odot ABC$ which is impossible since B_2 lies inside $\triangle ABC$. Thus we can apply radical axis theorem to the three circles to get AA_3, BB_3, CC_3 concur at a point Y which has equal power with respect to $\delta_A, \delta_B, \delta_C$.

We now make some technical observations before finishing.

- Let O be the centre of $\triangle ABC$. We have that

  $$\angle BA_1C = 480^\circ - \angle CB_1A - \angle AC_1B > 480^\circ - 180^\circ - 180^\circ = 120^\circ.$$

  so A_1 lies inside $\triangle BOC$. We have similar results for B_1, C_1 and thus $\triangle BA_1C$, $\triangle CB_1A$, $\triangle AC_1B$ have disjoint interiors. It follows that $A_1B_2C_1A_2B_1C_2$ is a convex hexagon thus X lies on segment A_1A_2 and therefore is inside δ_A.

- Since A_1 is the centre of A_2BC we have that $A_1A_2 = A_1A_3$ so, from cyclic quadrilateral $AA_2A_1A_3$ we get that lines AA_2 and $AA_3 \equiv AY$ are reflections in line AA_1. As X lies on segment A_1A_2, the only way $X \equiv Y$ is if A_1 and A_2 both lie on the perpendicular bisector of BC. But this forces B_1 and C_1 to also be reflections in this line meaning $A_1B_1 = A_1C_1$ contradicting the scalene condition.

Summarising, we have distinct points X, Y with equal power with respect to $\delta_A, \delta_B, \delta_C$ thus these circles have a common radical axis. As X lies inside δ_A (and similarly δ_B, δ_C), this radical axis intersects the circles at two points and so $\delta_A, \delta_B, \delta_C$ have two points in common.

Comment. An alternative construction for Y comes by observing that

$$\frac{\sin \angle BAA_2}{\sin \angle A_2AC} = \frac{\frac{A_2B}{AA_2} \sin \angle A_2BA}{\frac{A_2C}{AA_2} \sin \angle ACA_2} = \frac{A_2B}{A_2C} \cdot \frac{\sin \angle C_1BA}{\sin \angle ACB_1} = \frac{\sin \angle B_1CB}{\sin \angle CBC_1} \cdot \frac{\sin \angle C_1BA}{\sin \angle ACB_1}$$

and hence

$$\frac{\sin \angle BAA_2}{\sin \angle A_2AC} \cdot \frac{\sin \angle CBB_2}{\sin \angle B_2BA} \cdot \frac{\sin \angle ACC_2}{\sin \angle C_2CB} = 1$$

so by Ceva's theorem, AA_2, BB_2, CC_2 concur and thus we can construct the isogonal conjugate of this point of concurrency which turns out to be Y.

Solution 2. As in Solution 1, we establish the three distinct circles $\omega_A = \odot B_1B_2C_1C_2$, $\omega_B = \odot C_1C_2A_1A_2$ and $\omega_C = \odot A_1A_2B_1B_2$. Define

$$\Gamma_A = \frac{\operatorname{Pow}_{\omega_B}(A)}{\operatorname{Pow}_{\omega_C}(A)}, \quad \Gamma_B = \frac{\operatorname{Pow}_{\omega_C}(B)}{\operatorname{Pow}_{\omega_A}(B)}, \quad \Gamma_C = \frac{\operatorname{Pow}_{\omega_A}(C)}{\operatorname{Pow}_{\omega_B}(C)}.$$

By the coaxial lemma, $\delta_A = \odot AA_1A_2$ is the locus of all points Z such that $\operatorname{Pow}_{\omega_B}(Z) = \Gamma_A \operatorname{Pow}_{\omega_C}(Z)$.

Claim. $\Gamma_A\Gamma_B\Gamma_C = 1$.

Proof. Denote $\alpha = \angle A_1BC$, $\beta = \angle B_1CA$, $\gamma = \angle C_1AB$. The condition of the problem implies $\alpha + \beta + \gamma = 30°$. Let $A_B = AA_1 \cap \omega_C$ and $A_C = AA_1 \cap \omega_B$. Then $\angle A_1AB_1 = 30° - \beta$ and $\angle A_1A_BB_1 = \angle A_1A_2B_1 = \angle B_1CA_1 = 30° + \gamma$. Then $\angle AB_1A_B = \gamma + \beta = 30° - \alpha$.

Applying the sine rule, $\frac{AA_B}{AB_1} = \frac{\sin(30°-\alpha)}{\sin(30°+\gamma)}$. An analogous expression can be obtained for $\frac{AA_c}{AC_1}$. Thus

$$\Gamma_A = \frac{AA_c \cdot AA_1}{AA_B \cdot AA_1} = \frac{AC_1 \sin(30° + \gamma)}{AB_1 \sin(30° + \beta)}.$$

Writing similar expressions for Γ_B, Γ_C and using $BA_1 = CA_1$, etc., the claim follows.

Now, for any point Y that lies on both δ_A and δ_B,

$$\operatorname{Pow}_{\omega_B}(Y) \overset{y \in \delta_A}{=} \Gamma_A \operatorname{Pow}_{\omega_C}(Y) \overset{y \in \delta_B}{=} \Gamma_A\Gamma_B \operatorname{Pow}_{\omega_A}(Y)$$
$$\implies \quad \operatorname{Pow}_{\omega_A}(Y) = \Gamma_C \operatorname{Pow}_{\omega_B}(Y) \quad \implies \quad Y \in \delta_C.$$

Therefore Y also lies on δ_C. Finally, the solution can be completed by arguing that the circles δ_A and δ_B must intersect at two distinct points. This follows from the fact that $A_2B_1C_2A_1B_2C_1$ is convex.

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Elisa Lorenzo Garcia (SPE)　　Advisor
Nouzha El Yacoubi (MAR)　　Advisor
Taisuke Hoshino (星野　泰佑) (JPN)
Donghyun Kim (김동현) (KOR)
Ryotaro Kosuge (小菅　亮太朗) (JPN)
Yuliya Kryvitskaya (BLR)
Dae Jun Lee (이대준) (KOR)
Shanta Laishram (IND)
Zhao Yu Ma (SIN)
Masaki Nishimoto (西本　将樹) (JPN)
Ryosuke Odoi (大土井　亮祐) (JPN)
Heizo Sakamoto (坂本　平蔵)
Takashi Seki (関　典史) (JPN)
Jaka Vrhovnik (SLO)

Problem 2

Atsuo Yamauchi (山内　淳生)(JPN)　Captain
Géza Kós (HUN, IMOB)　　Advisor
Sam Bealing (UNK)　　Advisor
Tímea Csahók (HUN)
Horace Chaix (SUI)
Lovro Drofenik (SLO)
Shuho Kanda (神田　秀峰) (JPN)
Koichiro Kobayashi (小林　晃一良) (JPN)
Taiyo Kodama (兒玉　太陽) (JPN)
Anna Luchnikov (FRA)
Ryu Minegishi (峰　岸龍) (JPN)
Minkyu Choi (최민규) (KOR)
Kensuke Yoshida (吉田　健祐) (JPN)
Yuka Watanabe (渡部　由佳) (JPN)
Akos Zahorsky (SVK)

Problem 3

Yuya Matsumoto (松本　雄也)(JPN)　Captain
Dong Ryul Kim (KOR)　Advisor
Advisor: Geoff Smith (UNK, IMOB)　Advisor
Stijn Cambie (BEL)
Ildar Maratovich Gaisin (AUS)
Ahmed Ittihad Hasib (BGD)
Seongbin Jeon (서유완) (KOR)
Satoshi Hayakawa (早川　知志)
Dylan Toh Shan Hong (SIN)
Yui Hosaka (保坂　結) (JPN)
Naoki Kimura (木村　直記) (JPN)
Soh Kumabe (隈部　壮)(JPN)
James Lin (USA)
Yoshiki Oshima (大島　芳樹) (JPN)
Michael Ren (USA)

Problem 4

Hiroki Kodama (児玉　大樹) (JPN)　Captain
Arnaud Maret (SUI)　Advisor
Advisor: Shingo Saito (斎藤　新悟)(JPN)　Advisor
Youngjun Cho (조영준) (KOR)
Fuma Hirayama (平山　楓馬)(JPN)
Omar El Housni (MAR)
Yuhi Kamio (神尾　悠陽)(JPN)
Soji Kubota (窪田　壮児)(JPN)
Larry Lau (HKG)
Yuji Odaka (尾髙　悠志)(JPN)
Elena Madalina Persu (ROM)
Yuya Nakamura (中村　勇哉)(JPN)
Alec Sun (USA)
Stephan Wagner (SWE)
Tomoki Yoshida (吉田　智紀)(JPN)

Problem 5

Genki Shimizu (清水　元喜) (JPN) Captain
Paul Vaderlind (SWE) Advisor
Sung Jin Park (KOR) Advisor
Anubhab Ghosal (IND)
Zack Chroman (USA)
Takuya Inoue (井上　卓哉)(JPN)
Jimin Kim (서유완) (KOR)
Ander Lemaison (ESP)
Jeck Lim (SIN)
Shogo Murai (村井　翔悟)(JPN)
Joseph Myers (UNK)
Yuwan Seo (서유완) (KOR)
Mihir Singhal (USA)
Kota Tokoro (床呂　光太)(JPN)
Yuan Yao (USA)

Problem 6

Yuta Takaya (高谷　悠太)(JPN) Captain
Ivan Guo（AUS） Advisor
Charles Leytem (LUX) Advisor
Jeonghyun Ahn (안정현) (KOR)
Joseph Daniel Altamirano Pacheco (UNK)
Kei Irie (入江　慶)(JPN)
Milan Haiman (USA & HUN)
Junghun Ju (주정훈) (KOR)
Vesna Kadelburg (SRB)
Dain Kim (KOR)
Takuma Kitamura (北村　拓真)(JPN)
Hiromasa Kondo (近藤　滉将)(JPN)
Rikimaru Kurata (藏田　力丸)(JPN)
Naoki Kuroda (黒田　直樹)(JPN)
Ayato Shukuta (宿田　彩斗)(JPN)
Yuki Yoshida (吉田　雄紀)(JPN)

64th International Mathematical Olympiad 2023

Country Code

Question

IMO 2023
Chiba, JAPAN 64th

Contestant	Mark
1	
2	
3	
4	
5	
6	

Signature(Country)

Signature(Coordinator)

Coordination Schedule Day 1 10 July

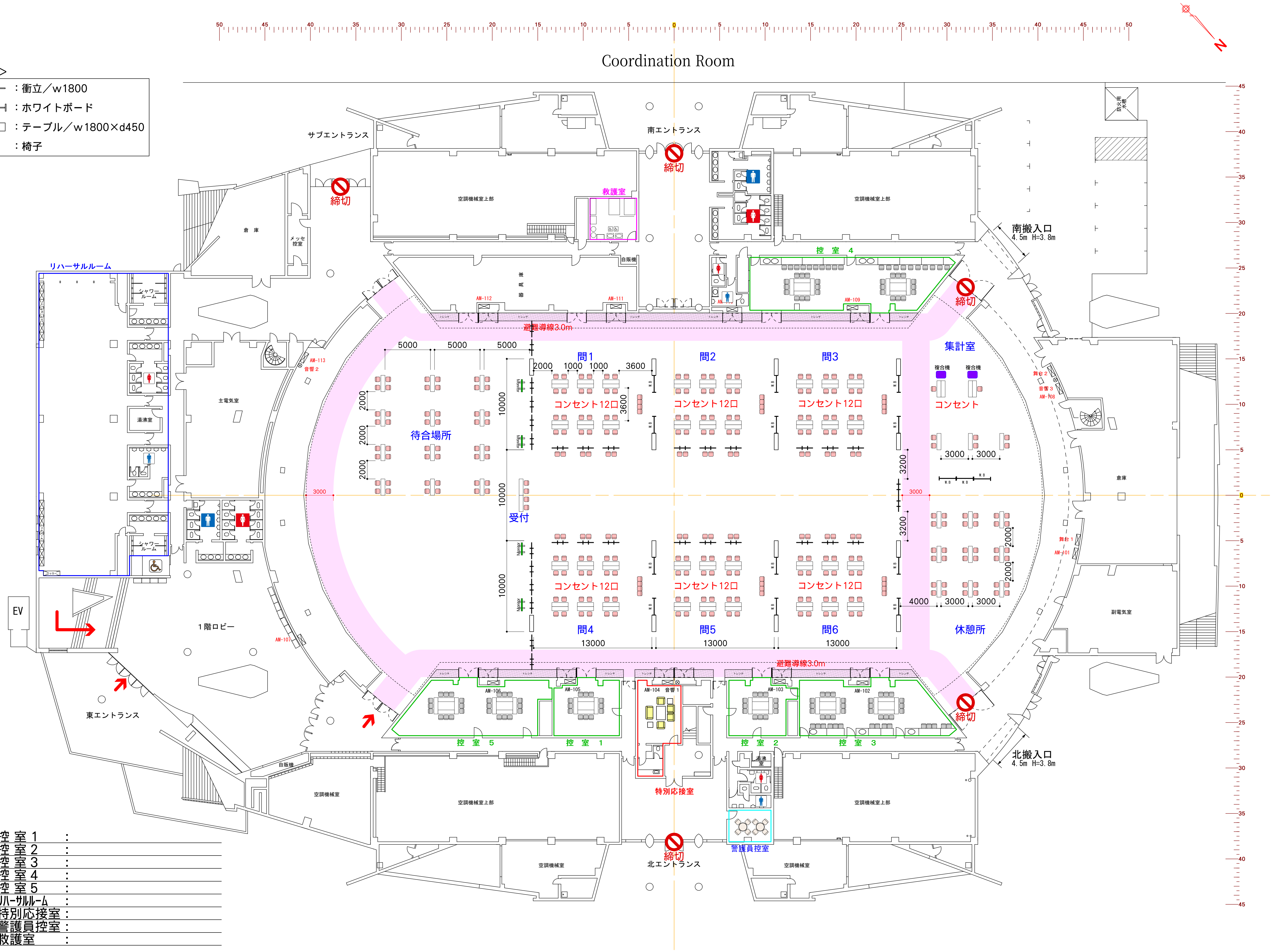
	Q1						Q2						Q3						Q4						Q5						Q6					
Table	A	B	C	D	E	F	A	B	C	D	E	F	A	B	C	D	E	F	A	B	C	D	E	F	A	B	C	D	E	F	A	B	C	D	E	F
9:00	AUS	SWE	RUS	NIC	LIE	KGZ	THA	FRA	TKM	BGD	MAR	LTU	GER	NLD	CHN	KEN	MRT	TJK																		
9:30	JPN	KOR	IRN	PER	ITA	MKD	USA	ALG	SVN	ESP	IRL	SUI	CUB	ALB	MNG	NZR	BRA	UKR																		
10:15	SRB	LUX	SVK	RWA	KSV	MNE	BGR	CAN	VEN	MAS	MDA	ISL	TWN	POL	CRI	LKA	OMN	HND																		
10:45	UZB	SGP	CHN	ARM	PHI	EST	VNM	SAF	POR	KAZ	BOL	COL	USA	TUR	MMR	GEO	ARG	HRV	AUS	SWE	RUS	NIC	LIE	KGZ	AUT	ROU	GHA	SYR	PRI	CYP	UNK	ISR	MAC	SLV	TUN	URY
11:30	GER	NLD	PAN	KEN	MRT	TJK	CHI	HUN	ECU	PAK	FIN	PAR	UAE	FRA	TKM	BGD	MAR	LTU	JPN	KOR	IRN	RWA	KSV	MNE	BEL	SAU	HKG	CZE	NOR	BWA	UGA	BLR	AZE	DEN	BIH	LVA
14:00	CUB	ALB	MNG	NZL	BRA	UKR	IDN	NPL	IND	HEL	TUN	MEX	THA	ALG	SVN	ESP	IRL	SUI	SRB	LUX	SVK	PER	ITA	MKD	BFA	TZA	DOM	GTM	IRQ	CMR	AUT	ROU	GHA	SYR	PRI	CYP
14:30	TWN	POL	CRI	LKA	OMN	HND	UNK	ISR	MAC	SLV	BIH	URY	BGR	CAN	VEN	MAS	MDA	ISL	UZB	SGP	PAN	ARM	PHI	EST	AUS	SWE	RUS	NIC	LIE	KGZ	BEL	SAU	HKG	CZE	NOR	BWA
15:00	USA	TUR	MMR	GEO	ARG	HRV	UGA	BLR	AZE	DEN	PRI	LVA	VNM	SAF	POR	KAZ	BOL	COL	GER	NLD	CHN	KEN	MRT	TJK	JPN	KOR	IRN	RWA	KSV	MNE	BFA	TZA	DOM	GTM	IRQ	CMR
15:45	UAE	FRA	TKM	BGD	MAR	LTU	AUT	ROU	GHA	SYR	NOR	CYP	CHI	HUN	ECU	PAK	FIN	PAR	CUB	ALB	MNG	NZL	BRA	UKR	SRB	LUX	SVK	PER	ITA	MKD	AUS	SWE	RUS	NIC	LIE	KGZ
16:15	THA	ALG	SVN	ESP	IRL	SUI	BEL	SAU	HKG	CZE	LIE	BWA	IDN	NPL	IND	HEL	TUN	MEX	TWN	POL	CRI	LKA	OMN	HND	UZB	SGP	PAN	ARM	PHI	EST	JPN	KOR	IRN	RWA	KSV	MNE
17:00	BGR	CAN	VEN	MAS	MDA	ISL	AUS	SWE	RUS	NIC	IRQ	KGZ	UNK	ISR	MAC	SLV	BIH	URY	USA	FRA	TKM	BGD	ARG	LTU	GER	NLD	CHN	KEN	MRT	TJK	SRB	LUX	SVK	PER	ITA	MKD
17:30	CHI	SAF	POR	KAZ	BOL	COL	BFA	TZA	DOM	GTM	ITA	CMR	UGA	BLR	AZE	DEN	PRI	LVA	UAE	TUR	MMR	GEO	MAR	HRV	CUB	ALB	MNG	NZL	BRA	UKR	UZB	SGP	PAN	ARM	PHI	EST

Day 2 11 July

	Q1						Q2						Q3						Q4						Q5						Q6					
Table	A	B	C	D	E	F	A	B	C	D	E	F	A	B	C	D	E	F	A	B	C	D	E	F	A	B	C	D	E	F	A	B	C	D	E	F
9:00	VNM	HUN	ECU	PAK	FIN	PAR	SRB	LUX	SVK	PER	KSV	MKD	AUT	ROU	GHA	SYR	LIE	CYP	THA	ALG	SVN	ESP	IRL	SUI	TWN	POL	CRI	LKA	OMN	HND	GER	NLD	CHN	KEN	MRT	TJK
9:30	IDN	NPL	IND	HEL	TUN	MEX	JPN	KOR	IRN	RWA	PHI	MNE	BEL	SAU	HKG	CZE	IRQ	BWA	BGR	CAN	VEN	MAS	MDA	ISL	USA	FRA	TKM	BGD	ARG	LTU	CUB	ALB	MNG	NZL	BRA	UKR
10:15	UNK	ISR	MAC	SLV	BIH	URY	UZB	SGP	PAN	ARM	MRT	EST	BFA	TZA	DOM	GTM	NOR	CMR	CHI	SAF	POR	KAZ	BOL	COL	UAE	TUR	MMR	GEO	MAR	HRV	TWN	POL	CRI	LKA	OMN	HND
10:45	UGA	BLR	AZE	DEN	PRI	LVA	GER	NLD	CHN	KEN	BRA	TJK	JPN	SWE	RUS	NIC	ITA	KGZ	VHM	HUN	ECU	PAK	FIN	PAR	THA	ALG	SVN	ESP	IRL	SUI	USA	FRA	TKM	BGD	ARG	LTU
11:30	AUT	ROU	GHA	SYR	NOR	CYP	CUB	ALB	MNG	NZL	OMN	UKR	SRB	LUX	SVK	PER	KSV	MKD	IDN	NPL	IND	HEL	TUN	MEX	BGR	CAN	VEN	MAS	MDA	ISL	UAE	TUR	MMR	GEO	MAR	HRV
12:00	BEL	SAU	HKG	CZE	IRQ	BWA	TWN	POL	CRI	LKA	ARG	HND	UZB	SGP	PAN	RWA	PHI	MNE	UNK	ISR	MAC	SLV	BIH	URY	CHI	SAF	POR	KAZ	BOL	COL	THA	ALG	SVN	ESP	IRL	SUI
14:00	BFA	TZA	DOM	GTM		CMR	UAE	TUR	MMR	GEO		HRV	AUS	KOR	IRN	ARM		EST	UGA	BLR	AZE	DEN	PRI	LVA	VNM	HUN	ECU	PAK	FIN	PAR	BGR	CAN	VEN	MAS	MDA	ISL
14:30																			AUT	ROU	GHA	SYR	NOR	CYP	IDN	NPL	IND	HEL	TUN	MEX	CHI	SAF	POR	KAZ	BOL	COL
15:15																			BEL	SAU	HKG	CZE	IRQ	BWA	UNK	ISR	MAC	SLV	BIH	URY	VNM	HUN	ECU	PAK	FIN	PAR
15:45																			BFA	TZA	DOM	GTM		CMR	UGA	BLR	AZE	DEN		LVA	IDN	NPL	IND	HEL		MEX
17:00																																				
17:30																																				

＜凡例＞

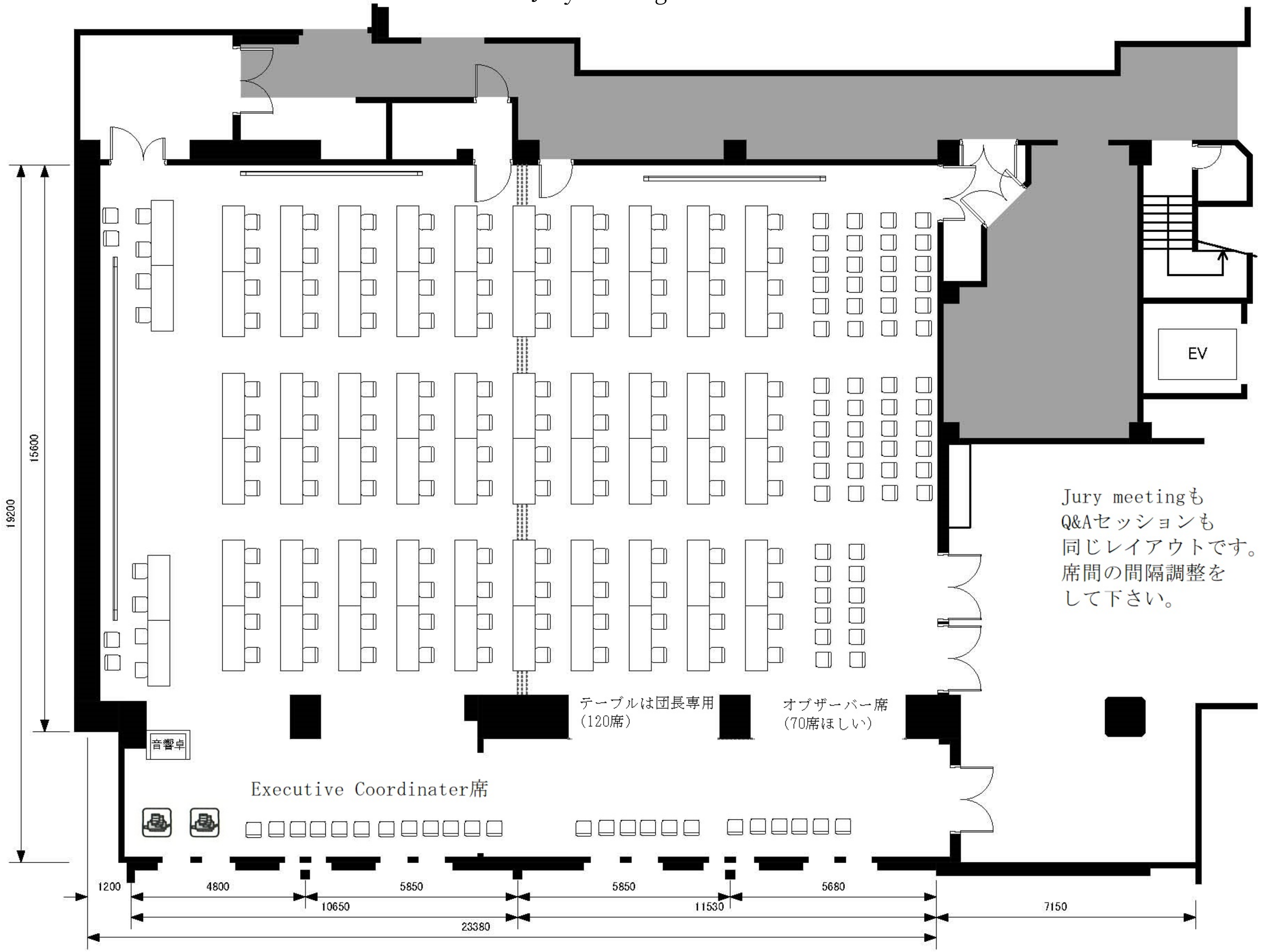
- ：衝立／w1800
- W.B ：ホワイトボード
- ：テーブル／w1800×d450
- ：椅子



控 室 1	：
控 室 2	：
控 室 3	：
控 室 4	：
控 室 5	：
ﾘﾊｰｻﾙﾙｰﾑ	：
特別応接室	：
警護員控室	：
救護室	：

PROJECT	TITLE	SCHEDULE	PLACE	DRAWING	UP DATED	SCALE	SIZE		No.
国際数学オリンピック	1階_平面図	'23.10.10-11	幕張イベントホール	'22.11.18	'23.06.13	1/400	A3size	文化が集まる、未来が集まる。 幕張メッセ Makuhari Messe	01

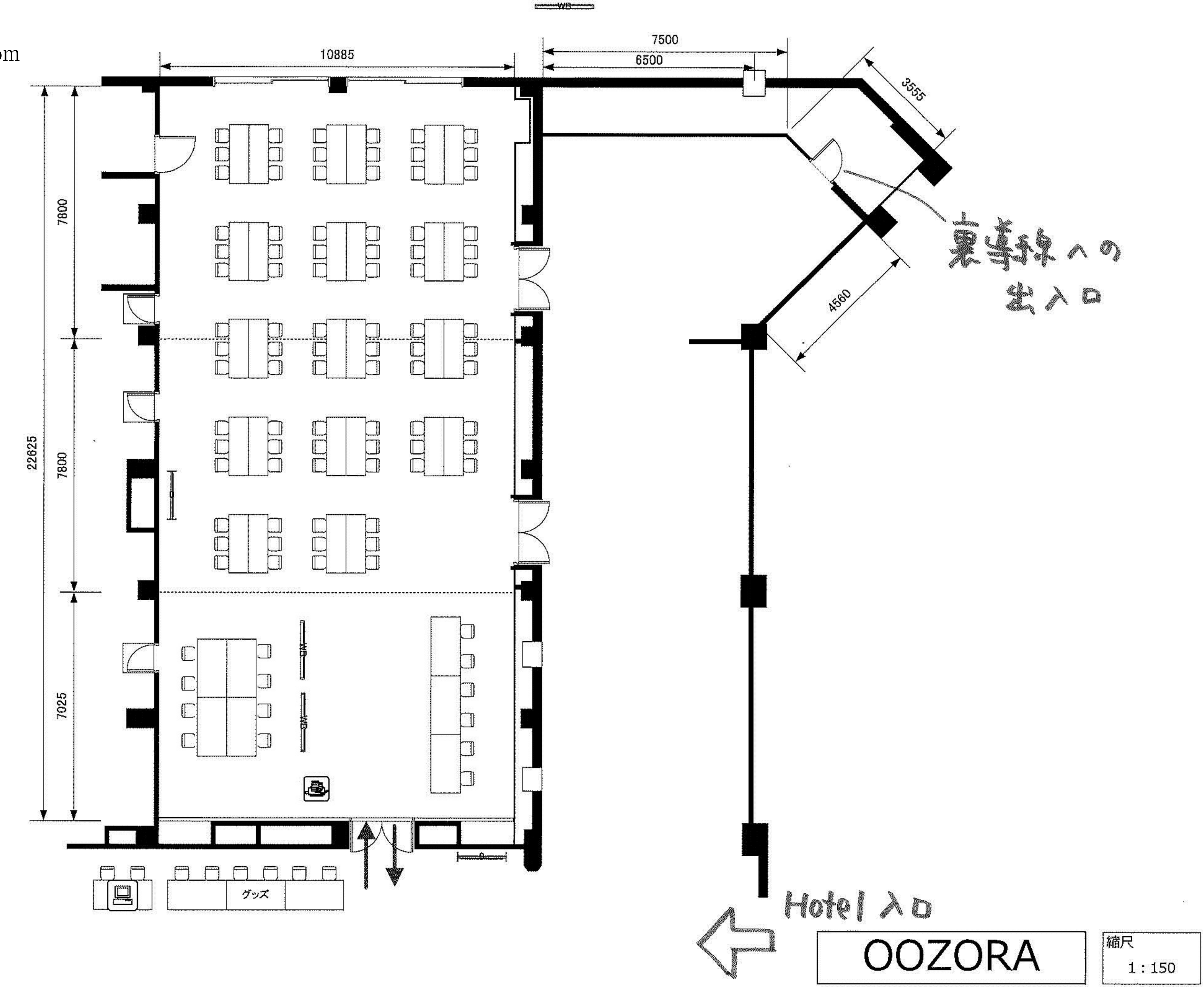
Jury Meeting Room



TSURU

縮尺 1 : 150

Working Room



Meeting Rooms