Some Cubic and Quartic Inequalities of Four Variables

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Abstract. Let $\mathcal{H} \subset \mathcal{H}_{n,d} := \mathbb{R}[x_1,\ldots, x_n]_d$ be a vector space, and A be a compact semialgebraic subset of $\mathbb{P}^{n-1}_{\mathbb{R}}$. We shall study some PSD cones $\mathcal{P} = \mathcal{P}(A, \mathcal{H}) := \{f \in \mathcal{H} \mid f(a) \geq 0\ (\forall a \in A)\}$. Our interests are (1) to determine the extremal elements of $\mathcal{P}$, (2) to determine discriminants of $\mathcal{P}$, (3) to describe $\mathcal{P}$ as a union of basic semialgebraic subsets, and (4) to find a nice test set when $\dim \mathcal{H}$ is low. In this article, we present (1), (2), (3) and (4) for $\mathcal{P}(\mathbb{R}^4, \mathcal{H}^{s0}_{4,4})$ and $\mathcal{P}(\mathbb{R}^4_+, \mathcal{H}^{s0}_{4,4})$, where $\mathcal{H}^{s0}_{n,d} := \{f \in \mathcal{H}_{n,d} \mid f$ is symmetric and $f(1,\ldots,1) = 0\}$. We also provide (1)—(4) for $\mathcal{P}(\mathbb{R}^4_+, \mathcal{H}^{c0}_{4,3})$, where $\mathcal{H}^{c0}_{n,d} := \{f \in \mathcal{H}_{n,d} \mid f$ is cyclic and $f(1,\ldots,1) = 0\}$.

§1. Introduction.

Let $\mathcal{H}_{n,d} := \mathbb{R}[x_1,\ldots, x_n]_d$ (the part of degree d), and $\mathcal{H} \subset \mathcal{H}_{n,d}$ be a vector subspace. For a semialgebraic subset A of $\mathbb{R}^n$,

$$\mathcal{P}(A, \mathcal{H}) := \{f \in \mathcal{H} \mid f(a) \geq 0 \text{ for all } a \in A\}$$

is called the PSD cone on A in $\mathcal{H}$. Our interests are:

(I1) To determine all the extremal elements of $\mathcal{P} := \mathcal{P}(A, \mathcal{H})$.

(I2) To determine all the discriminants of $\mathcal{P}$ (see Definition 2.6).

(I3) To describe $\mathcal{P}$ as a union of basic semialgebraic subsets using some inequalities.

(I4) Find a nice test set for $(A, \mathcal{H})$ when $\dim \mathcal{H}$ is low (see Definition 2.9).

In this article, we present (I1), (I2), (I3) and (I4) for PSD cones $\mathcal{P}^{s0}_{4,4}$, $\mathcal{P}^{s0+}_{4,4}$ and $\mathcal{P}^{c0+}_{4,3}$. We also treat some SOS problems relating these PSD cones. We shall explain these symbols. Let

$$\begin{aligned}
\mathcal{H}^c_{n,d} &:= \{f \in \mathcal{H}_{n,d} \mid f(x_2,\ldots,x_n,x_1) = f(x_1,\ldots,x_n)\},\\
\mathcal{H}^s_{n,d} &:= \{f \in \mathcal{H}_{n,d} \mid f(x_{\sigma(1)},\ldots,x_{\sigma(n)}) = f(x_1,\ldots,x_n) \text{ for all } \sigma \in \mathfrak{S}_n\},\\
\mathcal{H}^0_{n,d} &:= \{f \in \mathcal{H}_{n,d} \mid f(a,a,\ldots,a) = 0 \text{ for all } a \in \mathbb{R}\},\\
\mathcal{E}(\mathcal{P}) &:= \{f \in \mathcal{P} \mid f \text{ is a extremal element of } \mathcal{P}\},\\
\mathbb{R}_+ &:= \{x \in \mathbb{R} \mid x \geq 0\},
\end{aligned}$$

and $\mathcal{H}^{c0}_{n,d} := \mathcal{H}^c_{n,d} \cap \mathcal{H}^0_{n,d}$, $\mathcal{H}^{s0}_{n,d} := \mathcal{H}^s_{n,d} \cap \mathcal{H}^0_{n,d}$. We denote $\mathcal{P}_{n,d} := \mathcal{P}(\mathbb{R}^n, \mathcal{H}_{n,d})$, $\mathcal{P}^+_{n,d} := \mathcal{P}(\mathbb{R}^n_+, \mathcal{H}_{n,d})$, $\mathcal{P}^s_{n,d} := \mathcal{P}(\mathbb{R}^n, \mathcal{H}^s_{n,d})$, $\mathcal{P}^{s+}_{n,d} := \mathcal{P}(\mathbb{R}^n_+, \mathcal{H}^s_{n,d})$, $\mathcal{P}^{s0}_{n,d} := \mathcal{P}(\mathbb{R}^n, \mathcal{H}^{s0}_{n,d})$, $\mathcal{P}^{s0+}_{n,d} := \mathcal{P}(\mathbb{R}^n_+, \mathcal{H}^{s0}_{n,d})$, $\mathcal{P}^c_{n,d} := \mathcal{P}(\mathbb{R}^n, \mathcal{H}^c_{n,d})$, $\mathcal{P}^{c+}_{n,d} := \mathcal{P}(\mathbb{R}^n_+, \mathcal{H}^c_{n,d})$, $\mathcal{P}^{c0}_{n,d} := \mathcal{P}(\mathbb{R}^n, \mathcal{H}^{c0}_{n,d})$, and

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$\mathcal{P}^{c0+}_{n,d} := \mathcal{P}(\mathbb{R}^n_+, \mathcal{H}^{c0}_{n,d})$. The rule of indexing will be clear. "c" means cyclic, "s" means symmetric, "0" means an equality condition $f(a,\ldots, a) = 0$, and "+" means $A = \mathbb{R}^n_+$.

We have already completed (I1), (I2) and (I3) for the PSD cones $\mathcal{P}^{c+}_{3,3}$, $\mathcal{P}^{c0+}_{3,3}$, $\mathcal{P}^{c0}_{3,4}$, $\mathcal{P}^{c0+}_{3,4}$, $\mathcal{P}^{s}_{3,4}$ and $\mathcal{P}^{s0+}_{3,5}$. See [3], [2]and [1]. For $\mathcal{P}^{c0}_{3,4}$, see also [13]and [19]. (I4) for $\mathcal{P}^{c+}_{3,3}$ is provided in Example 2.11. (I1) for $\mathcal{P}^{+}_{3,3}$ is given in [4].

In §3, we study $\mathcal{P}^{s0}_{4,4}$ and $\mathcal{P}^{s0+}_{4,4}$. (I1)—(I4) for $\mathcal{P}^{s0}_{4,4}$ are given in Theorem 3.4, and these for $\mathcal{P}^{s0+}_{4,4}$ are given in Theorem 3.8. Here, we present (I3) for $\mathcal{P}^{s0}_{4,4}$ and $\mathcal{P}^{s0}_{4,4}$ slightly different style from Theorem 3.4 and 3.8.

Theorem 1.1. *Let* $\sigma_1 := a_0 + a_1 + a_2 + a_3$, $\sigma_2 := \sum_{0\le i<j\le 3} a_i a_j$, $\sigma_3 := \sum_{0\le i<j<k\le 3} a_i a_j a_k$, *and* $\sigma_4 := a_0a_1a_2a_3$. *Consider a family of quartic symmetric polynomials*

$$f(a_0, a_1, a_2, a_3) = \sigma_1^4 + p_1\sigma_1^2\sigma_2 + p_2\sigma_2^2 + p_3\sigma_1\sigma_3 - (256 + 96p_1 + 36p_2 + 16p_3)\sigma_4 \in \mathcal{H}^{s0}_{4,4}$$

$(p_1, p_2, p_3 \in \mathbb{R})$. *Then*

(1) $f(a_0, a_1, a_2, a_3) \ge 0$ *for all* $a_0,\ldots, a_3 \in \mathbb{R}$ *if and only if* $16 + 6p_1 + 2p_2 + p_3 \ge 0$ *and* $9p_1^2 \le 128 + 24p_1 + 36p_2 + 12p_3$.

(2) $f(a_0, a_1, a_2, a_3) \ge 0$ *for all* $a_0 \ge 0,\ldots, a_3 \ge 0$ *if and only if* "(i) *or* (ii)" *and* "(iii) *or* (iv)" *hold:*
   - (i) $p_1 \le -8$ *and* $p_1^2 \le 4p_2$.
   - (ii) $p_1 \ge -8$ *and* $4p_1 + p_2 + 16 \ge 0$.
   - (iii) $p_1 \le -14/3$ *and* $9p_1^2 \le 128 + 24p_1 + 36p_2 + 12p_3$.
   - (iv) $p_1 \ge -14/3$ *and* $27 + 9p_1 + 3p_2 + p_3 \ge 0$.

Next, we present (I1).

Theorem 1.2. *All the extremal elements of* $\mathcal{P}^{s0}_{4,4}$ *are positive multiples of the following polynomials:*

$$\mathfrak{g}_t(a_0, a_1, a_2, a_3) := \frac{1}{3}\big(3\sigma_1^4 - 2(t + 7)\sigma_1^2\sigma_2 + (t + 3)^2\sigma_2^2 - 2(t^2 - 9)\sigma_1\sigma_3 - 4(t + 3)^2\sigma_4\big),$$
$$\mathfrak{g}_\infty(a_0, a_1, a_2, a_3) := \sigma_2^2 - 2\sigma_1\sigma_3 - 4\sigma_4,$$
$$\mathfrak{p}(a_0, a_1, a_2, a_3) := \sigma_2^2 - 3\sigma_1\sigma_3 + 12\sigma_4.$$

Here, $t \in \mathbb{R}$. *Conversely, these are extremal elements of* $\mathcal{P}^{s0}_{4,4}$.

$\mathfrak{g}_t$ $(t \ne 1, -3)$ *is characterized by the equality conditions* $\mathfrak{g}_t(t, 1, 1, 1) = \mathfrak{g}_t(-1, -1, 1, 1) = 0$. $\mathfrak{g}_1$ *is characterized by the equality conditions* $\mathfrak{g}_1(x, x, 1, 1) = 0$ *for all* $x \in \mathbb{P}^1_{\mathbb{R}}$. $\mathfrak{g}_{-3}$ *is characterized by the equality conditions* $\mathfrak{g}_{-3}(a, b, c, -a - b - c) = 0$ *for all* $a, b, c \in \mathbb{R}$. $\mathfrak{g}_\infty$ *is characterized by the equality conditions* $\mathfrak{g}_\infty(0, 0, 0, 1) = \mathfrak{g}_\infty(-1, -1, 1, 1) = 0$.

$\mathfrak{p}$ *is characterized by the equality conditions* $\mathfrak{p}(0, 0, 0, 1) = 1$ *and* $\mathfrak{p}(s, 1, 1, 1) = 0$ *for all* $s \in \mathbb{R}$.

We say f is *characterized by the equality conditions* $f(\mathbf{x}_\lambda) = 0$ $(\lambda \in \Lambda)$ if

$$\mathbb{R}_+ \cdot f := \big\{g \in \mathcal{P} \;\big|\; g(\mathbf{x}_\lambda) = 0 \text{ for all } \lambda \in \Lambda\big\}.$$

Note that if $f \in \mathcal{P}$ is characterized by certain equality conditions, then f is extremal. About the converse, please read [4].

An elements $f \in \mathcal{P}_{n,2d}$ is called *SOS*, if there exists $r \in \mathbb{N}$ and $g_1,\dots, g_r \in \mathcal{P}_{n,d}$ such that $f = g_1^2 + \cdots + g_r^2$. The set of all the SOS elements in $\mathcal{P}_{n,2d}$ are written by the symbol $\Sigma_{n,2d}$, and is called a *SOS cone*. In this case, $\mathfrak{g}_t$, $\mathfrak{g}_\infty$, $\mathfrak{p} \in \Sigma_{4,4}$, since

$$
\begin{aligned}
3\mathfrak{g}_t(a,b,c,d) &= \big(a^2+b^2-c^2-d^2+(t+1)(cd-ab)\big)^2 \\
&\quad + \big(a^2-b^2+c^2-d^2+(t+1)(bd-ac)\big)^2 \\
&\quad + \big(a^2-b^2-c^2+d^2+(t+1)(bc-ad)\big)^2 \\
&= \frac{1}{16}\sum_{\tau\in\mathfrak{S}_4} \big(a_{\tau(0)}-a_{\tau(1)}\big)^2\big(2(a_{\tau(0)}+a_{\tau(1)})-(t+1)(a_{\tau(2)}+a_{\tau(3)})\big)^2, \\
\mathfrak{g}_\infty(a,b,c,d) &= (ab-cd)^2+(ac-bd)^2+(ad-bc)^2, \\
\mathfrak{p}(a,b,c,d) &= (1/2)\big((a-b)^2(c-d)^2+(a-c)^2(b-d)^2+(a-d)^2(b-c)^2\big).
\end{aligned}
$$

Here $(a_0, a_1, a_2, a_3) = (a, b, c, d)$. Moreover, $\mathfrak{g}_t$, $\mathfrak{p} \notin \mathcal{E}(\mathcal{P}^{s0+}_{4,4})$. Thus we obtain:

Corollary 1.3. *$\mathcal{P}^{s0}_{4,4} \subset \Sigma_{4,4}$, and $\mathcal{E}(\mathcal{P}^{s0}_{4,4}) \cap \mathcal{E}(\mathcal{P}_{4,4}) = \emptyset$.*

Remember that $\mathcal{E}(\mathcal{P}^{c0}_{3,4}) \cap \mathcal{E}(\mathcal{P}_{3,4}) = \emptyset$, because $f \in \mathcal{E}(\mathcal{P}^{c0}_{3,4})$ is not a square of a quadric polynomial (see [13]). The following theorem provides extremal elements which do not appear in [25].

Theorem 1.4. *All the extremal elements of $\mathcal{P}^{s0+}_{4,4}$ are positive multiples of the following polynomials:*

$$
\begin{aligned}
\mathfrak{f}^{ab}_t(a_0,a_1,a_2,a_3) &:= (1/3)\Big(3\sigma_1^4 - 2(t+7)\sigma_1^2\sigma_2 + 8(t+1)\sigma_2^2 \\
&\qquad + (t^2-6t+21)\sigma_1\sigma_3 - 16(t^2+3)\sigma_4\Big) \quad (0 \le t \le 5), \\
\mathfrak{f}^{c}_t(a_0,a_1,a_2,a_3) &:= (1/9)\Big(9\sigma_1^4 - 6(t+7)\sigma_1^2\sigma_2 + (t+7)^2\sigma_2^2 \\
&\qquad + 12(t-1)\sigma_1\sigma_3 - 12(t-1)(3t+13)\sigma_4\Big) \quad (t \ge 5), \\
\mathfrak{p}(a_0,a_1,a_2,a_3) &:= \sigma_2^2 - 3\sigma_1\sigma_3 + 12\sigma_4, \\
\mathfrak{q}_1(a_0,a_1,a_2,a_3) &:= \sigma_1^2\sigma_2 - 4\sigma_2^2 + 3\sigma_1\sigma_3 = \sum_{i<j} a_i a_j (a_i - a_j)^2, \\
\mathfrak{q}_2(a_0,a_1,a_2,a_3) &:= \sigma_1\sigma_3 - 16\sigma_4 = \frac{1}{4}\sum_{\tau\in\mathfrak{S}_4} a_{\tau(0)}a_{\tau(1)}\big(a_{\tau(2)}-a_{\tau(3)}\big)^2.
\end{aligned}
$$

Conversely, these are extremal elements of $\mathcal{P}^{s0+}_{4,4}$.

$\mathfrak{f}^{ab}_t$ *($0 \le t < 1$ or $1 < t \le 5$) is characterized by the equality conditions*

$$\mathfrak{f}^{ab}_t(t,1,1,1) = \mathfrak{f}^{ab}_t(0,0,1,1) = 0.$$

$\mathfrak{f}^{ab}_1$ *is characterized by the equality conditions*

$$\mathfrak{f}^{ab}_1(t,t,1,1) = 0 \quad \text{for all } t \ge 0 \text{ and } \quad \frac{\partial^2}{\partial a_0^2}\mathfrak{f}^{ab}_1(1,1,1,1) = 0.$$

$\mathfrak{f}^{c}_t$ *($t > 5$) is characterized by the equality conditions*

$$\mathfrak{f}^{c}_t(t,1,1,1) = \mathfrak{f}^{c}_t(0,0,u,1) = 0,$$

where $u \in \mathbb{R}_+$ *is any root of* $3u^2 - (t+1)u + 3 = 0$. $\mathfrak{p}$ *is characterized by the equality conditions*

$$\mathfrak{p}(0,0,0,1) = \mathfrak{p}_a(0,0,0,1) = \mathfrak{p}(x,1,1,1) = 0$$

for all $x \geq 0$. $\mathfrak{q}_1$ *is characterized by the equality conditions*

$$\mathfrak{q}_1(1,1,1,0) = \mathfrak{q}_1(1,1,0,0) = \mathfrak{q}_1(1,0,0,0) = 0.$$

$\mathfrak{q}_2$ *is characterized by the equality conditions* $\mathfrak{q}_2(s,1,0,0) = 0$ *for all* $s \geq 0$.

By the above representation, we have $\mathfrak{p}(a^2,\, b^2,\, c^2,\, d^2)$, $\mathfrak{q}_i(a^2,\, b^2,\, c^2,\, d^2) \in \Sigma_{4,8}$ $(i = 1,$ $2)$. But for $f = \mathfrak{f}_t^{ab}$ and $\mathfrak{f}_t^c$, we obtain:

Proposition 1.5. *If* $0 < t \leq 5$ *and* $t \neq 1$, *then* $\mathfrak{f}_t^{ab}(a^2,\, b^2,\, c^2,\, d^2) \notin \Sigma_{4,8}$. *If* $t > 5$, *then* $\mathfrak{f}_t^{c}(a^2,\, b^2,\, c^2,\, d^2) \notin \Sigma_{4,8}$.

It is clear that $\mathfrak{p}$, $\mathfrak{q}_1$, $\mathfrak{q}_2 \notin \boldsymbol{\mathcal{E}}(\boldsymbol{\mathcal{P}}_{4,4}^{+})$. But we have:

Proposition 1.6. *If* $t > 5$, *then* $\mathfrak{f}_t^c \in \boldsymbol{\mathcal{E}}(\boldsymbol{\mathcal{P}}_{4,4}^{s0+}) \cap \boldsymbol{\mathcal{E}}(\boldsymbol{\mathcal{P}}_{4,4}^{+})$.

Remember that if $f \in \boldsymbol{\mathcal{E}}(\boldsymbol{\mathcal{P}}_{3,4}^{s0})$, f can be written as $f = g\overline{g}$, where g is an imaginal quadric polynomial.

Proposition 1.7. (1) *If* $t \neq -3$, *then* $\mathfrak{g}_t$ *is irreducible in* $\mathbb{C}[a,b,c,d]$.
(2) *If* $0 \leq t \leq 5$, *then* $\mathfrak{f}_t^{ab}$ *is irreducible in* $\mathbb{C}[a,b,c,d]$.
(3) *If* $t > 5$, *then* $\mathfrak{f}_t^{c}$ *is irreducible in* $\mathbb{C}[a,b,c,d]$.

We should explain about the discriminants of $\boldsymbol{\mathcal{P}} = \boldsymbol{\mathcal{P}}(A,\, \boldsymbol{\mathcal{H}})$. Let s_0, $s_1, \ldots,$ s_N be a basis of the vector space $\boldsymbol{\mathcal{H}}$, and let $\Phi_{\boldsymbol{\mathcal{H}}} \colon A \to \cdots \mathbb{P}_{\mathbb{R}}^N$ be the rational map defined by $\Phi_{\boldsymbol{\mathcal{H}}}(\mathbf{a}) = \big(s_0(\mathbf{a}) \colon \cdots \colon s_N(\mathbf{a})\big)$. $X := \Phi_{\boldsymbol{\mathcal{H}}}(A)$ is called the *characteristic variety*. Let $\Delta(X) = \big\{D_1, \ldots,$ $D_r\big\}$ be the critical decomposition of X (see Definition 2.3). Each $D \in \Delta(X)$ is a smooth semialgebraic variety, and D has its dual variety $D^\vee$. Let $\mathrm{disc}(D)$ be the defining equation of the Zariski closure of $D^\vee$ in $\boldsymbol{\mathcal{H}}$, and let $V_{\boldsymbol{\mathcal{H}}}(\mathrm{disc}(D))$ be the zero locus of $\mathrm{disc}(D)$ in $\boldsymbol{\mathcal{H}}$. If $\dim \big(V_{\boldsymbol{\mathcal{H}}}(\mathrm{disc}(D)) \cap \partial \boldsymbol{\mathcal{P}}\big) = \dim \boldsymbol{\mathcal{P}} - 1$, we say $\mathrm{disc}(D)$ is a *discriminant* of $\boldsymbol{\mathcal{P}}$. For any $f \in \partial\boldsymbol{\mathcal{P}}$, there exists $D \in \Delta(X)$ such that $f \in V_{\boldsymbol{\mathcal{H}}}(\mathrm{disc}(D)$. Assume that a subset $B \subset A$ satisfies $\Phi_{\boldsymbol{\mathcal{H}}}(B) = D$. Then, for each $f \in V_{\boldsymbol{\mathcal{H}}}(\mathrm{disc}(D)) \cap \partial\boldsymbol{\mathcal{P}}$, there exists a point $\mathbf{a} \in B$ such that $f(\mathbf{a}) = 0$. In this case, we shall say that $\mathrm{disc}(D)$ is a discriminant *corresponding to* B.

Theorem 1.8. *Let's denote the elements of* $\boldsymbol{\mathcal{H}}_{4,4}^{s0}$ *as*

$$f(a_0, a_1, a_2, a_3) = p_0\sigma_1^4 + p_1\sigma_1^2\sigma_2 + p_2\sigma_2^2 + p_3\sigma_1\sigma_3 - (256p_0 + 96p_1 + 36p_2 + 16p_3)\sigma_4,$$

and use $(p_0, \ldots,\, p_3)$ *as a coordinate system of* $\boldsymbol{\mathcal{H}}_{4,4}^{s0}$.
(1) $\boldsymbol{\mathcal{P}}_{4,4}^{s0}$ *has the following two discriminants:*

$$d_1 := 128p_0^2 + 24p_0p_1 + 36p_0p_2 + 12p_0p_3 - 9p_1^2, \quad d_2 := 16p_0 + 6p_1 + 2p_2 + p_3.$$

d_1 *corresponds to* $\big\{(t,1,1,1) \in \mathbb{R}^4 \;\big|\; t \in \mathbb{R},\, t \neq -3,\, 1\big\}$, *and* d_2 *corresponds to a point* $(1,\, 1,\, -1,\, -1)$.
(2) $\boldsymbol{\mathcal{P}}_{4,4}^{s0+}$ *has the following five discriminants:*

$$d_1 := 128p_0^2 + 24p_0p_1 + 36p_0p_2 + 12p_0p_3 - 9p_1^2, \quad d_3 := 4p_0p_2 - p_1^2,$$
$$d_4 := 27p_0 + 9p_1 + 3p_2 + p_3, \quad d_5 := 16p_0 + 4p_1 + p_2, \quad d_6 := p_0.$$

d_3 corresponds to $\{(0,0,t,1) \in \mathbb{R}^4 \mid 0 < t < 1\}$. d_4, d_5, d_6 corresponds to points (1, 1, 1, 0), (1, 1, 0, 0), (1, 0, 0, 0) *respectively.*

We explain about (I4). Let $r_0 := \max\{2, \lfloor d/2 \rfloor\}$. For general $f \in \mathcal{H}^s_{n,d}$, Riener, Timofte and Harris proved that $f \in \mathcal{P}^s_{n,d}$ if $f(x) \geq 0$ for all $x \in \{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid \#\{x_1,\ldots, x_n\} \leq r_0\}$. Moreover, $f \in \mathcal{P}^{s+}_{n,d}$ if $f(x) \geq 0$ for all $x \in \{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid \#(\{x_1,\ldots, x_n\} - \{0\}) \leq r_0\}$. (See Corollary 1.3 of [22], Corollary 2.1 of [23]. See also [24], [25].)

In the case $f \in \mathcal{H}^s_{4,4}$, if $f(t, t, 1, 1) \geq 0$ and $f(t, 1, 1, 1) \geq 0$ for all $t \in \mathbb{R}$ then $f \in \mathcal{P}^s_{4,4}$. If $f(t, t, 1, 1) \geq 0$, $f(t, 1, 1, 1) \geq 0$, $f(0, t, 1, 1) \geq 0$ and $f(0, 0, t, 1) \geq 0$ for all $t \geq 0$ then $f \in \mathcal{P}^{s+}_{4,4}$.

We prove that the number of test conditions can be decreased as the following theorem in the cases of $\mathcal{P}^{s0}_{4,3}$ and $\mathcal{P}^{s0+}_{4,3}$.

Theorem 1.9. (1) *If $f \in \mathcal{H}^{s0}_{4,4}$ satisfies $f(-1, -1, 1, 1) \geq 0$ and $f(t, 1, 1, 1) \geq 0$ for all $t \in \mathbb{R}$, then $f(a, b, c, d) \geq 0$ for all $a, b, c, d \in \mathbb{R}$.*

(2) *If $f \in \mathcal{H}^{s0}_{4,4}$ satisfies $f(t, 1, 1, 1) \geq 0$ and $f(0, 0, t, 1) \geq 0$ for all $t \geq 0$, then $f(a, b, c, d) \geq 0$ for all $a, b, c, d \in \mathbb{R}_+$.*

In §4, we study the PSD cone of cyclic cubic polynomials $\mathcal{P}^{c0+}_{4,3}$. (I2) and (I3) for $\mathcal{P}^{c0+}_{4,3}$ are given in Theorem 4.1. $\mathcal{P}^{c0+}_{4,3}$ has 4 discriminants. Since one of them is very complicated polynomial, the structure of $\mathcal{P}^{c0+}_{4,3}$ is not simple. We also need somewhat strange algebraic numbers to state (I3). This is completely different from cases of $\mathcal{P}^{c0+}_{3,3}$ and $\mathcal{P}^{c+}_{3,3}$. (I1) and (I4) for $\mathcal{P}^{c0+}_{4,3}$ are as the following:

Theorem 1.10. (1) *All the elements of $\mathcal{E}(\mathcal{P}^{c0+}_{4,3})$ is the positive multiple of $\mathfrak{e}^h_{u,v,w}$ $((u{:}\,v{:}\,w) \in D^h_e)$ or $\mathfrak{e}^{P_2}_t$ $(t \in \mathbb{P}^1_{\mathbb{R}})$.*

(2) *If $f \in \mathcal{H}^c_{4,3}$ satisfies $f(1, 1, 1, 1) \geq 0$ and $f(0, s, t, 1) \geq 0$ for all $s, t \in \mathbb{R}_+$, then $f(a, b, c, d) \geq 0$ for all $a, b, c, d \in \mathbb{R}_+$.*

Definitions of $\mathfrak{e}^h_{u,v,w}$, $\mathfrak{e}^{P_2}_t$ and D^h_e are given in Proposition 4.3, Lemma 4.7 and Theorem 4.13 respectively. (1) is proved in Theorem 4.15, and (2) is proved in §4.2.

In [4], we have proved that $\mathcal{E}(\mathcal{P}^{c0+}_{3,3}) \subset \mathcal{E}(\mathcal{P}^{c+}_{3,3}) \subset \mathcal{E}(\mathcal{P}^{+}_{3,3})$. But $\mathcal{E}(\mathcal{P}^{c0+}_{4,3}) \not\subset \mathcal{E}(\mathcal{P}^{+}_{4,3})$. Relating SOS problem, $\mathfrak{e}^h_{u,v,w}$ satisfies:

Proposition 1.11. *Assume that $(u{:}\,v{:}\,w) \in D^h_e$, $u > 0$, $v > 0$, $w > 0$ and $v \neq u + w$. Then, $\mathfrak{e}^h_{u,v,w}(a^2, b^2, c^2, d^2) \in \mathcal{P}_{4.6} - \Sigma_{4,6}$.*

In §5, we will give an exact definition of semialgebraic varieties, and prove some basic general theorems. In this article, we use $\mathbb{P}^3_{\mathbb{R}}/\mathfrak{S}_4$ and $\mathbb{P}_+/\mathfrak{S}_4$. These are not real algebraic variety. $\mathbb{P}^3_{\mathbb{R}}/\mathfrak{S}_4$ does not agree with a real weighted projective space $\mathbb{P}_{\mathbb{R}}(1, 2, 3, 4)$. But we need to treat these with certain variety structure, i.e. semialgebraic varieties. So, the author think it will be better to give an exact definition of semialgebraic variety. For example, there exists continuous rational map which is not holomorphic (see Lemma 3.5). Such maps do not exist in complex algebraic geometry. Some results will be useful for studies of real algebraic varieties. Especially, Theorem 5.11 and Theorem 5.15 show that semialgebraic geometry is very different from complex algebraic geometry. In our theory of algebraic inequalities in this article, a phenomenon of Theorem 5.15 occurs. For example, $\Phi_{\mathcal{H}}\colon A \cdots \to X$ may include some exceptional set even if $A = \mathbb{P}^3_{\mathbb{R}}$ (see Lemma 3.5).

We shall explain a short history of study of PSD cones. Originally, $\mathcal{P}_{n,2d}$ is called a PSD cone. Hilbert proved, $\mathcal{P}_{n,2d} = \Sigma_{n,2d}$ if and only if $n \le 2$ or $2d = 2$ or $(n,\, 2d) = (3,\, 4)$ ([17]). History of studies till 1991 are written in §6.6 of [8]. So we don't explain them again. Choi and Lam found some extremal forms of $\mathcal{P}_{n,2d}$ which don't belong to $\Sigma_{n,2d}$ in [9]. In [21], Reznick studied the condition that $f \in \mathcal{P}_{n,2d}$ is included in $\Sigma_{n,2d}$. He also studied the condition that $f \in \mathcal{P}_{n,2d}$ is extremal. See also [10]. They implies that if $f \in \mathcal{E}(\mathcal{P}_{n,2d})$, then $V_{\mathbb{R}}(f)$ is larger set. This fact is formalized in Theorem 2.7 and Proposition 2.9 of [4].

An element $f \in \mathcal{H}_{n,2d}$ is called even, if $f \in \mathbb{R}[x_1^2, \ldots,\, x_n^2]$. Choi, Lam and Reznick studied $\mathcal{P}^{es}_{n,2d} := \mathcal{P}_{n,2d} \cap \mathbb{R}[x_1^2, \ldots,\, x_n^2]$ in [11]. They studied the condition for $\mathcal{P}^{es}_{n,2d} \subset \Sigma_{n,2d}$. Note that $\mathcal{P}^{es}_{n,2d} \cong \mathcal{P}^{s+}_{n,d}$, as is stated in [12]. Harris proved $\mathcal{P}^{es}_{3,8} \subset \Sigma_{3,8}$ in [15]. But $\mathcal{E}(\mathcal{P}^{+}_{3,3}) \cong \mathcal{E}(\mathcal{P}^{e}_{3,6}) \subset \mathcal{E}(\mathcal{P}_{3,6})$ and $\mathcal{E}(\mathcal{P}^{e}_{3,6}) \not\subset \Sigma_{3,6}$ (see [4]). The relations $\mathcal{P}^{s}_{n,2d}$ and $\Sigma^{s}_{n,2d}$ are studied by Goel, Kuhlmann and Reznick in [14]. A related study can be found in [7]. Our study of $\mathcal{E}(\mathcal{P}^{s0+}_{4,4})$ and $\mathcal{E}(\mathcal{P}^{c0+}_{4,3})$ will give a small contribution for it.

About discriminants of $\mathcal{P}(A,\, \mathcal{H})$, Nie shown some interesting results in [20]. He treated the case that A is an affine real algebraic variety. In this article, we only treat the cases that A is a compact semialgebraic variety. But they have very close relation. [6] provides many nice ideas to treat algebraic inequalities using complex algebraic geometry.

About $\mathcal{P}_{3,6}$, $\Sigma_{3,6}$, $\mathcal{P}_{4,4}$ and $\Sigma_{4,4}$, very important results are obtained in [5]. It provides relation with theory of K3 surfaces.

$\dim \mathcal{H}_{4,3} = 20$ and $\dim \mathcal{H}_{4,4} = 35$ are somewhat large to proceed precise numerical analysis. It will not be insignificant to study some lower dimensional subspaces $\mathcal{H} \subset \mathcal{H}_{n,d}$.

To check many calculations in this article, we will need Mathematica or a similar tool. The author provides a file for Mathematica in the authors WEB and in arXiv's anc folder. It will be useful for experimentation of inequalities.

§2. General theories

2.1. Known results.

By studies in [3], we have better to use $\mathbb{P}^{n-1}_{\mathbb{R}}$ and $\mathbb{P}^{n-1}_{+}$ instead of $\mathbb{R}^n$ and $\mathbb{R}^n_+$ where

$$\mathbb{P}^n_+ := (\mathbb{R}^{n+1}_+ - \{0\})/\mathbb{R}^\times_+ = \big\{(x_0 \colon \cdots \colon x_n) \in \mathbb{P}^n_{\mathbb{R}} \;\big|\; x_0 \ge 0, \ldots,\, x_n \ge 0\big\}.$$

The merits are that $\mathbb{P}^{n-1}_{\mathbb{R}}$ is compact and $\dim \mathbb{P}^{n-1}_{\mathbb{R}} < \dim \mathbb{R}^n$. But $f \in \mathcal{H}_{n,d}$ is not a function on $\mathbb{P}^{n-1}_{\mathbb{R}}$. So, we must treat $\mathcal{H}_{n,d}$ as a signed linear system on $\mathbb{P}^{n-1}_{\mathbb{R}}$. We need some more generalizations. About the exact definition of a semialgebraic variety, please see §5. We may understand here that a semialgebraic variety $(A,\, \mathcal{R}_A)$ is a locally ringed space with semialgebraic set A and a sheaf of rings $\mathcal{R}_A$ which represent real holomorphic functions on open subsets of A. We only use $\mathcal{R}_A$ to define singularities of A, regular maps between semialgebraic varieties, and signed linear systems. The author apologizes that Definition 1.7 of [3] must be corrected as the following:

Definition 2.1. Let $(A,\, \mathcal{R}_A)$ be a semialgebraic variety, and $\mathcal{C}^0_A$ be the sheaf of germs of real continuous functions on A.

(1) Let $\mathcal{I}$ be an invertible $\mathcal{R}_A$-sheaf. $\mathcal{I}$ is called a *signed invertible sheaf* on A if

(i) there exists $\mathfrak{C}^0_A$-invertible sheaf $\mathfrak{J}$ such that $\mathfrak{I} \otimes_{\mathfrak{R}_A} \mathfrak{C}^0_A = \mathfrak{J} \otimes_{\mathfrak{C}^0_A} \mathfrak{J}$, and

(ii) there exists $e \in \mathfrak{J}(A)$ such that $e^2 \in \mathfrak{I}(A)$ and $\mathfrak{I}(A) = \mathfrak{R}_A(A) \cdot e^2$.

Then, for $f \in H^0(A, \mathfrak{I})$, there exists $g \in H^0(A, \mathfrak{R}_A)$ such that $f = ge^2$. We define $\operatorname{sign}(f(P)) \in \{0, \pm 1\}$ by $\operatorname{sign}(f(P)) = \operatorname{sign}(g(P))$ for $P \in A$.

(2) Let $\mathfrak{I}$ be a signed invertible $\mathfrak{R}_A$-sheaf. A finite dimensional vector subspace $\mathfrak{H} \subset H^0(A, \mathfrak{I})$ is called a *signed linear system* on A. For $f \in \mathfrak{H}$, we say f is *PSD* on A if $f(P) \geq 0$ for all $P \in A$.

(3) $\mathfrak{P} = \mathfrak{P}(A, \mathfrak{H}) := \{ f \in \mathfrak{H} \mid f(P) \geq 0 \text{ for all } P \in X \}$ is called the *PSD cone* on A in $\mathfrak{H}$. Note that $\mathfrak{P}_{n,d} = \mathfrak{P}(\mathbb{P}^{n-1}_{\mathbb{R}}, \mathfrak{H}_{n,d})$ and $\mathfrak{P}^+_{n,d} = \mathfrak{P}(\mathbb{P}^{n-1}_{+}, \mathfrak{H}_{n,d})$ and so on.

(4) $\operatorname{Bs}\mathfrak{H} := \{ P \in A \mid f(P) = 0 \text{ for all } f \in \mathfrak{H} \}$ is called the *base locus* of $\mathfrak{H}$. When $\mathfrak{P}$ is non-degenerate in $\mathfrak{H}$, we define $\operatorname{Bs}\mathfrak{P} := \operatorname{Bs}\mathfrak{H}$.

If $\dim \operatorname{Bs}\mathfrak{P} < \dim A$, we can define a rational map $\Phi_{\mathfrak{H}} : A \cdots \to \mathbb{P}_{\mathbb{R}}(\mathfrak{H}^\vee)$, using a basis of $\mathfrak{H}$. $X = X(A, \mathfrak{H}) := \operatorname{Cls}(\Phi_{\mathfrak{H}}(A))$ (Euclidian closure) is called the *characteristic variety* of A.

For example,

$$\mathfrak{H}_{n+1,d} := \{ f(x_0, \ldots, x_n) \mid f \text{ is a homogeneous polynomial of degree } d \} \cup \{0\}$$

is a signed linear system on $\mathbb{P}^n_+$. For $f \in \mathfrak{H}_{n+1,d}$ and $P \in \mathbb{P}^n_+$, we cannot define the value $f(P)$ but can define $\operatorname{sign}(f(P))$. If d is even, $\mathfrak{H}_{n+1,d}$ is also a signed linear system on $\mathbb{P}^n_{\mathbb{R}}$.

Proposition 2.2. *Let $X := X(A, \mathfrak{H})$, and let Y be the convex closure of X in $\mathbb{P}(\mathfrak{H}^\vee)$. Then*

$$\mathfrak{P}(A, \mathfrak{H}) = \mathfrak{P}(X, \mathfrak{H}_{N+1,1}) = \mathfrak{P}(Y, \mathfrak{H}_{N+1,1}),$$

where $\mathfrak{H}_{N+1,1}$ is the set of linear polynomials on $\mathbb{P}(\mathfrak{H}^\vee)$.

Proof. $\mathfrak{P}(A, \mathfrak{H}) = \mathfrak{P}(X, \mathfrak{H}_{N+1,1})$ is proved at Proposition 1.13 in [3]. $\mathfrak{P}(X, \mathfrak{H}_{N+1,1}) = \mathfrak{P}(Y, \mathfrak{H}_{N+1,1})$ is clear since every element of $\mathfrak{H}_{N+1,1}$ is linear. □

Assume that a semialgebraic set B is a subset of a complete real algebraic variety V. The minimal closed algebraic subset which contains B is called the Zariski closure of B and is denoted by $\operatorname{Zar}_V(B)$. We denote the Euclidian closure of B in V by $\operatorname{Cls}_V(B)$ or $\overline{B}$. Assume that $\operatorname{Zar}_V(B) = V$. The interior of B is defined by $\operatorname{Int}(B) := V - \operatorname{Cls}_V(V - B)$. The boundary of B is defined by $\partial B := B - \operatorname{Int}(B)$. Do not confuse with $\partial_V B := \operatorname{Cls}_V(B) - \operatorname{Int}(B)$. Note that $\operatorname{Int}(B)$ and ∂B do not depend on the choice of V. But $\operatorname{Cls}_V(B)$ and $\partial_V B$ depend on V.

Definition 2.3.(Critical decomposition. See Definition 1.5 of [3]) Let A be a reduced semialgebraic variety with $\dim A = n$. We shall define $\Delta^i(A)$ $(i = 0, \ldots, n)$ by induction on n. If $\dim A = 0$, then $A = \{P_1, \ldots, P_m\}$ where P_i are points. In this case we put $\Delta^0(A) = \{P_1, \ldots, P_m\}$, and put $\Delta^i(A) = \emptyset$ for $i \neq 0$.

Assume that $n = \dim A \geq 1$. Let $Z_1, \ldots, Z_r$ be all the irreducible components of A with $\dim Z_i = n$. Put $A_i := \operatorname{Int}(Z_i - \operatorname{Sing}(A))$, and $\Delta^n(A) := \{A_1, \ldots, A_r\}$. Note that $Z_i \cap Z_j \cap \operatorname{Int}(A) \subset \operatorname{Sing}(A)$ for $i \neq j$.

Let $Y_1, \ldots, Y_k$ be all the irreducible components of A with $\dim Y_j \leq n-1$, and let $B_j := Y_j - (A_1 \cup \cdots \cup A_r)$. Put

$$B := \operatorname{Sing}(A) \cup \partial A \cup B_1 \cup \cdots \cup B_k.$$

Then, we can regard B to be a semialgebraic subvariety of A with the reduced structure. Note that $\dim B < \dim A$. Thus we put $\Delta^i(A) := \Delta^i(B)$ for $i \neq n$.

We denote $\Delta(A) := \Delta^0(A) \cup \Delta^1(A) \cup \cdots \cup \Delta^n(A)$, and is called a *critical decomposition* of A. Each element $D \in \Delta(A)$ is called a *critical set* of A. Note that D is a non-singular semialgebraic variety with $\partial D = \emptyset$.

Example 2.4. Consider the case $A = \mathbb{P}^2_+$. This is homeomorphic to a triangle. Let $P_x := (1{:}\,0{:}\,0)$, $P_y := (0{:}\,1{:}\,0)$, and $P_z := (0{:}\,0{:}\,1)$. For two points P, $Q \in \mathbb{P}^2_+$, we denote the open line segment connecting P and Q as (PQ). Then, the critical decomposition of $\mathbb{P}^2_+$ is $\Delta^0(\mathbb{P}^2_+) = \{P_x,\, P_y,\, P_z\}$, $\Delta^1(\mathbb{P}^2_+) = \{(P_xP_y),\, (P_yP_z),\, (P_zP_x)\}$, $\Delta^2(\mathbb{P}^2_+) = \{\operatorname{Int}(\mathbb{P}^2_+)\}$.

On the other hand, if $A = \mathbb{P}^n_{\mathbb{R}}$, then $\Delta^n(\mathbb{P}^n_{\mathbb{R}}) = \{\mathbb{P}^n_{\mathbb{R}}\}$, and $\Delta^r(\mathbb{P}^n_{\mathbb{R}}) = \emptyset$ for $r \neq n$.

Definition 2.5. (1) Let X be a subset of $\mathbb{R}^n$ or $\mathbb{P}^n_{\mathbb{R}}$. $e \in X$ is said to be *extremal* in X, if $a > 0$, $b > 0$ and x, $y \in X$ satisfy $e = ax + by$ then $x = y = e$. Let $\mathcal{P}$ be a closed convex cone which contain no lines. $0 \neq f \in \mathcal{P}$ is called *extremal* in $\mathcal{P}$, if g, $h \in X$ satisfy $f = g + h$ then g and h are multiples of f. For both cases $Y = X$ and $Y = \mathcal{P}$, we denote that

$$\mathcal{E}(Y) := \{ y \in Y \mid y \text{ is extremal in } Y \}.$$

(2) For a semialgebraic variety A and $a \in A - \operatorname{Bs}\mathcal{H}$ and a signed linear system $\mathcal{H}$ on A, we put

$$\mathcal{H}_a := \{ f \in \mathcal{H} \mid f(a) = 0 \}, \quad \mathcal{P}_a := \mathcal{P} \cap \mathcal{H}_a = \mathcal{P}(A, \mathcal{H}_a).$$

$\mathcal{P}_a$ is called the *local cone* of $\mathcal{P}$ at a.

Even if $a \in \operatorname{Bs}\mathcal{H}$, we can define $\mathcal{P}_a$ as Definition 2.6 of [4]. But we don't use it in this article.

Definition 2.6. (See Definition 1.15 and 1.17 of [3]) (1) Let $\mathbb{P} = \mathbb{P}^N_{\mathbb{R}}$ and $\mathbb{P}^\vee$ be the set of all the hyperplanes in $\mathbb{P}$. Assume that $D \subset \mathbb{P}$ is a non-singular semialgebraic variety with $\partial D = \emptyset$ (i.e. $\Delta(D) = \{D\}$). For $x \in D$, let $T_{D,x} := T_{\operatorname{Zar}(D),x} \subset \mathbb{P}$ be the tangent space of $\operatorname{Zar}(D)$ at x. Then,

$$D^\vee := \{ H \in \mathbb{P}^\vee \mid H \supset T_{D,x} \text{ for a certain } x \in D \}$$

is called the *dual variety* of D. Since D is irreducible and non-singular, $D^\vee$ is irreducible. Thus $D^\vee$ is a semialgebraic variety.

(2) Under the same notation with Definition 2.1, let $\pi : (\mathcal{H} - \{0\}) \to \mathbb{P}(\mathcal{H})$ be the natural surjection. For $D \in \Delta(X)$, we denote

$$\mathcal{F}(D) := \operatorname{Cls}_{\mathcal{H}}(\pi^{-1}(D^\vee) \cap \partial\mathcal{P}).$$

If $\dim \mathcal{F}(D) = \dim(\partial\mathcal{P})$, then $\mathcal{F}(D)$ is called a *face component* of $\mathcal{P}$ or of $\partial\mathcal{P}$, and an irreducible defining equation of the Zariski closure $\operatorname{Zar}(\mathcal{F}(D))$ is called a *discriminant* of $\mathcal{P}$, and denoted by disc_D or $\operatorname{disc}(D)$.

Especially, if $D \in \Delta^{\dim X}(X)$ and $\mathcal{F}(D)$ is a face component, then $\mathcal{F}(D)$ is called a *main component* of $\mathcal{P}$, and $\operatorname{disc}(D)$ is called a *main discriminant* of $\mathcal{P}$.

For example, if $X \cong \mathbb{P}^n_{\mathbb{R}} = A$, then $\mathcal{P}$ has unique discriminant which is a main discriminant.

In the case $D \in \Delta^0(X)$, $\operatorname{disc}(D)$ is linear. That is, if $\Phi_{\mathcal{H}}$ is defined by basis $\{s_0, \ldots, s_N\}$ of $\mathcal{H}$, and if we represent $f \in \mathcal{H}$ as $f = p_0s_0 + \cdots + p_Ns_N$, and $D = (b_0{:}\cdots{:}\,b_N) \in \mathbb{P}(\mathcal{H}^\vee)$, then $\operatorname{disc}(D) = b_0p_0 + \cdots + b_Np_N$.

Theorem 2.7.(Theorem 1.18 of [3]) *We use the same notation as Definition 2.1 and the above.*

(1) Let

$$\mathcal{D} := \{D \in \Delta(X) \mid \mathcal{F}(D) \text{ is a face component of } \mathcal{P}\}.$$

Then $\partial\mathcal{P} = \bigcup_{D\in\mathcal{D}} \mathcal{F}(D)$.

(2) *For $D \in \Delta(X)$, take a subset $B \subset A$ such that $\Phi_{\mathcal{H}}(B) \subset D$ and $\mathrm{Cls}_D\left(\Phi_{\mathcal{H}}(B)\right) = D$. Put $B_0 := B - \mathrm{Bs}\,\Phi_{\mathcal{H}}$. Then,*

$$\mathcal{F}(D) = \mathrm{Cls}_{\mathcal{H}}\left(\bigcup_{a\in B_0} \mathcal{P}_a\right).$$

(3) *Assume that $\mathcal{P} := \mathcal{P}(X, \mathcal{H}_{N+1,1})$ is non-degenerate in $\mathcal{H}_{N+1,1}$. Take $x \in D \in \Delta^r(X)$. Then $\dim\mathcal{P}_x \leq N - r$.*

The author should apologize for that Proposition 1.27 of [3] is not correct. It should be corrected as (3) of the above theorem. We present a corrected proof of (3).

Proof. (3) For $f \in \mathcal{H}$, let H_f be the hyperplane in $\mathbb{P}(\mathcal{H}^\vee)$ defined by $f = 0$. Since $\mathcal{P}$ is non-degenerate, $\dim(U \cap \mathcal{P}) = N + 1$ for any Euclidean open neighborhood U of x. Let $\mathcal{L} := \{f \in \mathcal{H} \mid T_{D,x} \subset H_f\}$. Note that $\dim T_{D,x} = \dim D = r \leq N + 1$, since D is non-singular. The condition $T_{D,x} \subset H_f$ means that f passes through independent $r + 1$ points. Thus, $\dim\mathcal{L} = \dim\mathcal{H} - (r + 1) = N - r$. Since $\mathcal{P}_x = \mathcal{P} \cap \mathcal{L}$, we have $\dim\mathcal{P}_x \leq N - r$. □

Even if we determine all the discriminants of $\mathcal{P}$, the signature of $\mathrm{disc}(D)$ may not be constant in $\mathrm{Int}(\mathcal{P})$. To describe $\mathcal{P}$ as a union of basic semialgebraic sets of $\mathcal{H}$ using some inequalities, we need some more inequalities to cut off extra parts or to avoid the interior zero locus $\mathrm{Int}(\mathcal{P}) \cap V_{\mathcal{H}}(\mathrm{disc}(D))$. Such inequalities are called *separators*. Note that discriminants are unique up to multiplication by non-zero constant, but there may be many possibilities of the choice of separators.

About extremality of $f \in \mathcal{P}$, the following theorem is useful. About the definition of infinitesimal local cone, please see Definition 2.9 and 2.12 of [4].

Theorem 2.8. (Theorem 2.11, Proposition 2.13 of [4]) *Let $\mathcal{P} = \mathcal{P}(A, \mathcal{H})$. Assume that $\dim\mathcal{P} \geq 2$.*

(1) *If $f \in \mathcal{E}(\mathcal{P})$, then there exists local cones or infinitesimal local cones $\mathcal{P}_1, \ldots, \mathcal{P}_r \subset \mathcal{P}$ with respect to f which satisfy $\mathcal{P}_1 \cap \cdots \cap \mathcal{P}_r = \mathbb{R}_+ \cdot f$.*

(2) *Let $f \in \mathcal{P}$. If there exists local cones or infinitesimal local cones $\mathcal{P}_1, \ldots, \mathcal{P}_r \subset \mathcal{P}$ such that $\mathcal{P}_1 \cap \cdots \cap \mathcal{P}_r = \mathbb{R}_+ \cdot f$. Then, $f \in \mathcal{E}(\mathcal{P})$.*

In the above theorem, infinitesimal local cones appear for special $f \in \mathcal{E}(\mathcal{P})$. In ordinary case, there exists points $\mathbf{a}_1, \ldots, \mathbf{a}_r \in A$ such that

$$\mathbb{R}_+ \cdot f = \{g \in \mathcal{P} \mid g(\mathbf{a}_1) = \cdots = g(\mathbf{a}_r) = 0\}.$$

We can choose each $\mathbf{a}_i$ so that $\Phi_{\mathcal{H}}(\mathbf{a}_i) \in \mathcal{E}(X)$. Infinitesimal local cones appears when not less than two zero points of f become infinitely near points.

Definition 2.9. Let $\mathcal{H}$ be a signed linear system on a semialgebraic variety A. A subset $\Omega \subset A$ is called a *test set* for $(A, \mathcal{H})$, if $f(\mathbf{a}) \geq 0$ for all $\mathbf{a} \in \Omega$, then $f(\mathbf{a}) \geq 0$ for all $\mathbf{a} \in A$.

The following theorem will be trivial.

Theorem 2.10. *Let $\mathcal{H}$ be a signed linear system on a compact semialgebraic variety A with $\dim \mathcal{H} \geq 3$, and let $X := \mathrm{Cls}(\Phi_{\mathcal{H}}(A))$ be the characteristic variety. Take a subset $\Omega \subset A$. If $\mathcal{E}(X) \subset \mathrm{Cls}(\Phi_{\mathcal{H}}(\Omega))$, then Ω is a test set for $\mathcal{H}$.*

Some artices add the following condition for a test set:
(Additional condition) For any $\mathbf{a} \in \Omega$, there exixts $f \in \mathcal{H}$ such that $f(\mathbf{a}) = 0$.
Under this definition, $\mathcal{E}(X) \subset \mathrm{Cls}(\Phi_{\mathcal{H}}(\Omega))$ must be replaced by $\mathcal{E}(X) = \Phi_{\mathcal{H}}(\Omega)$.

Example 2.11. Consider the case $A = \mathbb{P}^2_+$, $\mathcal{H} = \mathcal{H}^c_{3,3}$. Then

$$\Omega := \{(1\colon 1\colon 1)\} \cup \bigl\{(0\colon t\colon 1) \in \mathbb{P}^2_+ \bigm| t \geq 0\bigr\}$$

is a test set for $\mathcal{H}^c_{3,3}$ (see Theorem 3.1 of [3]). Thus, if $f \in \mathcal{H}^c_{3,3}$ satisfies $f(1, 1, 1) \geq 0$ and $f(0, t, 1) \geq 0$ for all $t \geq 0$, then $f(a, b, c) \geq 0$ for all $a, b, c \in \mathbb{R}_+$.

2.2. Some more general theorems.

Let V and W be non-singular semialgebraic varieties with $\dim V = n$, $\dim W = m$, and $\varphi\colon V \to W$ be a regular map. Take a point $a \in V$ and put $b := \varphi(a)$. We can take open neighborhoods $a \in U_V \subset V$ and $b \in U_W \subset W$ such that $\varphi(U_V) \subset U_W$ and that U_V, U_W have local coordinate systems $(x_1,\ldots, x_n)$ and $(y_1,\ldots, y_m)$ whose origins are a, b. φ can be represented by functions $y_j = \varphi_j(x_1,\ldots, x_n)$ $(j = 1,\ldots, m)$. Let $J_a := \left(\dfrac{\partial y_j}{\partial x_i}\right)\Bigg|_{(x_1,\ldots,x_n)=a}$ be the Jacobian matrix of φ at a. Note that $\operatorname{rank} J_a$ does not depend on the choice of $(x_1,\ldots, x_n)$ and $(y_1,\ldots, y_m)$. We denote

$$\mathrm{Sing}(\varphi) := \bigl\{a \in V \bigm| \operatorname{rank} J_a < \dim \varphi(V)\bigr\}.$$

Proposition 2.12. *If V is a non-singular complete real algebraic variety, then $\partial\bigl(\varphi(V)\bigr) \subset \varphi\bigl(\mathrm{Sing}(\varphi)\bigr)$.*

Proof. Put $r := \dim \varphi(V)$, and assume that $\operatorname{rank} J_a = r$. We may assume that

$$\det \left(\frac{\partial y_j}{\partial x_i}\right)_{1 \leq i \leq r,\, 1 \leq j \leq r} \neq 0$$

at a. Let $U' := \bigl\{(x_1,\ldots, x_n) \in U_V \bigm| x_{r+1} = \cdots = x_n = 0\bigr\}$. If U_V is sufficiently small Euclidean open set, $\varphi\big|_{U'}\colon U' \longrightarrow \varphi(U')$ is an isomorphism. Thus $b \notin \partial\bigl(\varphi(V)\bigr)$. □

When V and W are open subsets of $\mathbb{P}^r_{\mathbb{R}}$, and φ is given by $y_j = \varphi_j(x_0\colon \cdots \colon x_r)$ $(j = 0,\ldots, r)$ using homogeneous coordinate system, the condition $\operatorname{rank} J_a = r$ can be replaced by

$$\det \left(\frac{\partial y_j}{\partial x_i}\right)_{0 \leq i \leq r,\, 0 \leq j \leq r} \neq 0.$$

When V has singularities, we put $\mathrm{Sing}(\varphi) := \mathrm{Sing}\left(\varphi|_{\mathrm{Reg}(V)}\right)$.

Corollary 2.13. *Assume that A is a compact semialgebraic variety, then,*

$$\partial\big(\varphi(A)\big) \subset \varphi\big(\,\mathrm{Sing}(\varphi) \cup \mathrm{Sing}(A) \cup \partial A\big).$$

Proposition 2.14. *Let $X^{s+}_{3,d} := X(\mathbb{P}^2_+,\, \mathcal{H}^s_{3,d})$. If $d \geq 4$, then $X^{s+}_{3,d} \cong \mathbb{P}^2_+/\mathfrak{S}_3$.*

Proof. We denote the coordinate system of $\mathbb{P}^2_+$ by $(a\colon b\colon c)$, and put $S_1 := a + b + c$. $\Phi_{3,d} := \Phi_{\mathcal{H}^s_{3,d}} : \mathbb{P}^2_+ \to X^{s+}_{3,d}$ is decomposed as $\Phi_{3,d}\colon \mathbb{P}^2_+ \xrightarrow{\sigma} \mathbb{P}^2_+/\mathfrak{S}_3 \xrightarrow{\Psi_{3,d}} X^{s+}_{3,d}$. By Proposition 2.13, 2.14 and §4.5 in [3], $\Psi_{3,4}\colon \mathbb{P}^2_+/\mathfrak{S}_3 \longrightarrow X^{s+}_{3,4}$ is an isomorphism. Since $\mathrm{Bs}\, S_1 \cap \mathbb{P}^2_+ = \emptyset$, the multiplication map $\times S_1\colon \mathcal{H}^s_{s,d} \longrightarrow \mathcal{H}^s_{s,d+1}$ induces an isomorphism $X^{s+}_{3,d+1} \to X^{s+}_{3,d}$. □

In the cyclic case $X^{c+}_{n,d} := X(\mathbb{P}^{n-1}_+,\, \mathcal{H}^c_{n,d})$, we know that $X^{c+}_{n,d} \cong \mathbb{P}^{n-1}_+/\mathfrak{C}_n$ if $d \geq n$, here $\mathfrak{C}_n = \mathbb{Z}/n\mathbb{Z}$ (see Proposition 1.36 in [3]). When $n = 3$, $\Delta^1(X^{c+}_{3,d})$ has a unique element $C^{c+}_{3,d} := \left\{\Phi^c_{3,d}(0\colon s\colon 1) \;\middle|\; s > 0\right\}$. We call $\mathrm{disc}(C^{c+}_{3,d})$ to be the *edge discriminant* of $\mathcal{P}^{c+}_{3,d}$ (see Definition 2.7 in [3]). The following Theorem is a replacement of Proposition 2.10, Theorem 5.9 and Theorem 6.8 in [3].

We denote the discriminant of $c_n x^n + c_{n-1}x^{n-1} + \cdots + c_1 x + c_0$ by

$$\mathrm{Disc}_n(c_n, c_{n-1}, \ldots, c_1, c_0).$$

Theorem 2.15. *Let's denote the coordinate system of $\mathbb{P}^2_+$ by $(a\colon b\colon c)$, and put $S_{m,n} = S_{m,n}(a,b,c) := a^m b^n + b^m c^n + c^m a^n$, $S_n = S_n(a,b,c) := a^n + b^n + c^n$, and $U = U(a,b,c) := abc$. Take the basis of $\mathcal{H}^c_{3,d}$ so that $s_0 = S_d$, $s_1 = S_{d-1,1}$, $s_2 = S_{d-2,2}, \ldots$, $s_{d-1} = S_{1,d-1}, \ldots$. Here, if $i \geq d$, then s_i is a multiple of abc. We represent $f \in \mathcal{H}^c_{3,d}$ as $f = \sum p_i s_i$. Then, the edge discriminant of $\mathcal{P}^{c+}_{3,d}$ agrees with $\mathrm{Disc}_d(p_0, p_1, \ldots, p_{d-1}, p_0)$.*

Proof. Let $\mathcal{L}^{c+}_{0,t}$ be the local cone of $\mathcal{P}^{c+}_{3,s}$ at $(0\colon t\colon 1) \in \mathbb{P}^2_+$. Take $f \in \mathcal{L}^{c+}_{0,t} \subset \mathcal{F}(C^+_{n,d})$ ($p_0 > 0$ and $t > 0$). Then $f(0,t,1) = 0$. Since $f(0,x,1) \geq 0$ for all $x > 0$, the equation $f(0,x,1) = 0$ has a multiple root at $x = t$. Thus, the discriminant of f is equal to 0. Since $S_{i,d-1}(0,x,1) = x^i$ $(1 \leq i \leq d-1)$, $S_d(0,x,1) = x^d + 1$ and $U(0,x,1) = 0$, we have $f(0,x,1) = p_0 x^d + p_1 x^{d-1} + \cdots + p_{d-1}x + p_0$.

Since Disc_d and $\mathrm{disc}(C^{c+}_{3,d})$ are irreducible, we have the conclusion. □

Theorem 2.16. *Consider the cases $A = \mathbb{P}^{n-1}_{\mathbb{R}}$ or $\mathbb{P}^{n-1}_+$, and $\mathcal{H} = \mathcal{H}^s_{n,d}$ or $\mathcal{H}^{s0}_{n,d}$. Let $\mathcal{P} := \mathcal{P}(A,\, \mathcal{H})$, $X := X(A,\, \mathcal{H}_{n,d})$, and $\Phi := \Phi_{\mathcal{H}}\colon A \cdots \to X$. Let $\sigma\colon \mathbb{P}^{n-1}_{\mathbb{R}} \longrightarrow \mathbb{P}^{n-1}_{\mathbb{R}}/\mathfrak{S}_n \subset \mathbb{P}_{\mathbb{R}}(1,2,\ldots,n)$ be the natural surjection, and $\Psi\colon \mathbb{P}^{n-1}_{\mathbb{R}}/\mathfrak{S}_n \cdots \to X$ be the rational map such that $\Psi \circ \pi = \Psi$. Assume that Ψ is a birational map. Let $D \in \Delta^r(X)$ with $r \geq \max\{2, \lfloor d/2 \rfloor\}$. Then $\mathcal{F}(D)$ is not a face component of $\mathcal{P}$.*

Proof. Let $r_0 := \max\{2,\, \lfloor d/2 \rfloor\}$, and take $D \in \Delta^r(X)$ with $r_0 \leq r \leq n-1$. Assume that $\mathcal{F}(D)$ is a face component of $\mathcal{P}$. Then $\dim \mathcal{F}(D) = n-1$.

(1) Consider the case $A = \mathbb{P}^{n-1}_{\mathbb{R}}$.

Let $\Omega := \left\{(x_1\colon \cdots\colon x_n) \in \mathbb{P}^{n-1}_{\mathbb{R}} \;\middle|\; \#\{x_1, \ldots, x_n\} \leq r_0\right\}$. Here $\#\{x_1, \ldots, x_n\} \leq r_0$ means that at most r_0 members of $x_1, \ldots, x_n$ are distinct. Ω is a test set by [22].

Ω is included in a union of some (r_0-1)-dimensional linear subspace of $\mathbb{P}^{n-1}_{\mathbb{R}}$. Take general $f \in \mathcal{F}(D)$. There exists a semialgebraic subset $E \subset A$ such that $\Phi(E) = D$, and $\mathbf{a} \in E$ such that $f(\mathbf{a}) = 0$. Since $\mathcal{F}(D)$ is a face component, we may assume that the hyperplane $H_f \subset \mathbb{P}(\mathcal{H}^\vee)$ corresponding to f, tangents to X only at the unique point $\Phi(\mathbf{a})$, if $\mathbf{a} \in E$ is a general point. This means that if $\mathbf{b} \in A - \mathrm{Bs}\,\mathcal{H}$ satisfies $f(\mathbf{b}) = 0$, then $\Phi(\mathbf{b}) = \Phi(\mathbf{a})$. We can choose such f and $\mathbf{a}$.

By Corollary 1.3 of [22] or Corollary 2.1 of [23], there exists $\mathbf{b} \in \Omega$ such that $f(\mathbf{b}) = 0$. We denote this $\mathbf{b}$ by $\mathbf{b}(\mathbf{a})$. $\mathbf{a}$ can move a certain r-dimensional subset of E. But $\dim \Omega = r_0 - 1 < r$. Thus, there exists $\mathbf{a} \in E$ such that $\Phi(\mathbf{b}(\mathbf{a})) \neq \Phi(\mathbf{a})$. A contradiction. Thus $\mathcal{F}(D)$ is not a face component of $\mathcal{P}$.

(2) Consider the case $A = \mathbb{P}^{n-1}_{+}$.

Let $\Omega' := \{(x_1 \colon \cdots \colon x_n) \in \mathbb{P}^{n-1}_{+} \mid \#(\{x_1, \ldots, x_n\} - \{0\}) \leq r_0\}$. If $f \in \mathcal{H}^s_{n,d}$ satisfies $f(\mathbf{a}) \geq 0$ for all $\mathbf{a} \in \Omega'$, then $f \in \mathcal{P}^{s+}_{n,d}$ by [22]. Ω' is also included in a union of some (r_0-1)-dimensional linear subspace of $\mathbb{P}^{n-1}_{\mathbb{R}}$.

The left part is same as (1). □

If $\mathcal{F}(D)$ is not a face component, then, for each $f \in \mathcal{F}(D)$, there exist $D_1, \ldots, D_r \in \Delta(X) - \{D\}$ such that $f \in \mathcal{F}(D_1) \cap \cdots \cap \mathcal{F}(D_r)$, and that all $\mathcal{F}(D_i)$ are face components.

Section 3. Quartic Inequalities of Four Variables

In this section, we shall study $\mathcal{P}^{s0}_{4,4}$ and $\mathcal{P}^{s0+}_{4,4}$. We write the homogeneous coordinate system of $A = \mathbb{P}^3_{\mathbb{R}}$ or $A = \mathbb{P}^3_{+}$ by $(a \colon b \colon c \colon d)$ or $(a_0 \colon a_1 \colon a_2 \colon a_3)$. We regard $a_{4n+i} = a_i$ for $n \in \mathbb{Z}$. We denote

$$S_d := \sum_{i=0}^{3} a_i^d, \quad T_{p,q} := \sum_{i=0}^{3} a_i^p (a_{i+1}^q + a_{i+2}^q + a_{i+3}^q), \quad S_{p,p} := \sum_{0 \leq i < j \leq 3} a_i^p a_j^p,$$

$$T_{p,q,q} := \sum_{i=0}^{3} a_i^p (a_{i+1}^q a_{i+2}^q + a_{i+1}^q a_{i+3}^q + a_{i+2}^q a_{i+3}^q), \quad U := a_0 a_1 a_2 a_3.$$

A polynomial $f \in \mathcal{H}^s_{n,d}$ or $\mathcal{H}^c_{n,d}$ is called *monic*, if the coefficient of $S_d = a_0^d + \cdots + a_{n-1}^d$ is equal to 1. For a subset $V \subset \mathcal{H}^c_{n,d}$, we denote

$$\check{V} := \{ f \in V \mid f \text{ is monic} \}.$$

We denote as $\mathbb{P}^n_{\mathbb{R}} : (a_0 \colon \cdots \colon a_n)$ when we treat $\mathbb{P}^n_{\mathbb{R}}$ with a homogeneous coordinate system $(a_0 \colon \cdots \colon a_n)$. Similarly we denote as $\mathbb{R}^n : (x_1, \ldots, x_n)$ when we study $\mathbb{R}^n$ with a coordinate system $(x_1, \ldots, x_n)$.

3.1. Structure of $\mathbb{P}^3_{\mathbb{R}}/\mathfrak{S}_4$

Let $(a_0 \colon \cdots \colon a_n)$ be the homogeneous coordinate system of $\mathbb{P}^n_{\mathbb{R}}$, and $\sigma_k = \sigma_k(a_0, \ldots, a_n)$ be the k-th symmetric function of $a_0, \ldots, a_n$ $(0 \leq k \leq n+1)$. The sequence of functions $(\sigma_1, \ldots, \sigma_{n+1})$ defines the regular map $\sigma \colon \mathbb{P}^n_{\mathbb{P}} \longrightarrow \mathbb{P}_{\mathbb{R}}(1, 2, \ldots, n+1)$, where $\mathbb{P}_{\mathbb{R}}(1, 2, \ldots, n+1)$ is the real weighted projective space which is defined as the real part of the complex weighted projective space $\mathbb{P}_{\mathbb{C}}(1, 2, \ldots, n+1)$. The image $\sigma(\mathbb{P}^n_{\mathbb{R}})$ is isomorphic to $\mathbb{P}^n_{\mathbb{R}}/\mathfrak{S}_{n+1}$ as semialgebraic varieties. Note that $\mathbb{P}^n_{\mathbb{C}}/\mathfrak{S}_{n+1} \cong \mathbb{P}_{\mathbb{C}}(1, 2, \ldots, n+1)$, but $\mathbb{P}^n_{\mathbb{R}}/\mathfrak{S}_{n+1} \not\cong \mathbb{P}_{\mathbb{R}}(1, 2, \ldots, n+1)$. In general, for two points $P, Q \in \mathbb{P}^n_{\mathbb{R}}$, (PQ) represents an open line segment, $[PQ] := (PQ) \cup \{P, Q\}$ represents a closed line segment, and PQ represents a line.

Definition 3.1. Assume that a finite group G acts on a semialgebraic variety A. Let $\sigma: A \to A/G$ be the natural surjection. A closed semialgebraic subset $A_0 \subset A$ is called a fundamental domain of A/G, if $\sigma(A_0) = A/G$ and $\sigma : \mathrm{Int}(A_0) \longrightarrow \sigma(\mathrm{Int}(A_0)) \subset A/G$ is an isomorphism.

Example 3.2. (1) Let $A = \mathbb{P}^n_{\mathbb{R}}$ and $G = \mathbb{Z}/(n+1)\mathbb{Z}$. Then $(\mathbb{P}^n_{\mathbb{R}})^G = \{\mathbf{1}\}$, and $\mathrm{Sing}(\mathbb{P}^n_{\mathbb{R}}/G) = \sigma\big((\mathbb{P}^n_{\mathbb{R}})^G\big) = \{\sigma(\mathbf{1})\}$, here $\mathbf{1} = (1{:}\,1{:}\cdots{:}\,1) \in A$. The following A_c is a fundamental domain.

$$A_c := \left\{ (a_0{:}\cdots{:}\,a_{n-1}{:}\,1) \in \mathbb{P}^n_{\mathbb{R}} \;\middle|\; \begin{array}{l} a_0 + a_1 + \cdots + a_{n-1} + 1 \geq 0, \\ a_0 \leq 1,\, a_1 \leq 1, \ldots, a_{n-1} \leq 1 \end{array} \right\}.$$

(2) Let $A = \mathbb{P}^n_{+}$ and $G = \mathbb{Z}/(n+1)\mathbb{Z}$. Then $(\mathbb{P}^n_{\mathbb{R}})^G = \{\mathbf{1}\}$, and

$$A_c^+ := \big\{ (a_0{:}\cdots{:}\,a_{n-1}{:}\,1) \in \mathbb{P}^n_{\mathbb{R}} \;\big|\; 0 \leq a_0 \leq 1, \ldots, 0 \leq a_{n-1} \leq 1 \big\}$$

is a fundamental domain.

(3) Let $A = \mathbb{P}^n_{\mathbb{R}}$ and $G = \mathfrak{S}_{n+1}$. Then

$$A_s := \big\{ (-1 \leq a_0{:}\cdots{:}\,a_{n-1}{:}\,1) \in A_c \;\big|\; a_0 \leq a_1 \leq \cdots \leq a_{n-1} \big\}$$

is a fundamental domain.

(4) Let $A = \mathbb{P}^n_{+}$ and $G = \mathfrak{S}_{n+1}$. Then

$$A_s^+ := \big\{ (a_0{:}\cdots{:}\,a_{n-1}{:}\,1) \in \mathbb{P}^n_{\mathbb{R}} \;\big|\; 0 \leq a_0 \leq a_1 \leq \cdots \leq a_{n-1} \leq 1 \big\}$$

is a fundamental domain.

Note that $\mathbb{P}^3_{\mathbb{C}}/\mathfrak{S}_4 \cong \mathbb{P}_{\mathbb{C}}(1,2,3,4)$ has cyclic quotients singularities at $\tilde{P}_0 := (0{:}\,1{:}\,0{:}\,0)$, $\tilde{P}'_0 := (0{:}\,0{:}\,1{:}\,0)$ and $\tilde{P}''_0 := (0{:}\,0{:}\,0{:}\,1)$.

Proposition 3.3. *About the structures of $\mathbb{P}^3_{\mathbb{R}}/\mathfrak{S}_4$ and $\mathbb{P}^3_{+}/\mathfrak{S}_4$, we have the following:*

(1) *Let $\sigma: \mathbb{P}^3_{\mathbb{R}} \longrightarrow \mathbb{P}^3_{\mathbb{R}}/\mathfrak{S}_4 \overset{\subset}{\longrightarrow} \mathbb{P}_{\mathbb{R}}(1,2,3,4)$ be the natural map. Then $\sigma^{-1}(\tilde{P}'_0) = \emptyset$, $\sigma^{-1}(\tilde{P}''_0) = \emptyset$, and $\sigma(-1,\,0,\,0,\,1) = \tilde{P}_0$.*

(2) *$\Delta^2(\mathbb{P}^3_{\mathbb{R}}/\mathfrak{S}_4) = \{\tilde{D}_1\}$, $\Delta^1(\mathbb{P}^3_{\mathbb{R}}/\mathfrak{S}_4) = \{\tilde{C}_1,\, \tilde{C}_2\}$, and $\Delta^0(\mathbb{P}^3_{\mathbb{R}}/\mathfrak{S}_4) = \{\tilde{P}_0,\, \tilde{P}_1,\, \tilde{P}_2\}$, where $\tilde{D}_1$, $\tilde{C}_i$ and $\tilde{P}_i$ are as follows:*

$$\begin{aligned}
\tilde{D}_1 &:= \big\{ \sigma(s{:}\,t{:}\,u{:}\,u) \in \mathbb{P}_{\mathbb{R}}(1,2,3,4) \;\big|\; s < t,\ s \neq u,\ t \neq u \big\}, \\
\tilde{C}_1 &:= \big\{ \sigma(s{:}\,1{:}\,1{:}\,1) \in \mathbb{P}_{\mathbb{R}}(1,2,3,4) \;\big|\; s \in \mathbb{P}^1_{\mathbb{R}},\ s \neq -3,\ 1 \big\}, \\
\tilde{C}_2 &:= \big\{ \sigma(s{:}\,s{:}\,1{:}\,1) \in \mathbb{P}_{\mathbb{R}}(1,2,3,4) \;\big|\; -1 < s < 1 \big\}, \\
\tilde{P}_1 &:= \sigma(1{:}\,1{:}\,1{:}\,1) = (4{:}\,6{:}\,4{:}\,1) \in \mathbb{P}_{\mathbb{R}}(1,2,3,4), \\
\tilde{P}_2 &:= \sigma(-1{:}\,-1{:}\,1{:}\,1) = (0{:}\,-2{:}\,0{:}\,1) \in \mathbb{P}_{\mathbb{R}}(1,2,3,4).
\end{aligned}$$

(3) *$\Delta^2(\mathbb{P}^3_{+}/\mathfrak{S}_4) = \{\tilde{D}_1^+,\, \tilde{D}_0\}$, $\Delta^1(\mathbb{P}^3_{+}/\mathfrak{S}_4) = \{\tilde{C}_1^+,\, \tilde{C}_2^+,\, \tilde{C}_3,\, \tilde{C}_4\}$, and $\Delta^0(\mathbb{P}^3_{+}/\mathfrak{S}_4) = \{\tilde{P}_1,$ $\tilde{P}_3,\, \tilde{P}_4,\, \tilde{P}_5\}$, where $\tilde{D}_1^+$, $\tilde{D}_0$, $\tilde{C}'_1$, $\tilde{C}_i$ and $\tilde{P}_i$ are as follows:*

$$\begin{aligned}
\tilde{D}_1^+ &:= \big\{ \sigma(s{:}\,t{:}\,1{:}\,1) \in \mathbb{P}_{\mathbb{R}}(1,2,3,4) \;\big|\; 0 < s < t,\ s \neq 1,\ t \neq 1 \big\}, \\
\tilde{D}_0 &:= \big\{ \sigma(0{:}\,s{:}\,t{:}\,1) \in \mathbb{P}_{\mathbb{R}}(1,2,3,4) \;\big|\; 0 < s < t < 1 \big\}, \\
\tilde{C}_1^+ &:= \big\{ \sigma(s{:}\,1{:}\,1{:}\,1) \in \mathbb{P}_{\mathbb{R}}(1,2,3,4) \;\big|\; 0 < s < 1 \text{ or } s > 1 \big\}, \\
\tilde{C}_2^+ &:= \big\{ \sigma(s{:}\,s{:}\,1{:}\,1) \in \mathbb{P}_{\mathbb{R}}(1,2,3,4) \;\big|\; 0 < s < 1 \big\}, \\
\tilde{C}_3 &:= \big\{ \sigma(0{:}\,s{:}\,1{:}\,1) \in \mathbb{P}_{\mathbb{R}}(1,2,3,4) \;\big|\; 0 < s < 1 \text{ or } 1 < s \big\},
\end{aligned}$$

$$\tilde{C}_4 := \{\sigma(0\colon 0\colon s\colon 1) \in \mathbb{P}_{\mathbb{R}}(1,2,3,4) \mid 0 < s < 1\},$$
$$\tilde{P}_3 := \sigma(0\colon 1\colon 1\colon 1) = (3\colon 3\colon 1\colon 0) \in \mathbb{P}_{\mathbb{R}}(1,2,3,4),$$
$$\tilde{P}_4 := \sigma(0\colon 0\colon 1\colon 1) = (2\colon 1\colon 0\colon 0) \in \mathbb{P}_{\mathbb{R}}(1,2,3,4),$$
$$\tilde{P}_5 := \sigma(0\colon 0\colon 0\colon 1) = (1\colon 0\colon 0\colon 0) \in \mathbb{P}_{\mathbb{R}}(1,2,3,4).$$

(4) $\mathrm{disc}(\tilde{D}_1) = \mathrm{Disc}_4$, *and* $\tilde{C}_1 \cup \tilde{C}_2 \subset \mathrm{Sing}(V(\mathrm{Disc}_4))$, *here* $V(f)$ *is the zero locus of* f *in* $\mathbb{P}_{\mathbb{R}}(1,2,3,4)$.

(5) $\mathrm{Cls}\,\tilde{C}_1$ *is isomorphic to a cubic curve on* $\mathbb{P}^2_{\mathbb{R}}$ *with a cusp at* $\tilde{P}_1$.

(6) $\tilde{C}_2 = (\tilde{P}_1\tilde{P}_2)$ *is isomorphic to an open line segment with ends* $\tilde{P}_1$ *and* $\tilde{P}_2$.

(7) $\mathbb{P}^3_{\mathbb{R}}/\mathfrak{S}_4$ *is the semialgebraic subset of* $\mathbb{P}_{\mathbb{R}}(1,2,3,4)$ *defined by* $\mathrm{Disc}_4(1, \sigma_1, \sigma_2, \sigma_3, \sigma_4) \geq 0$, $8\sigma_2 \leq 3\sigma_1^2$, *and* $64\sigma_4 - 16\sigma_2^2 + 16\sigma_1^2\sigma_2 - 16\sigma_1\sigma_3 - 3\sigma_1^4 \leq 0$. *Here,* σ_i *is the elementary symmetric polynomials of* a_0, a_1, a_2, a_3 *of degree* i.

Proof. (1) is clear.

(2) and (3) follows from the critical decompositions of fundamental domains A_s and A_s^+ in the above example.

(4) This follows from conditions that a quartic equation has a double root, a triple root or two double roots.

(5) Eliminate t from $x = \sigma_2(t,1,1,1)/\sigma_1(t,1,1,1)^2$, $y = \sigma_3(t,1,1,1)/\sigma_1(t,1,1,1)^3$, $z = \sigma_4(t,1,1,1)/\sigma_1(t,1,1,1)^4$, then we obtain $32(x-3/8)^3 + 27(x-3/8)^2 - 108(x-3/8)(y-1/16) + 108(y-1/16)^2 = 0$ and $x^2 = 3y - 12z$. This cuve is isomorphic to a cubic curve on $\mathbb{P}^2_{\mathbb{R}}$, and have a cusp at $(x,\, y,\, z) = (3/8,\, 1/16,\, 1/256) = \tilde{P}_1$.

(6) Eliminate t from $x = \sigma_2(t,t,1,1)/\sigma_1(t,t,1,1)^2$, $y = \sigma_3(t,t,1,1)/\sigma_1(t,t,1,1)^3$, $z = \sigma_4(t,t,1,1)/\sigma_1(t,t,1,1)^4$, then we obtain $4x - 8y = 1$ and $y^2 = z$. This is a non-singular rational curve.

(7) This follow from theory of quartic equations. $g(a,b,c,d) := 64\sigma_4 - 16\sigma_2^2 + 16\sigma_1^2\sigma_2 - 16\sigma_1\sigma_3 - 3\sigma_1^4$ is a separator. Note that
$$g(a,a,c,d) = -(c-d)^2(8a^2 - 8ac + 3c^2 - 8ad + 2cd + 3d^2),$$
$$g(a,a,a,d) = -3(a-d)^4.$$
Thus, $V(g)$ pass through $\tilde{C}_2$. ☐

3.2 The PSD cone $\boldsymbol{\mathcal{P}}^{s0}_{4,4}$

In this subsection, we shall study $\boldsymbol{\mathcal{P}}^{s0}_{4,4} := \boldsymbol{\mathcal{P}}(\mathbb{P}^3_{\mathbb{R}},\, \boldsymbol{\mathcal{H}}^{s0}_{4,4})$. We choose
$$s_0 := S_4 - 4U, \quad s_1 := T_{3,1} - 12U, \quad s_2 := S_{2,2} - 6U, \quad s_3 := T_{2,1,1} - 12U$$
as a basis of $\boldsymbol{\mathcal{H}}^{s0}_{4,4}$. The aim of this subsection is to prove the following theorem.

Theorem 3.4. (1) *For a monic* $f = s_0 + ps_1 + qs_2 + rs_3 \in \check{\boldsymbol{\mathcal{H}}}^{s0}_{4,4}$, $f(\mathbf{a}) \geq 0$ *for all* $\mathbf{a} \in \mathbb{R}^4$ *if and only if*
$$p + r \geq 0 \quad \textit{and} \quad -9p^2 + 12p + 12q + 12r + 8 \geq 0.$$

(2) *All the extremal elements of* $\boldsymbol{\mathcal{P}}^{s0}_{4,4}$ *are positive multiples of* $\mathfrak{g}_t$ $(t \in \mathbb{P}^1_{\mathbb{R}} = \mathbb{R} \cup \{\infty\})$ *or* $\mathfrak{p}$.

(3) *All the discriminants of* $\boldsymbol{\mathcal{P}}^{s0}_{4,4}$ *are* $\mathrm{disc}_{C_1} = 9p^2 + 12p + 12q + 12r + 8$ *and* $\mathrm{disc}_{P_2} = p + r$.

(4) $\{(t\colon 1\colon 1\colon 1) \in \mathbb{P}^3_+ \mid t \geq 0\} \cup \{(-1\colon -1\colon 1\colon 1)\}$ *is a test set for* $\boldsymbol{\mathcal{P}}^{s0}_{4,4}$.

This theorem will be proved after Lemma 3.7.

For $f \in \mathbb{C}[x_1, \ldots, x_n]_d$ and $K = \mathbb{R}$ or $\mathbb{C}$, we denote

$$V_K(f) := \{a \in \mathbb{P}^n_K \mid f(a) = 0\}, \quad V_+(f) := V_{\mathbb{R}}(f) \cap \mathbb{P}^n_+.$$

In some articles, $V_K(f)$ are also denoted by $\mathfrak{Z}(f)$. The symbol $V_K(f)$ is rather popular in algebraic geometry.

We define $\Phi^{s0}_{4,4} : \mathbb{P}^3_{\mathbb{R}} \cdots \to \mathbb{P}^3_{\mathbb{R}}$ by $\Phi^{s0}_{4,4}(a) = (s_0(a) : s_1(a) : s_2(s) : s_3(a))$. Let

$$X^{s0}_{4,4} := \Phi^{s0}_{4,4}(\mathbb{P}^3_{\mathbb{R}}) = X(\mathbb{P}^3_{\mathbb{R}}, \mathcal{H}^{s0}_{4,4}) \subset \mathbb{P}((\mathcal{H}^{s0}_{4,4})^\vee),$$

and let $\Psi\colon \mathbb{P}^3_{\mathbb{R}}/\mathfrak{S}_4 \cdots \to X^{s0}_{4,4}$ be the rational map such that $\Phi^{s0}_{4,4} = \Psi \circ \sigma$. Let

$$\begin{aligned}
C_1 &:= \mathrm{Cls}(\Psi(\tilde{C}_1)) = \{\Phi^{s0}_{4,4}(t\colon 1\colon 1\colon 1) \mid t \in \mathbb{P}^1_{\mathbb{R}}\}, \\
C_2 &:= \Psi(\tilde{C}_2) = \{\Phi^{s0}_{4,4}(t\colon t\colon 1\colon 1) \mid -1 < t < 1\}, \\
P_0 &:= \Psi(\tilde{P}_0) = \Phi^{s0}_{4,4}(-1\colon 0\colon 0\colon 1) = (2\colon -2\colon 1\colon 0), \\
P_2 &:= \Psi(\tilde{P}_2) = \Phi^{s0}_{4,4}(-1\colon -1\colon 1\colon 1) = (0\colon 1\colon 0\colon 1) \\
P_{-3} &:= \Phi^{s0}_{4,4}(-3\colon 1\colon 1\colon 1) = (2\colon -1\colon 1\colon 1).
\end{aligned}$$

Moreover let

$$\begin{aligned}
E_0 &:= \{(a\colon b\colon c\colon d) \in \mathbb{P}^4_{\mathbb{R}} \mid a,\, b,\, c,\, d \in \mathbb{R},\ a + b + c + d = 0\}, \\
D'_1 &:= \{(a\colon b\colon c\colon c) \in \mathbb{P}^4_{\mathbb{R}} \mid a,\, b,\, c \in \mathbb{R}\},
\end{aligned}$$

$L_0 := \Phi^{s0}_{4,4}(E_0)$ and $D_1 := \Psi(\tilde{D}_1) = \Phi^{s0}_{4,4}(D'_1)$. Note that $\Psi(\tilde{D}_1) = \Phi^{s0}_{4,4}(D'_1)$. Since $\mathrm{Bs}\,\mathcal{H}^{s0}_{4,4} = \{(1\colon 1\colon 1\colon 1)\}$, Ψ is not holomorphic at $\tilde{P}_1$.

Note that $\mathrm{Bs}\,\mathcal{H}^{s0}_{4,4} = \{(1\colon 1\colon 1\colon 1)\}$, and Ψ is not holomorphic at $\tilde{P}_1$.

Lemma 3.5.

(1) $\Psi : \mathbb{P}^3_{\mathbb{R}}/\mathfrak{S}_4 \longrightarrow X^{s0}_{4,4}$ *is continuous map and* $\Psi : (\mathbb{P}^3_{\mathbb{R}}/\mathfrak{S}_4 - \{\tilde{P}_1\}) \longrightarrow X^{s0}_{4,4}$ *is a birational morphism. All the exceptinal set of* $\Phi^{s0}_{4,4} : \mathbb{P}^3_{\mathbb{R}} \cdots \to \mathbb{P}^3_{\mathbb{R}}$ *is* E_0.

(2) $\partial \mathcal{P}^{s0}_{4,4} = \mathcal{F}(C_1) \cup \mathcal{F}(P_2)$, *and* $\mathcal{E}(X^{s0}_{4,4}) \subset C_1 \cup \{P_2\}$.

Proof. We denote the coordinate system of $\mathbb{P}((\mathcal{H}^s_{4,4})^\vee) = \mathbb{P}^3_{\mathbb{R}}$ by $(x_0\colon x_1\colon x_2\colon x_3)$, $\Phi^{s0}_{4,4}$ is defined by $x_i = s_i(\mathbf{a})$.

(0) Let $P_1 := (2\colon 3\colon 1\colon 1)$. When $a,\, b,\, c \to 0$,

$$\Phi^{s0}_{4,4}(1\colon 1{+}a\colon 1{+}b\colon 1{+}c) = (3a^2{-}2ab{+}3b^2{-}2ac{-}2bc{+}3c^2)(2\colon 3\colon 1\colon 1){+}(\text{higher terms of } a,\, b,\, c).$$

Thus $\Phi^{s0}_{4,4}(1\colon 1\colon 1\colon 1) = \Psi(\tilde{P}_1) = P_1$, and Ψ is continuoius at $\tilde{P}_1$.

(1) We take A_s as Example 3.2(3). It is easy to see that $\Phi^{s0}_{4,4} \otimes_{\mathbb{R}} \mathbb{C} : \mathbb{P}^3_{\mathbb{C}} \longrightarrow \mathbb{P}^3_{\mathbb{C}}$ is a generically finite rational map of degree 24. Thus $\Phi^{s0}_{4,4}\colon A_s \longrightarrow X^{s0}_{4,4}$ is generically one to one. Using PC, we have

$$J_P := \det\left(\frac{\partial s_i(a_0, a_1, a_2, a_3)}{\partial a_j}\right)_{0 \le i,j \le 3} = 16 S_1^2 (3S_2 - 2S_{1,1})^2 \prod_{i<j} (a_i - a_j).$$

$J_p \neq 0$ on $\mathrm{Int}(A_s) - \{(1\colon 1\colon 1\colon 1)\}$. Thus $\Phi^{s0}_{4,4}\colon \mathrm{Int}(A_s) \longrightarrow X^{s0}_{4,4}$ is injective. Since $\partial A_s \subset E_0 \cup \bigcup_{\tau \in \mathfrak{S}_4} \tau(D'_1)$, we have $\partial X^{s0}_{4,4} = L_0 \cup D_1$. So, $\Psi : (\mathbb{P}^3_{\mathbb{R}}/\mathfrak{S}_4 - \{\tilde{P}_1\}) \longrightarrow X^{s0}_{4,4}$ is a birational morphism. Let

$$\begin{aligned}f^{s0}_{4,4}(x_0,x_1,x_2,x_3) := &-3x_0x_1^4+4x_1^5+6x_0^2x_1^2x_2-24x_0x_1^3x_2+14x_1^4x_2-3x_0^3x_2^2\\ &+20x_0^2x_1x_2^2-48x_0x_1^2x_2^2+16x_1^3x_2^2+34x_0^2x_2^3+16x_0x_1x_2^3+8x_1^2x_2^3+44x_0x_2^4\\ &-48x_1x_2^4-72x_2^5+12x_0^2x_1^2x_3+12x_0x_1^3x_3-36x_1^4x_3-12x_0^3x_2x_3+20x_0^2x_1x_2x_3\\ &+120x_0x_1^2x_2x_3-56x_1^3x_2x_3-76x_0^2x_2^2x_3-32x_0x_1x_2^2x_3-64x_1^2x_2^2x_3\\ &-32x_0x_2^3x_3+112x_1x_2^3x_3+144x_2^4x_3-12x_0^3x_3^2-40x_0^2x_1x_3^2-112x_1x_2^3x_3\\ &+144x_2^4x_3-12x_0^3x_3^2-40x_0^2x_1x_3^2-18x_0x_1^2x_3^2+104x_1^3x_3^2+14x_0^2x_2x_3^2\\ &-104x_0x_1x_2x_3^2+84x_1^2x_2x_3^2+64x_0x_2^2x_3^2+16x_1x_2^2x_3^2-152x_2^3x_3^2+28x_0^2x_3^3\\ &+12x_0x_1x_3^3-136x_1^2x_3^3+8x_0x_2x_3^3-56x_1x_2x_3^3+32x_2^2x_3^3-3x_0x_3^4+84x_1x_3^4\\ &+14x_2x_3^4-20x_3^5.\end{aligned}$$

Since

$$f^{s0}_{4,4}(s_0,s_1,s_2,s_3)=16(a_0+a_1+a_2+a_3)^4\left(\prod_{i<j}(a_i-a_j)^2\right)\left(\sum_{i<j}(a_i-a_j)^2\right)^2,$$

we have $\partial X^{s0}_{4,4}=\Phi^{s0}_{4,4}(E_0\cup D_1')\subset V_{\mathbb{R}}(f^{s0}_{4,4})\subset\mathbb{P}^3_{\mathbb{R}}$ by Corollary 2.13. Since $f^{s0}_{4,4}$ is irreducible, we have $\mathrm{Zar}(\partial X^{s0}_{4,4})=V_{\mathbb{R}}(f^{s0}_{4,4})$. Note that $f^{s0}_{4,4}\geq 0$ on $X^{s0}_{4,4}$.

It is easy to see that L_0 is a closed line segment $[P_2P_{-3}]$ defined by $x_0=2x_2$, $x_0-x_1+x_3=0$ and $x_1/x_0\leq -1/2$. This also means that E_0 is an exceptinal set of $\Phi^{s0}_{4,4}$.

Similarlym C_2 is an open line segment (P_1P_2) defined by $x_0=2x_2$, $x_0-x_1+x_3=0$ and $x_1/x_0<3/2$. Note that L_0, $C_2\subset\partial X^{s0}_{4,4}$.

Next we consider C_1. Let

$$g_2(x_0,x_1,x_2,x_3):=(x_1-x_3)^2+2x_2^2-3x_2x_0.$$

Then C_1 is the conic defined by $x_2=x_3$ and $g_2(x_0,\,x_1,\,x_2,\,x_3)=0$. Note that $x_2-x_3\geq 0$ on $X^{s0}_{4,4}$, because $s_2-s_3=\mathfrak{p}\geq 0$ on A_s.

Let B be the ellise domain on the plane $x_2=x_3$ defined by $g_2(1,x_1,x_2,x_2)\leq 0$, and let Y be the cone with the base B and the virtex P_2.

(2) We shall show that Y is the convex closure of $X^{s0}_{4,4}$.

A point on C_1 can be written as

$$P(t)=\Phi^{s0}_{4,4}(t,1,1,1)=(t^2+2t+3\colon 3(t+2)\colon 3\colon 3)$$

where $t\in\mathbb{P}^1_{\mathbb{R}}$. $P(1)=P_1$ and $P(-3)=P_{-3}$. Let $L(t):=(P_2P(t))$ be an open line segment. Note that $L(-3)=(P_2P_{-3})\subset L_0$, and $L(1)=(P_2P_1)=C_2$. A point on $L(t)$ can be written as $P(t,s)=P(t)+sP_2$ by $s>0$. Using PC, we have

$$f^{s0}_{4,4}(P(t,s))=-12s^2(s-1)^2(t+3)^4.$$

This implies $L(t)\cap X^{s0}_{4,4}=\emptyset$, if $t\neq -3,1$. This means $X^{s0}_{4,4}\subset Y$. Since $C_1\cup\{P_2\}\subset X^{s0}_{4,4}\subset Y$, we conclude that Y is the convex closure of $X^{s0}_{4,4}$. This also implies $\boldsymbol{\mathcal{E}}(X^{s0}_{4,4})\subset C_1\cup\{P_2\}$. Since $X^{s0}_{4,4}\cap\partial Y=C_1\cup C_2\cup L_0\cup\{P_2\}$, we have $\partial\boldsymbol{\mathcal{P}}^{s0}_{4,d}=\boldsymbol{\mathcal{F}}(C_1)\cup\boldsymbol{\mathcal{F}}(C_2)\cup\boldsymbol{\mathcal{F}}((P_2P_{-3}))\cup\boldsymbol{\mathcal{F}}(P_2)$ by Theorem 2.7(1). But $\boldsymbol{\mathcal{F}}(C_2)$ and $\boldsymbol{\mathcal{F}}((P_2P_{-3}))$ are not face components, because dual varieties of $\mathrm{Zar}(C_2)$ and $\mathrm{Zar}((P_2P_{-3}))$ are linear subspaces of $cH^{s0}_{4,4}$ of codimension 2. Thus all the face components of $\boldsymbol{\mathcal{P}}^{s0}_{4,4}$ are $\boldsymbol{\mathcal{F}}(C_1)$ and $\boldsymbol{\mathcal{F}}(P_2)$. Therefore $\partial\boldsymbol{\mathcal{P}}^{s0}_{4,4}=\boldsymbol{\mathcal{F}}(C_1)\cup\boldsymbol{\mathcal{F}}(P_2)$. □

Proof of Theorem 1.9(1). Put $\Omega := \{(-1\colon -1\colon 1\colon 1)\} \cup \{(t\colon 1\colon 1\colon 1) \in \mathbb{P}^3_{\mathbb{R}} \mid t \in \mathbb{R}\}$. By Theorem 2.10, it is enough to show that $\Phi^{s0}_{4,4}(\Omega) \supset C_1 \cup \{P_2\} = \mathcal{E}(X^{s0}_{4,4})$. But this is clear. $\square$

We regard $\mathcal{H}^{s0}_{4,3} = \mathbb{R}^4$, by identifying $f = \sum_{i=0}^{3} p_i s_i \in \mathcal{H}^{s0}_{4,3}$ and $(p_0,\, p_1,\, p_2,\, p_3) \in \mathbb{R}^4$. We also use $(p_0,\, p_1,\, p_2,\, p_3)$ as a coordinate system of $\mathcal{H}^{s0}_{4,3} = \mathbb{R}^4$. We denote the local cone of $\mathcal{P}^{s0}_{4,4}$ at $(t\colon 1\colon 1\colon 1) \in \mathbb{P}^3_{\mathbb{R}}$ by $\mathcal{L}^{s0}_t$. Note that if $f \in \mathcal{F}(C_1)$, there exists $t \in \mathbb{R}$ such that $f(t,\, 1,\, 1,\, 1) = 0$. Thus $f \in \mathcal{L}^{s0}_t$. For $t = \infty \in \mathbb{P}^1_{\mathbb{R}}$, we denote the local cone of $\mathcal{P}^{s0}_{4,4}$ at $(1\colon 0\colon 0\colon 0) \in \mathbb{P}^3_{\mathbb{R}}$ by $\mathcal{L}^{s0}_{\infty}$.

We shall observe $\mathfrak{g}_t$, $\mathfrak{g}_\infty$ and $\mathfrak{p} \in \mathcal{P}^{s0}_{4,4}$. Note that

$$\begin{aligned}
3\mathfrak{g}_t(a,b,c,d) &= 3s_0 - 2(t+1)(s_1 - s_3) + (t^2 + 2t - 1)s_2 \\
&= \big(a^2 + b^2 - c^2 - d^2 + (t+1)(cd - ab)\big)^2 \\
&\quad + \big(a^2 - b^2 + c^2 - d^2 + (t+1)(bd - ac)\big)^2 \\
&\quad + \big(a^2 - b^2 - c^2 + d^2 + (t+1)(bc - ad)\big)^2, \\
\mathfrak{g}_\infty(a,b,c,d) &= s_2 = (ab - cd)^2 + (ac - bd)^2 + (ad - bc)^2, \\
\mathfrak{p}(a,b,c,d) &= s_2 - s_3 = (a-b)^2(c-d)^2 + (a-c)^2(b-d)^2 + (a-d)^2(b-c)^2.
\end{aligned}$$

Espacially, $\mathfrak{g}_t$, $\mathfrak{g}_\infty$, $\mathfrak{p} \in \Sigma_{4,4}$.

If $f \in \mathcal{E}(\mathcal{P}_{4,4}) \cap \Sigma_{4,4}$, then there exists $g \in \mathcal{P}_{2,4}$ such that $f = g^2$. Therefore $\mathfrak{g}_t$, $\mathfrak{g}_\infty$, $\mathfrak{p} \notin \mathcal{E}(\mathcal{P}_{4,4})$. But $\mathfrak{g}_t$, $\mathfrak{g}_\infty$, $\mathfrak{p} \notin \mathcal{E}(\mathcal{P}^{s0}_{4,4})$ as the following Lemma.

For $f(a,b,c,d) \in \mathbb{R}[a,b,c,d]$, we dnote $\dfrac{\partial}{\partial a} f$ by f_a, $\dfrac{\partial^2}{\partial a^2} f$ by f_{aa}, and so on.

Lemma 3.6. $\mathfrak{g}_t \in \mathcal{E}(\mathcal{P}^{s0}_{4,4})$ *for all* $t \in \mathbb{P}^1_{\mathbb{R}}$, *and* $\mathfrak{p} \in \mathcal{E}(\mathcal{P}^{s0}_{4,4})$. *These are characterized as the following:*

(1) *Let* $t \in \mathbb{R} - \{1, -3\}$. *If* $f \in \mathcal{P}^{s0}_{4,4}$ *satisfies* $f(t,\, 1,\, 1,\, 1) = 0$ *and* $f(-1,\, -1,\, 1,\, 1) = 0$, *then there exists* $\alpha \geq 0$ *such that* $f = \alpha\mathfrak{g}_t$.

(2) *If* $f \in \mathcal{P}^{s0}_{4,4}$ *satisfies* $f(x,\, x,\, 1,\, 1) = 0$ *for all* $x \in \mathbb{R}$, *then there exists* $\alpha \geq 0$ *such that* $f = \alpha\mathfrak{g}_1$.

(3) *If* $f \in \mathcal{P}^{s0}_{4,4}$ *satisfies* $f(x,\, y,\, z,\, -x-y-z) = 0$ *for all* $x,\, y,\, z \in \mathbb{R}$, *then there exists* $\alpha \geq 0$ *such that* $f = \alpha\mathfrak{g}_{-3}$.

(4) *If* $f \in \mathcal{P}^{s0}_{4,4}$ *satisfies* $f(0,\, 0,\, 0,\, 1) = 0$ *and* $f(-1,\, -1,\, 1,\, 1) = 0$, *then there exists* $\alpha \geq 0$ *such that* $f = \alpha\mathfrak{g}_\infty$.

(5) *If* $f \in \mathcal{P}^{s0}_{4,4}$ *satisfies* $f(0,\, 0,\, 0,\, 1) = 0$ *and* $f(x,\, 1,\, 1,\, 1) = 0$ *for all* $x \in \mathbb{R}$, *then there exists* $\alpha \in \mathbb{R}_+$ *such that* $f = \alpha\mathfrak{p}$.

Proof. Note that of $f \in \mathcal{P}^{s0}_{4,4}$ satisfies $f(a,b,c,d) = 0$, then $f_a(a,b,c,d) = 0$. Similarly, if $f_{aa}(a,b,c,d) = 0$, then $f_{aaa}(a,b,c,d) = 0$. Otherwise, f will be negative at a certain point near $(a,\, b,\, c,\, d)$. $f \in \mathcal{H}^s_{4,4}$ can be written as $f = p_0 s_0 + p_1 s_1 + p_2 s_2 + p_3 s_3)$ by p_0, p_1, p_2, $p_3) \in \mathbb{R}$.

(1) Take $t \in \mathbb{R} - \{1,\, -3\}$. Let's consider a system of equations

$$f(t,1,1,1) = 0, \quad f_a(t,1,1,1) = 0, \quad f(-1,-1,1,1) = 0. \qquad (*)$$

Let $a_{0,j} := s_j(t,1,1,1)$, $a_{1,j} := (s_j)_a(t,1,1,1)$, $a_{2,j} := s_j(1,1,-1,-1)$, and $A := (a_{i,j}) \in M_{3,4}(\mathbb{R})$. Then, $(*)$ is equivalent to $A\mathbf{p} = \mathbf{0}$. That is

$$\begin{pmatrix} (t-1)^2(t^2+2t+3) & 3(t-1)^2(t+2) & 3(t-1)^2 & 3(t-1)^2 \\ 4(t^3-1) & 9(t^2-1) & 6(t-1) & 6(t-1) \\ 0 & -16 & 0 & -16 \end{pmatrix} \begin{pmatrix} p_0 \\ p_1 \\ p_2 \\ p_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

Using Mathmatica, we can soon check that $\operatorname{Ker} A = \mathbb{R} \cdot \mathfrak{g}_t$. If $f \in \boldsymbol{\mathcal{P}}^{s0}_{4,4}$ satisfies $f(t,1,1,1) = 0$, then $f_a(t,1,1,1) = 0$ always holds. Thus, if $f \in \boldsymbol{\mathcal{P}}^{s0}_{4,4}$ satisfies $f(t,1,1,1) = 0$ and $f(-1,-1,1,1) = 0$, then $f = \alpha\mathfrak{g}_t$ by a certain $\alpha > 0$.

(2) Consider a system of equations $f(0,0,1,1) = 0$, $f(2,2,1,1) = 0$ instead of $(*)$. Then $\dim \operatorname{Ker} A = 2$, and $\mathfrak{g}_1$ and $g := s_1 - 2s_2 - s_3$ is a basis of $\operatorname{Ker} A$. g is not PSD. Since $\mathfrak{g}_1(x,1,1,1) + cg(x,1,1,1) = (x-1)^3(x-1+3c)$, $\mathfrak{g}_1 + cg$ is PSD only if $c = 0$.

(3) Consider $f(1,2,3,-6) = 0$, $f_a(1,2,3,-6) = 0$, $f(1,2,4,-7) = 0$.

(4) Consider $f(0,0,0,1) = 0$, $f_a(0,0,0,1) = 0$, $f(-1,-1,1,1) = 0$.

(5) Consider $f(2,1,1,1) = 0$, $f(0,0,0,1) = 0$, $f_a(0,0,0,1) = 0$.

Each A of the cases (2)—(5) are as follows:

$$(2) \quad A = \begin{pmatrix} 18 & 26 & 9 & 8 \\ 24 & 34 & 12 & 10 \\ 24 & 18 & 0 & 0 \end{pmatrix}, \qquad (3) \quad A = \begin{pmatrix} 1538 & -962 & 769 & 576 \\ 148 & 248 & 314 & 516 \\ 2898 & -2002 & 1449 & \end{pmatrix},$$

$$(4) \quad A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 16 & 0 & -16 \end{pmatrix}, \qquad (5) \quad A = \begin{pmatrix} 11 & 12 & 3 & 3 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}.$$

□

$\mathfrak{g}_t$ ($t \in \mathbb{P}^1_{\mathbb{R}}$) degenerates when $t = 1, -3$. Note that

$$\mathfrak{g}_{-3} = S_1^2(3S_3 - 2T_{1,1}).$$

Thus $\boldsymbol{\mathcal{F}}(L_0) = \mathbb{R}_+ \cdot \mathfrak{g}_{-3}$.

Since $\mathfrak{g}_1(x,\ x,\ 1,\ 1) = 0$ for all $x \in \mathbb{P}^1_{\mathbb{R}}$, we have $\boldsymbol{\mathcal{F}}(C_2) = \mathbb{R}_+ \cdot \mathfrak{g}_1$. These also implies that $\boldsymbol{\mathcal{F}}(L_0)$ and $\boldsymbol{\mathcal{F}}(C_2)$ are not a face component of $\boldsymbol{\mathcal{P}}^{s0}_{4,4}$, and we can omit $\{(x\colon x\colon 1\colon 1) \in \mathbb{P}^3_{\mathbb{R}} \mid x \in \mathbb{R}_+\}$ from the test set.

Lemma 3.7. *$\boldsymbol{\mathcal{L}}^{s0}_t = \mathbb{R}_+ \cdot \mathfrak{g}_t + \mathbb{R}_+ \cdot \mathfrak{p}$, and the discriminant of $\boldsymbol{\mathcal{F}}(C_1)$ and $\boldsymbol{\mathcal{F}}(P_2)$ are*

$$\operatorname{disc}_{C_1}(p_0,p_1,p_2,p_3) = 8p_0^2 - 9p_1^2 + 12p_0p_1 + 12p_0p_2 + 12p_0p_3,$$
$$\operatorname{disc}_{P_2}(p_0,p_1,p_2,p_3) = p_1 + p_3.$$

Proof. Since $P_2 = (0\colon 1\colon 0\colon 1)$, $\operatorname{disc}_{P_2}(p_0,\ p_1,\ p_2,\ p_3) = p_1 + p_3$, by Remark 1.28 of [4].

Since $\mathfrak{g}_t$, $\mathfrak{p} \in \boldsymbol{\mathcal{L}}^{s0}_t$ ($t \in \mathbb{P}^1_{\mathbb{R}}$), we have $\dim \boldsymbol{\mathcal{L}}^{s0}_t \geq 2$. On the other hand, since $\dim \boldsymbol{\mathcal{L}}^{s0}_t < \dim \boldsymbol{\mathcal{P}}^{s0}_{4,4} = 3$, we have $\dim \boldsymbol{\mathcal{L}}^{s0}_t = \dim \boldsymbol{\mathcal{L}}^{s0}_\infty = 2$ ($t \neq 1$). Since $\mathfrak{g}_t$, $\mathfrak{p} \in \boldsymbol{\mathcal{E}}(\boldsymbol{\mathcal{P}}^{s0}_{4,4})$, we have $\boldsymbol{\mathcal{L}}^{s0}_t = \mathbb{R}_+ \cdot \mathfrak{g}_t + \mathbb{R}_+ \cdot \mathfrak{p}$ for all $t \in \mathbb{P}^1_{\mathbb{R}}$.

Using PC, we can check that $\mathfrak{g}_t$ ($\forall t \in \mathbb{P}^1_{\mathbb{R}}$) and $\mathfrak{p}$ exists on the hypersurface in $\boldsymbol{\mathcal{H}}^{s0}_{4,4}$ defined by $8p_0^2 - 9p_1^2 + 12p_0p_1 + 12p_0p_2 + 12p_0p_3$. This equation is also the defining equation of the dual variety of C_1. So, this is $\operatorname{disc}_{C_1}$. □

Proof of Theorem 3.4. By the above lemma, we have

$$\mathcal{F}(P_2) = \left\{ \sum_{i=0}^{3} p_i s_i \in \mathcal{H}_{4,4}^{s0} \;\middle|\; p_1 + p_3 = 0,\ p_0 \geq 0,\ -9p_1^2 + 12p_0p_2 + 8p_0^2 \geq 0 \right\},$$

$$\mathcal{F}(C_1) = \left\{ \sum_{i=0}^{3} p_i s_i \in \mathcal{H}_{4,4}^{s0} \;\middle|\; \begin{array}{l} p_1 + p_3 \geq 0,\ p_0 \geq 0, \\ -9p_1^2 + 12p_0p_1 + 12p_0p_2 + 12p_0p_3 + 8p_0^2 = 0 \end{array} \right\}.$$

Thus, all the extremal elements of $\mathcal{P}_{4,4}^{s0}$ are $\mathfrak{g}_t$ ($t \in \mathbb{P}_{\mathbb{R}}^1$) and $\mathfrak{p}$.

Thus, for $f = s_0 + ps_1 + qs_2 + rs_3 \in \mathcal{H}_{4,4}^{s0}$, $f(a) \geq 0$ for all $a \in \mathbb{P}_{\mathbb{R}}^3$ if and only if $p + r \geq 0$ and $-9p^2 + 12p + 12q + 12r + 8 \geq 0$.

(4) follow from $\partial\mathcal{P}_{4,4}^{s0} = \mathcal{F}(C_1) \cup \mathcal{F}(P_2)$. ∎

Proof of Theorem 1.1(1), 1.2 and 1.8(1). Let $t_0 := \sigma_1^4 - 256\sigma_4$, $t_1 := \sigma_1^2\sigma_2 - 96\sigma_4$, $t_2 := \sigma_2^2 - 36\sigma_4$, $t_3 := \sigma_1\sigma_3 - 16\sigma_4$. Then $s_0 = t_0 - 4t_1 + 2t_2 + 4t_3$, $s_1 = t_1 - 2t_2 - t_3$, $s_2 = t_2 - 2t_3$ and $s_3 = t_3$. Using these substitution for $\mathfrak{g}_t$, $\mathfrak{g}_\infty$ and $\mathfrak{p}$, we obtain Theorem 1.2.

Take $f = p_0s_0 + p_1s_1 + p_2s_2 + p_3s_3 = q_0t_0 + q_1t_1 + q_2t_2 + q_3t_3 \in \mathcal{H}_{4,4}^{s0}$. Since $t_0 = s_0 + 4s_1 + 6s_2 + 12s_3$, $t_1 = s_1 + 2s_2 + 5s_3$, $t_2 = s_2 + 2s_3$ and $t_3 = s_3$, we have $p_0 = q_0$, $p_1 = 4q_0 + q_1$, $p_2 = 6q_0 + 2q_1 + q_2$, and $p_3 = 12q_0 + 5q_1 + 2q_2 + q_3$. Substitute these for p_i in disc_{C_1} and disc_{P_2} of Lemma 3.7, we obtain d_1 and d_2 of Theorem 1.8(1). Theorem 1.1(1) follows from these. ∎

Proof of Proposition 1.7(1). Let $f(x, y) := \mathfrak{g}_t(x, y, 1, -x - y - 1)/(t + 3)^3$ for $t \in \mathbb{P}_{\mathbb{R}}^1 - \{-3\}$. If $\mathfrak{g}_t$ is reducible, then f is also reducible. By

$$\frac{\partial}{\partial x} f(x, y) = 2(2x + y + 1)(x^2 + xy + y^2 + x + 3y + 1)$$

and so on, we have

$$\mathrm{Sing}(V_{\mathbb{C}}(f)) = \{(-1\colon -1\colon 1),\ (-1\colon 0\colon 1),\ (0\colon 1\colon 1)\}.$$

Moreover, these are acnodes. Assume that $f = gh$. If $\deg g = 1$, then $\#\,\mathrm{Sing}(V_{\mathbb{C}}(f)) = 4$ or $\#\,\mathrm{Sing}(V_{\mathbb{C}}(f)) \subset V_{\mathbb{C}}(g)$. This cannot occur. Thus, g and h are irreducible quadric curves which intersect transversally. Then, $\#\,\mathrm{Sing}(V_{\mathbb{C}}(f)) = 4$. Therefore, $V_{\mathbb{C}}(f)$ must be an irreducible rational quartic curve. ∎

Proof of Corollary 1.3. $\mathcal{E}(\mathcal{P}_{4,4}^{s0}) \subset \Sigma_{4,4}$ is already proved. Since, any element of $\mathcal{P}_{4,4}^{s0}$ can be written as a sum of some elements in $\mathcal{E}(\mathcal{P}_{4,4}^{s0})$, we have $\mathcal{P}_{4,4}^{s0} \subset \Sigma_{4,4}$.

Assume that $\exists f \in \mathcal{E}(\mathcal{P}_{4,4}^{s0}) \cap \mathcal{E}(\mathcal{P}_{4,4}) \neq \emptyset$. f is SOS, since $\mathcal{E}(\mathcal{P}_{4,4}^{s0}) \subset \Sigma_{4,4}$. Since, $f \in \mathcal{E}(\mathcal{P}_{4,4})$, we have $f \in \mathcal{E}(\Sigma_{4,4})$. Thus, there exists $g \in \mathcal{H}_{4,2}$ such that $f = g^2$. Then $V_{\mathbb{R}}(g) = V_{\mathbb{R}}(f)$. Since $\#V_{\mathbb{R}}(\mathfrak{g}_t) \geq 2$ and $\#V_{\mathbb{R}}(\mathfrak{p}) \geq 2$, we have $\#V_{\mathbb{R}}(g) \geq 2$. Such conic g satisfies $\dim_{\mathbb{R}} V_{\mathbb{R}}(g) \geq 1$. But, $V_{\mathbb{R}}(f)$ is a finite set. ∎

3.3. The PSD cone $\mathcal{P}_{4,4}^{s0+}$

In this subsection, we shall study $\mathcal{P}_{4,4}^{s0+} := \mathcal{P}(\mathbb{P}_+^3, \mathcal{H}_{4,4}^{s0})$. The aim of this subsection is to prove the following theorem.

Theorem 3.8. (I) *For a monic*

$$f = s_0 + ps_1 + qs_2 + rs_3 \in \breve{\mathcal{H}}^{s0}_{4,4},$$

$f(\mathbf{a}) \geq 0$ *for all* $\mathbf{a} \in \mathbb{R}^4_+$ *if and only if the following "(1) or (2)" and "(3) or (4)" hold:*
(1) $p \leq -4$ *and* $p^2 \leq 4q - 8$.
(2) $p \geq -4$ *and* $2p + q + 2 \geq 0$.
(3) $p \leq -2/3$ *and* $9p^2 \leq 12p + 12q + 12r + 8$.
(4) $p \geq -2/3$ *and* $3q + 3r \geq 1$.

(II) *All the extremal elements of* $\mathcal{P}^{s0+}_{4,4}$ *are positive multiples of* $\mathfrak{f}^{ab}_t$ $(0 \leq t \leq 5)$, $\mathfrak{f}^c_t$ $(5 < t < \infty)$, $\mathfrak{p} = s_2 - s_3$, $\mathfrak{q}_1 = s_1 - 2s_2$ *or* $\mathfrak{q}_2 = s_3$.

(III) *The following set is a test set for* $(\mathbb{P}^3_+, \mathcal{H}^{s0+}_{4,4})$.

$$\left\{(t\colon 1\colon 1\colon 1) \in \mathbb{P}^3_+ \;\middle|\; t \geq 0\right\} \cup \left\{(0\colon 0\colon t\colon 1) \in \mathbb{P}^3_+ \;\middle|\; t \geq 0\right\}.$$

This theorem will be proved after Lemma 3.16.

Essentially, we use the same symbols as the previous subsection, but there are some changes. Let $A := \mathbb{P}^3_+ : (a_0\colon a_1\colon a_2\colon a_3)$, $X^{s0+}_{4,4} := \Phi^{s0}_{4,4}(\mathbb{P}^3_+) = X(\mathbb{P}^3_+, \mathcal{H}^{s0}_{4,4}) \subset \mathbb{P}\big((\mathcal{H}^{s0}_{4,4})^\vee\big)$. As §3.2, put $D_0 := \Psi(\tilde{D}_0)$, $D^+_1 := \Psi(\tilde{D}^+_1) \subset D_1$, $P_1 := (2\colon 3\colon 1\colon 1)$, $C^+_1 := \Psi(\tilde{C}^+_1) \cup \{P_1\} \subset C_1$, $C^+_2 := \Psi(\tilde{C}^+_2) \subset C_2$, $C_i := \Psi(\tilde{C}_i)$ for $i = 3, 4$ and $P_j := \Psi(\tilde{P}_j)$ for $j = 3, 4, 5$. Note that

$$\begin{aligned}
P_3 &= (1\colon 2\colon 1\colon 1) = \Phi^{s0}_{4,4}(0, 1, 1, 1), \\
P_4 &= (2\colon 2\colon 1\colon 0) = \Phi^{s0}_{4,4}(0, 0, 1, 1), \\
P_5 &= (1\colon 0\colon 0\colon 0) = \Phi^{s0}_{4,4}(0, 0, 0, 1).
\end{aligned}$$

Lemma 3.9. *Let* $Z := \mathbb{P}^3_+/\mathfrak{S}_4 - \tilde{C}^+_1 - \tilde{C}^+_2 - \{\tilde{P}_1, \tilde{P}_3, \tilde{P}_4, \tilde{P}_5\}$.
(1) $\Psi\colon \mathbb{P}^3_+/\mathfrak{S}_4 \longrightarrow X^{s0+}_{4,4}$ *is continuous bijective map and* $\Psi : Z \longrightarrow \Psi(Z)$ *is an isomprohism.*
(2) $\Delta^0(X^{s0+}_{4,4}) = \{P_3, P_4, P_5\}$, $\Delta^1(X^{s0+}_{4,4}) = \{C^+_1, C^+_2, C_3, C_4\}$, $\Delta^2(X^{s0+}_{4,4}) = \{D_0, D^+_1\}$, $\Delta^3(X^{s0+}_{4,4}) = \{\mathrm{Int}(X^{s0+}_{4,4})\}$.

Proof. (1) We use tha same symbols with the proof of Lemma 3.5. Note that $E_0 \cap A^+_s = \emptyset$. So, it is enough to show that $\Phi^{s0}_{4,4}$ is injective on $A^+_s \cap \bigcup_{\tau \in \mathfrak{S}_4} \tau(D'_1)$. It is enough to show that $\Phi^{s0}_{4,4}$ is injective on $\mathbb{P}^2_+ \cap D'_1$. It's Jacobian is equal to $J(x_1, x_2) := \det\left(\dfrac{\partial h_i}{\partial x_j}\right)_{1 \leq i,j \leq 1}$, where $h_i(x_1, x_2) := s_1(x_1, x_2, 1, 1)/s_0(x_1, x_2, 1, 1)$ $(i = 1, 2)$. Using PC, we have

$$\begin{aligned}
J(x, y) &= \frac{4(x-1)(y-1)(x-y)w(x, y)}{s_0(x, y, 1, 1)^3}, \\
w(x, y) &:= \frac{1}{8}\Big((x+y-2)^4(x+y+2)^2 + (x-y)^4(3(x+y)^2 + 28(x+y) + 12) \\
&\qquad + 4(x-y)^2(x+y-2)^2((x+y)^2 + 6(x+y) + 4)\Big).
\end{aligned}$$

Thus $J(x, y) \geq 0$ on $\mathbb{P}^2_+ \cap D'_1$, and $J(x, y) = 0$ only at points of the form $(x\colon 1\colon 1\colon 1)$ or $(1\colon x\colon 1\colon 1)$ or $(x\colon x\colon 1\colon 1)$. Thus $\Psi : Z \longrightarrow \Psi(Z)$ is an isomprohism.

(2) follows from Proposition 3.3(3) and the proof of Lemma 3.5. □

Lemma 3.10. (1) $\partial\mathcal{P}^{s0+}_{4,4} = \mathcal{F}(C^+_1) \cup \mathcal{F}(C_3) \cup \mathcal{F}(C_4) \cup \mathcal{F}(P_3) \cup \mathcal{F}(P_4) \cup \mathcal{F}(P_5)$.

(2) *Take* $f \in \mathcal{H}^{s0}_{4,4}$. *If* $f(x,1,1,1) \geq 0$, $f(0,x,1,1) \geq 0$, $f(0,0,x,1) \geq 0$ *for all* $x \geq 0$, *then* $f \in \mathcal{H}^{s0}_{4,4}$.

Proof. (1) $\mathrm{Int}(X^{s0+}_{4,4})$, $\mathcal{F}(D_0)$ and $\mathcal{F}(D_1^+)$ are not face components of $\mathcal{P}^{s0+}_{4,4}$ by Theorem 2.16. $\mathcal{F}(C_2^+)$ is not also a face component of $\mathcal{P}^{s0+}_{4,4}$, because C_2^+ is an open line segment (P_1, P_4). Thus, we have (1).

(2) Let

$$\begin{aligned} A_1^+ &:= \{(t\colon 1\colon 1\colon 1) \in \mathbb{P}^3_+ \mid t \geq 0\}, \\ A_2^+ &:= \{(t\colon t\colon 1\colon 1) \in \mathbb{P}^3_+ \mid 0 \leq t \leq 1\}, \\ A_3 &:= \{(0\colon t\colon 1\colon 1) \in \mathbb{P}^3_+ \mid t \geq 0\}, \\ A_4 &:= \{(0\colon 0\colon t\colon 1) \in \mathbb{P}^3_+ \mid t \geq 0\}. \end{aligned}$$

Note that $\Phi^{s0}_{4,4}(A_i^+) \supset C_i^+$ ($i = 1, 2$), and $\Phi^{s0}_{4,4}(A_j) \supset C_j$ ($j = 3, 4$). By Corollary 1.3 of [22] or Corollary 2.1 of [23], we can choose $A_1^+ \cup A_2^+ \cup A_3 \cup A_4$ as a test set for ($\mathbb{P}^3_+$, $\mathcal{H}^{s0}_{4,4}$). Since $\mathcal{F}(C_2^+)$ is not a face component of $\mathcal{P}^{s0+}_{4,4}$ and $P_1 \in C_1^+$, $P_4 \in \mathrm{Cls}(C_3) \cap \mathrm{Cls}(C_4)$, we can omit A_2^+ from the test set. Thus, if $f \in \mathcal{H}^{s0}_{4,4}$ satisfies $f(x, 1, 1, 1) \geq 0$, $f(0, x, 1, 1) \geq 0$ and $f(0, 0, x, 1) \geq 0$ for all $x \geq 0$, then $f \in \mathcal{P}^{s0+}_{4,4}$. ∎

In fact, $\mathcal{F}(C_3)$ is not a face component, and we can omit A_3 from the test set. But it will be proved later. We summarize here what C_1^+, C_3 and C_4 are.

Lemma 3.11.
(1) $\mathrm{Zar}(C_1^+)$ *is a conic defined by* $x_1^2 - 2x_1x_2 - 3x_0x_2 + 3x_2^2 = 0$, $x_2 - x_3 = 0$. *Especially,* $\mathrm{Zar}(C_1^+)$ *is nonsingular. The ends of* C_1^+ *are* P_3 *and* P_5.
(2) $\mathrm{Zar}(C_3)$ *has a cusp at* P_3. *The ends of* C_3 *are* P_4 *and* P_5.
(3) $\mathrm{Zar}(C_4)$ *is a conic defined by* $x_1^2 - 2x_2^2 - x_0x_2 = 0$ *on the plane* $V_{\mathbb{R}}(x_3)$. *The ends of* C_4 *are* P_4 *and* P_5.

Next, we shall study $\mathfrak{f}^{ab}_t$ ($0 \leq t \leq 5$), $\mathfrak{f}^c_t$ ($5 \leq t < \infty$), $\mathfrak{p} = s_2 - s_3$, $\mathfrak{q}_1 = s_1 - 2s_2$, and $\mathfrak{q}_2 = s_3$. Note that

$$\begin{aligned} \mathfrak{f}^{ab}_t &= \frac{1}{3}\big(3s_0 - 2(t+1)s_1 + 2(2t-1)s_2 + (t^2+3)s_3\big), \\ \mathfrak{f}^{c}_t &= \frac{1}{9}\big(9s_0 - 6(t+1)s_1 + (t^2+2t+19)s_2 + 2(t^2+5t-8)s_3\big), \end{aligned}$$

and $\mathfrak{f}^{ab}_5 = \mathfrak{f}^c_5$. Put $\mathfrak{f}^c_\infty := s_2 + 2s_3$. Since $\mathfrak{f}^c_\infty = \mathfrak{p} + 3\mathfrak{q}_2$, $\mathfrak{f}^c_\infty$ is not extremal. The author studied $\Phi^{c0}_{4,4}(t\colon 1\colon 1\colon 1) \in \mathcal{F}(C_1^+)$, dividing three cases (a) $0 \leq t < 1$, (b) $1 < t \leq 5$ and (c) $t > 5$. The symbol $\mathfrak{f}^{ab}_t$ stands for cases (a) and (b). For $u \geq 0$, let

$$\mathfrak{h}^c_u := 3u^2s_0 - 6u(u^2+1)s_1 + 3(u^4+4u^2+1)s_2 + 2(3u^4+3u^3+2u^2+3u+3)s_3.$$

If $t = (3u^2 - u + 3)/u$, then $\mathfrak{h}^c_u = 3u^2\mathfrak{f}^c_t$. So, $\mathfrak{h}^c_u$ is not a new polynomial, but it is convenient to study $\mathcal{F}(C_4)$ for the property $\mathfrak{h}^c_u(0, 0, u, 1) = 0$.

We shall denote the local cone of $\mathcal{P}^{c0+}_{4,4}$ at the point $(t\colon 1\colon 1\colon 1) \in \mathbb{P}^3_+$ by $\mathcal{L}^{C_1}_t$, and the local cone at the point $(0\colon 0\colon t\colon 1)$ by $\mathcal{L}^{C_4}_t$.

Lemma 3.12. $\mathfrak{f}^{ab}_t$ ($0 \leq t \leq 5$), $\mathfrak{f}^c_t$ ($5 < t < \infty$), $\mathfrak{p}$, $\mathfrak{q}_1$, *and* $\mathfrak{q}_2$ *are extremal elements of* $\mathcal{P}^{s0+}_{4,4}$. *These are characterized as follows:*

(1) *Let* $0 \le t < 1$ *or* $1 < t \le 5$. *If* $f \in \mathcal{P}^{s0+}_{4,4}$ *satisfies* $f(t,\ 1,\ 1,\ 1) = 0$ *and* $f(0,\ 0,\ 1,\ 1) = 0$, *then there exists* $\alpha \in \mathbb{R}_+$ *such that* $f = \alpha \mathfrak{f}^{ab}_t$.
(2) *If* $f \in \mathcal{P}^{s0}_{4,4}$ *satisfies* $f_{aa}(1,\ 1,\ 1,\ 1) = 0$ *and* $f(x,\ x,\ 1,\ 1) = 0$ *for all* $x \ge 0$, *then there exists* $\alpha \in \mathbb{R}_+$ *such that* $f = \alpha \mathfrak{f}^{ab}_1$.
(3) *Assume that* $t,\ u \in \mathbb{R}_+$ *satisfy* $3u^2 - (t+1)u + 3 = 0$. *If* $f \in \mathcal{P}^{s0}_{4,4}$ *satisfies* $f(t,\ 1,\ 1,\ 1) = 0$ *and* $f(0,\ 0,\ u,\ 1) = 0$, *then there exists* $\alpha \in \mathbb{R}_+$ *such that* $f = \alpha \mathfrak{f}^{c}_t$.
(4) *If* $f \in \mathcal{P}^{s0}_{4,4+}$ *satisfies* $f(0,\ 0,\ 0,\ 1) = 0$, $f_a(0,\ 0,\ 0,\ 1) = 0$ *and* $f(x,\ 1,\ 1,\ 1) = 0$ *for all* $x \ge 0$, *then there exists* $\alpha \in \mathbb{R}_+$ *such that* $f = \alpha \mathfrak{p}$.
(5) *If* $f \in \mathcal{P}^{s0}_{4,4}$ *satisfies* $f(0,\ 1,\ 1,\ 1) = 0$, $f(0,\ 0,\ 1,\ 1) = 0$ *and* $f(0,\ 0,\ 0,\ 1) = 0$, *then there exists* $\alpha \in \mathbb{R}_+$ *such that* $f = \alpha \mathfrak{q}_1$.
(6) *If* $f \in \mathcal{P}^{s0}_{4,4}$ *satisfies* $f(0,\ 0,\ x,\ y) = 0$ *for all* $x,\ y \in \mathbb{R}_+$, *then there exists* $\alpha \in \mathbb{R}_+$ *such that* $f = \alpha \mathfrak{q}_2$.

Proof. We shall show that $\mathfrak{f}^{ab}_t$ $(0 \le 1 \le 5)$, $\mathfrak{f}^{c}_t$ $(t \ge 5)$, $\mathfrak{p}$, $\mathfrak{q}_1$ and $\mathfrak{q}_2$ belong to $\mathcal{P}^{s0+}_{4,4}$. Since

$$\mathfrak{f}^{ab}_t(0, x, 1, 1) = \frac{1}{3}x(x+2)\left(\left(t - \frac{2(x-1)^2}{(x+2)}\right)^2 + \frac{x(16-x)(x-1)^2}{(x+2)^2}\right),$$

we have $\mathfrak{f}^{ab}_t(0,\ x,\ 1,\ 1) \ge 0$ if $x \le 16$. On the other hand

$$\begin{aligned}\mathfrak{f}^{ab}_t(0, x, 1, 1) = \frac{1}{3}x\Big(&18(25-t)^2 + \big(t^2 + 120(5-t) + 1575\big)(x-16) \\ &+ \big(4(5-t) + 120\big)(x-16)^2 + 3(x-16)^3\Big),\end{aligned}$$

we have $\mathfrak{f}^{ab}_t(0,\ x,\ 1,\ 1) \ge 0$ for $x \ge 16$. Similarly,

$$\mathfrak{f}^{ab}_t(x, 1, 1, 1) = (x-t)^2(x-1)^2 \ge 0,$$
$$\mathfrak{f}^{ab}_t(0, 0, x, 1) = \frac{1}{3}(x-1)^2\left(3\left(x - \frac{t-2}{3}\right)^2 + \frac{1}{3}(5-t)(1+t)\right) \ge 0,$$
$$\mathfrak{f}^{c}_t(x, 1, 1, 1) = (x-t)^2(x-1)^2 \ge 0,$$
$$\mathfrak{f}^{c}_t(0, x, 1, 1) = \frac{1}{9}(2x+1)^2\left(\left(t - \frac{(x-1)^2(6x+5)}{(2x+1)^2}\right)^2 + \frac{24x(x-1)^2(x+2)(3x+2)}{(2x+1)^4}\right) \ge 0,$$
$$\mathfrak{f}^{c}_t(0, 0, x, 1) = \frac{1}{9}(3x^2 - (t+1)x + 3)^2 \ge 0,$$
$$\mathfrak{h}^{c}_u(0, 0, x, 1) = 3(x-u)^2(ux-1)^2 \ge 0,$$
$$\mathfrak{q}_1(x, 1, 1, 1) = 3x(x-1)^2 \ge 0,$$
$$\mathfrak{q}_1(0, x, 1, 1) = 2x(x-1)^2 \ge 0,$$
$$\mathfrak{q}_1(0, 0, x, 1) = x(x-1)^2 \ge 0,$$
$$\mathfrak{q}_2(x, 1, 1, 1) = 3(x-1)^2 \ge 0,$$
$$\mathfrak{q}_2(0, x, 1, 1) = x(x+2) \ge 0,$$
$$\mathfrak{q}_2(0, 0, x, 1) = 0.$$

Thus $\mathfrak{f}^{ab}_t$, $\mathfrak{f}^{c}_t$, $\mathfrak{q}_1$, $\mathfrak{q}_2 \in \mathcal{P}^{s0+}_{4,4}$.

The left part can be proved similarly as the proof of Lemmma 3.6.

(1) Consider a system of equations $f(t,1,1,1)=0$, $f_a(t,1,1,1)=0$, $f(0,0,1,1)=0$ instead of $(*)$ in Lemmma 3.6. Then $A\mathbf{p}=\mathbf{0}$ become

$$\begin{pmatrix} (t-1)^2(t^2+2t+3) & 3(t-1)^2(t+2) & 3(t-1)^2 & 3(t-1)^2 \\ 4(t^3-1) & 9(t^2-1) & 6(t-1) & 6(t-1) \\ 2 & 2 & 1 & 0 \end{pmatrix} \begin{pmatrix} p_0 \\ p_1 \\ p_2 \\ p_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

Using Mathematica, we can check $\operatorname{Ker} A = \mathbb{R}\cdot \mathfrak{f}_t^{ab}$ if $t \neq 1$. $f_a(t,1,1,1)=0$ follows from $f(t,1,1,1)=0$ if $f \in \boldsymbol{\mathcal{P}}_{4,4}^{s0+}$.

(2) Consider $f_{aaa}(1,1,1,1)=0$, $f(0,0,1,1)=0$, $f_a(0,0,1,1)=0$.

(3) This case is slightly complicated. Let $t=(3u^2-u+3)/u$ and consider the system of equations $f\big((3u^2-u+3)/u,1,1,1\big)=0$, $f_a\big((3u^2-u+3)/u,1,1,1\big)=0$, $f(0,0,u,1)=0$. Then $A\mathbf{p}=\mathbf{0}$ become

$$\begin{pmatrix} (t-1)^2(t^2+2t+3) & 3(t-1)^2(t+2) & 3(t-1)^2 & 3(t-1)^2 \\ 4(t^3-1) & 9(t^2-1) & 6(t-1) & 6(t-1) \\ u^4+1 & u(u^2+1) & u^2 & 0 \end{pmatrix} \begin{pmatrix} p_0 \\ p_1 \\ p_2 \\ p_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

Using Mathematica, we can check $\operatorname{Ker} A = \mathbb{R}\cdot \mathfrak{f}_t^{c}$.

(4) Same with (5) of Lemmma 3.6.

(5) Consider $f(0,1,1,1)=0$, $f(0,0,1,1)=0$, $f(0,0,0,1)=0$.

(6) Consider $f(0,0,0,1)=0$, $f(0,0,1,1)=0$, $f(0,0,1,2)=0$.

Each A of the cases (2), (5), (6) are as follows:

$$(2)\ A=\begin{pmatrix} 24 & 18 & 0 & 0 \\ 2 & 2 & 1 & 0 \\ 0 & 2 & 0 & 2 \end{pmatrix}, \quad (5)\ A=\begin{pmatrix} 3 & 6 & 3 & 3 \\ 2 & 2 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \quad (6)\ A=\begin{pmatrix} 1 & 0 & 0 & 0 \\ 2 & 2 & 1 & 0 \\ 17 & 10 & 4 & 0 \end{pmatrix}.$$

□

Lemma 3.13.

(1) $\mathfrak{f}_t^{ab} \in \boldsymbol{\mathcal{F}}(C_1)\cap\boldsymbol{\mathcal{F}}(P_4)$ *and* $\boldsymbol{\mathcal{L}}_t^{C_1} = \mathbb{R}_+\cdot\mathfrak{f}_t^{ab} + \mathbb{R}_+\cdot\mathfrak{p}$ *for* $0<t<1$ *or* $1<t\le 5$.

(2) $\mathfrak{f}_t^{c} \in \boldsymbol{\mathcal{F}}(C_1)\cap\boldsymbol{\mathcal{F}}(C_4)$ *and* $\boldsymbol{\mathcal{L}}_t^{C_1} = \mathbb{R}_+\cdot\mathfrak{f}_t^{c} + \mathbb{R}_+\cdot\mathfrak{p}$ *for* $t>5$. *Moreover,* $\boldsymbol{\mathcal{L}}_u^{C_4} = \mathbb{R}_+\cdot\mathfrak{h}_u^{c} + \mathbb{R}_+\cdot\mathfrak{q}_2$ *for* $u\ge 0$ *with* $t=(3u^2-u+3)/u$.

(3) $\mathfrak{f}_0^{ab} \in \boldsymbol{\mathcal{F}}(C_1)\cap\boldsymbol{\mathcal{F}}(P_3)\cap\boldsymbol{\mathcal{F}}(P_4)$.

(4) $\mathfrak{f}_5 := \mathfrak{f}_5^{ab} = \mathfrak{f}_5^{c} \in \boldsymbol{\mathcal{F}}(C_1)\cap\boldsymbol{\mathcal{F}}(C_4)\cap\boldsymbol{\mathcal{F}}(P_4)$.

(5) $\mathfrak{f}_\infty^{c} \in \boldsymbol{\mathcal{F}}(C_1)\cap\boldsymbol{\mathcal{F}}(C_4)\cap\boldsymbol{\mathcal{F}}(P_5)$.

(6) $\mathfrak{p} \in \boldsymbol{\mathcal{F}}(C_1)\cap\boldsymbol{\mathcal{F}}(P_3)\cap\boldsymbol{\mathcal{F}}(P_5)$.

(7) $\mathfrak{q}_1 \in \boldsymbol{\mathcal{F}}(P_3)\cap\boldsymbol{\mathcal{F}}(P_4)\cap\boldsymbol{\mathcal{F}}(P_5)$.

(8) $\mathfrak{q}_2 \in \boldsymbol{\mathcal{F}}(C_4)\cap\boldsymbol{\mathcal{F}}(P_4)\cap\boldsymbol{\mathcal{F}}(P_5)$.

Proof. If $\boldsymbol{\mathcal{F}}(D)$ ($D \in \Delta(X_{4,4}^{c0+})$) is a face component of $\boldsymbol{\mathcal{P}}_{4,4}^{c0+}$, then $\dim\boldsymbol{\mathcal{F}}(D) = \dim(\partial\boldsymbol{\mathcal{P}}_{4,4}^{c0+}) = \dim\boldsymbol{\mathcal{P}}_{4,4}^{c0+} - 1 = 3$. So, if D_1, D_2, D_3 are distinct elements of $\Delta(X_{4,4}^{c0+})$, and $\boldsymbol{\mathcal{F}}(D_i)$ ($i=1$, 2, 3) are face components, then $\dim\big(\boldsymbol{\mathcal{F}}(D_1)\cap\boldsymbol{\mathcal{F}}(D_2)\big)=2$ and $\dim\big(\boldsymbol{\mathcal{F}}(D_1)\cap\boldsymbol{\mathcal{F}}(D_2)\cap\boldsymbol{\mathcal{F}}(D_3)\big)=1$.

Now, we shall prove (1)—(8).

(1) Assume that $0\le t<1$ or $1<t\le 5$. By previous lemma, we have $\mathfrak{f}_t^{ab} \in \boldsymbol{\mathcal{L}}_t^{C_1}\cap\boldsymbol{\mathcal{F}}(P_4)$ for $0\le t\le 5$. Since $\dim\boldsymbol{\mathcal{L}}_t^{C_1}=2$, we have $\boldsymbol{\mathcal{L}}_t^{C_1} = \mathbb{R}_+\cdot\mathfrak{f}_t^{ab} + \mathbb{R}_+\cdot\mathfrak{p} \subset \boldsymbol{\mathcal{F}}(C_1)$,

(2) Let $u > 0$ and $t = (3u^2 - u + 3)/u \geq 5$. By previous lemma, $\mathfrak{f}_t^c \in \boldsymbol{\mathcal{F}}(C_1) \cap \boldsymbol{\mathcal{F}}(C_4)$. Since $\dim \boldsymbol{\mathcal{L}}_t^{C_4} = 2$, $\boldsymbol{\mathcal{L}}_u^{C_4} = \mathbb{R}_+ \cdot \mathfrak{h}_u^c + \mathbb{R}_+ \cdot \mathfrak{q}_2$. As (1), we have $\boldsymbol{\mathcal{L}}_u^{C_1} = \mathbb{R}_+ \cdot \mathfrak{f}_t^c + \mathbb{R}_+ \cdot \mathfrak{p}$.

(3)—(8) can be proved similarly. ∎

Note that $\mathfrak{f}_1^{ab} \in \boldsymbol{\mathcal{F}}(C_2^+)$, because $\mathfrak{f}_1^{ab}(x, x, 1, 1) = 0$ for all $x \in \mathbb{R}$. By Lemmma 3.6(2), we have $\boldsymbol{\mathcal{F}}(C_2^+) = \mathbb{R}_+ \cdot \mathfrak{f}_1^{ab}$. This also implies that $\boldsymbol{\mathcal{F}}(C_2^+)$ is not a face component.

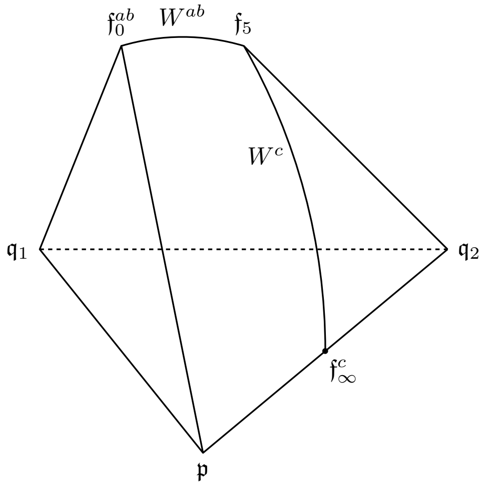
Using the above lemma, we shall determine the structure of the face components $\boldsymbol{\mathcal{F}}(C_1^+)$, $\boldsymbol{\mathcal{F}}(C_4)$, $\boldsymbol{\mathcal{F}}(P_3)$, $\boldsymbol{\mathcal{F}}(P_4)$ and $\boldsymbol{\mathcal{F}}(P_5)$.

Lemma 3.14. *For $f, g \in \boldsymbol{\mathcal{H}}_{4,4}^{s0}$, let $\mathrm{Fan}(f, g) := \mathbb{R}_+ \cdot f + \mathbb{R}_+ \cdot g$ be the fan whose edges are f and g. Put*

$$W^{ab} := \mathbb{R}_+ \cdot \left\{\mathfrak{f}_t^{ab} \;\middle|\; 0 \leq t \leq 5\right\} \subset \boldsymbol{\mathcal{H}}_{4,4}^{s0}, \quad W^c := \mathbb{R}_+ \cdot \left\{\mathfrak{f}_t^c \;\middle|\; t \geq 5\right\} \cup \mathbb{R}_+ \cdot \mathfrak{f}_\infty^c.$$

Then the following hold.

(1) $\partial\boldsymbol{\mathcal{F}}(C_1^+) = W^{ab} \cup W^c \cup \mathrm{Fan}(\mathfrak{f}_\infty^c, \mathfrak{p}) \cup \mathrm{Fan}(\mathfrak{p}, \mathfrak{f}_0^{ab})$.
(2) $\partial\boldsymbol{\mathcal{F}}(C_4) = W^c \cup \mathrm{Fan}(\mathfrak{f}_5, \mathfrak{q}_2) \cup \mathrm{Fan}(\mathfrak{q}_2, \mathfrak{f}_\infty^c)$.
(3) $\partial\boldsymbol{\mathcal{F}}(P_3) = \mathrm{Fan}(\mathfrak{f}_0^{ab}, \mathfrak{q}_1) \cup \mathrm{Fan}(\mathfrak{q}_1, \mathfrak{p}) \cup \mathrm{Fan}(\mathfrak{p}, \mathfrak{f}_0^{ab})$. *That is, $\boldsymbol{\mathcal{F}}(P_3)$ is a triangle cone with edges $\mathfrak{f}_0^{ab}$, $\mathfrak{q}_1$ and $\mathfrak{p}$.*
(4) $\partial\boldsymbol{\mathcal{F}}(P_4) = W^{ab} \cup \mathrm{Fan}(\mathfrak{f}_5, \mathfrak{q}_2) \cup \mathrm{Fan}(\mathfrak{q}_2, \mathfrak{q}_1) \cup \mathrm{Fan}(\mathfrak{q}_1, \mathfrak{f}_0^{ab})$.
(5) *$\boldsymbol{\mathcal{F}}(P_5)$ is a triangle cone with edges $\mathfrak{p}$, $\mathfrak{q}_1$ and $\mathfrak{q}_2$. Note that $\mathfrak{f}_\infty^c \in \mathrm{Fan}(\mathfrak{p}, \mathfrak{q}_2)$, and $\mathrm{Fan}(\mathfrak{p}, \mathfrak{f}_\infty^c) = \boldsymbol{\mathcal{F}}(P_5) \cap \boldsymbol{\mathcal{F}}(C_1)$, $\mathrm{Fan}(\mathfrak{f}_\infty^c, \mathfrak{q}_2) = \boldsymbol{\mathcal{F}}(P_5) \cap \boldsymbol{\mathcal{F}}(C_4)$.*



$\boldsymbol{\mathcal{F}}(C_1) = \langle \mathfrak{f}_0^{ab}, \mathfrak{f}_5, \mathfrak{f}_\infty^c, \mathfrak{p} \rangle$

$\boldsymbol{\mathcal{F}}(C_4) = \langle \mathfrak{f}_5, \mathfrak{f}_\infty^c, \mathfrak{q}_2 \rangle$

$\boldsymbol{\mathcal{F}}(P_3) = \langle \mathfrak{f}_0^{ab}, \mathfrak{q}_1, \mathfrak{p} \rangle$

$\boldsymbol{\mathcal{F}}(P_4) = \langle \mathfrak{f}_0^{ab}, \mathfrak{f}_5, \mathfrak{q}_2, \mathfrak{q}_1 \rangle$

$\boldsymbol{\mathcal{F}}(P_5) = \langle \mathfrak{p}, \mathfrak{q}_1, \mathfrak{q}_2 \rangle$

Fig.3.1. $\boldsymbol{\mathcal{P}}_{4,4}^{s0+}$

By the above lemma, we know that $\partial\boldsymbol{\mathcal{P}}_{4,4}^{s0+}$ is enclosed by $\boldsymbol{\mathcal{F}}(C_1^+)$, $\boldsymbol{\mathcal{F}}(C_4)$, $\boldsymbol{\mathcal{F}}(P_3)$, $\boldsymbol{\mathcal{F}}(P_4)$ and $\boldsymbol{\mathcal{F}}(P_5)$. We don't need $\boldsymbol{\mathcal{F}}(C_3)$. See Fig.3.1. Thus, we have:

Lemma 3.15. *$\partial\boldsymbol{\mathcal{P}}_{4,4}^{s0+} = \boldsymbol{\mathcal{F}}(C_1^+) \cup \boldsymbol{\mathcal{F}}(C_4) \cup \boldsymbol{\mathcal{F}}(P_3) \cup \boldsymbol{\mathcal{F}}(P_4) \cup \boldsymbol{\mathcal{F}}(P_5)$, and $\boldsymbol{\mathcal{E}}(X_{4,4}^{s0+}) \subset C_1^+ \cup C_4 \cup \{P_3, P_4, P_5\}$. Especially, $\boldsymbol{\mathcal{F}}(C_3)$ is not a face component of $\boldsymbol{\mathcal{P}}_{4,4}^{s0+}$.*

Proof of Theorem 1.9(2). Put $\Omega_+ := A_1^+ \cup A_4$. By Theorem 2.10, it is enough to show that $\Phi_{4,4}^{s0}(\Omega_+) \supset C_1^+ \cup C_4 \cup \{P_3, P_4, P_5\}$. But this is clear. ∎

Geometrically, $C_3 - \{P_3,\ P_4,\ P_5\}$ is included in the interior of the convex closure of $X^{s0+}_{4,4}$. So, any $f \in \mathfrak{P}^{s0+}_{4,4}$ cannot satisfy $f(0,\,x,\,1,\,1) = 0$ for $x > 0$, $x \neq 1$.

Theorem 1.4 is also proved from the above results.

Finally, we shall study discriminants $\mathrm{disc}_D = \mathrm{disc}(D)$ for $D = C_1^+,\ C_4,\ P_3,\ P_4$ and P_5. We use $(p_0,\,p_1,\,p_2,\,p_3)$ as a coordinate system of $\mathfrak{H}^{s0}_{4,4}$ as before. $(p_0,\,p_1,\,p_2,\,p_3)$ corresponds to $\sum_{i=0}^{3} p_i s_i \in \mathfrak{H}^{s0}_{4,4}$.

Lemma 3.16.

$$\begin{aligned}
\mathrm{disc}(C_1^+) &= 8p_0^2 - 9p_1^2 + 12p_0p_1 + 12p_0p_2 + 12p_0p_3,\\
\mathrm{disc}(C_4) &= -8p_0^2 - p_1^2 + 4p_0p_2,\\
\mathrm{disc}(P_3) &= p_0 + 2p_1 + p_2 + p_3,\\
\mathrm{disc}(P_4) &= 2p_0 + 2p_1 + p_2,\\
\mathrm{disc}(P_5) &= p_0.
\end{aligned}$$

Proof. $\mathrm{disc}(C_1^+) = \mathrm{disc}(C_1)$, since $\mathrm{Zar}(C_1^+) = \mathrm{Zar}(C_1)$.

If $P = (c_0 \colon c_1 \colon c_2 \colon c_3) \in \Delta^0(\mathfrak{P}^{s0+}_{4,4})$, then $\mathrm{disc}(P) = \sum_{i=1}^{3} c_i p_i$. Thus we have $\mathrm{disc}(P_i)$ $(i = 3,\,4,\,5)$.

We shall study $\mathrm{disc}(C_4)$. Take $f = (1/3u^3)\mathfrak{h}^c_u + v\mathfrak{q}_2 \in \mathfrak{F}(C_4)$ $(u > 0,\ v \geq 0)$. The coefficients of f are $p_1/p_0 = -2(u^2+1)/u$, $p_2/p_0 = (u^4 + 4u^2 + 1)/u^2$, $p_3/p_0 = 2(3u^4 + 3u^3 + 2u^2 + 3u + 3)/(3u^2) + v$. Eliminate u and v from these relations. Then we have $\mathrm{disc}(C_4) = -8p_0^2 - p_1^2 + 4p_0p_2 = 0$. □

Proof of Theorem 3.8. This proof is almost completed. What we should do is only to observe the signature of discriminants. Then, we find that we can use $p + 4$ and $p + 2/3$ as separators to describe $\mathfrak{P}^{c0+}_{4,4}$ as a union of basic semialgebraic sets as (1)—(4) of Theorem 3.8(I). □

Proof of Theorem 1.1(2), 1.4 and 1.8(2). This is same as the proof of Theorem 1.1(1), 1.2 and 1.8(1). □

Proof of Proposition 1.7(2), (3).(2-i) Consider the case $0 \leq t < 1$ or $1 < t \leq 5$. Let $F(x,y,z) := 3\mathfrak{f}^{ab}_t(x,\,y,\,z-x-y,\,-z)$, and $f(x,y) := F(x,y,1)$. If $\mathfrak{f}^{ab}_t$ is reducible, then f is also reducible. Consider the real curve $\Gamma := V_{\mathbb{R}}(F) \subset \mathbb{P}^2_{\mathbb{R}}$. Note that $f(y,\,x) = f(x,\,y)$. Since

$$\begin{aligned}
f(x,0) = f(0,x) &= 8(t+1)(x^2 - x + 1)^2 > 0,\\
F(1,0,z) = F(0,1,z) &= 8(t+1)(z^2 - z + 1)^2 > 0,\\
f(1,1) = f(1,-1) = f(-1,1) &= -16(t-1)^2 < 0,
\end{aligned}$$

Γ has at least three connected components Γ_1, Γ_2, Γ_4 in the 1-st, 2-nd and 4-th quadrant. Γ_1, Γ_2, Γ_4 are all bounded. This implies Γ cannot contain a line. Moreover, Γ cannot be a union of two quadric curves. Thus $V_{\mathbb{C}}(F)$ must be an irreducible curve.

(3) Consider the case $t > 5$. Let $G(x, y, z) := 9\mathfrak{f}_t^c(x,\ y,\ z - x - y,\ -z)$, and $g(x, y) := G(x, y, 1)$. Then,

$$\begin{aligned} g(x,0) &= g(0,x) = (t+7)^2(x^2 - x + 1)^2 > 0, \\ G(1,0,z) &= G(0,1,z) = (t+7)^2(z^2 - z + 1)^2 > 0, \\ g(1,1) &= g(1,-1) = g(-1,1) = -32(t^2 + 2t - 11) < 0. \end{aligned}$$

Thus $V_{\mathbb{C}}(G)$ must be an irreducible curve.

(2-ii) Consider the case $t = 1$. Assume that $\mathfrak{f}_1^{ab}$ is reducible. Since

$$\mathfrak{f}_1^{ab}(x, y, 1, 1) = \frac{1}{3}(x - y)^2(3x^2 + 2xy + 3y^2 - 8x - 8y + 8),$$

$\mathfrak{f}_1^{ab}$ must be product of two real quadrics. But this is impossible. since $(\mathfrak{f}_1^{ab})_{aa}(1, 1, 1, 1) = 0$. □

Proof of Proposition 1.5. For $f_t = \mathfrak{f}_t^{ab}$ ($0 < t < 1$ or $1 < t \le 5$) or $f_t = \mathfrak{f}_t^c$ ($t > 5$), let $F_t(a, b, c, d) := f_t(a^2,\ b^2,\ c^2,\ d^2)$, and consider the zero point set $Z_t := V_{\mathbb{R}}(F_t) \subset \mathbb{P}^3_{\mathbb{R}}$.

Let u be a positive root of $t = (3u^2 - u + 3)/u$ if $t > 5$, and $u := 1$ if $0 < t \le 5$. Remember that $f_t(1, 1, 1, 1) = f_t(t, 1, 1, 1) = f_t(0, 0, u, 1) = 0$. Let $s := \sqrt{t}$ and $v := \sqrt{u}$. Then $F_t(\pm 1,\ \pm 1,\ \pm 1,\ 1) = F_t(\pm s,\ \pm 1,\ \pm 1,\ 1) = F_t(0,\ 0,\ \pm v,\ 1) = 0$. Thus, if $0 < t < 1$ or $1 < t \le 5$, then $\#Z_t = 52$. If $t > 5$, then $\#Z_t = 64$.

Assume that $F_t \in \Sigma_{4,8}$. Then, there exists $r \in \mathbb{N}$ and $g_1, \dots, g_r \in \mathcal{H}_{4,4}$ such that $F_t = g_1^2 + \cdots + g_r^2$. If $\mathbf{a} \in Z_t$, then $g_1(\mathbf{a}) = \cdots = g_r(\mathbf{a}) = 0$, since $F_t(\mathbf{x}) \ge 0$ for all $\mathbf{x} \in \mathbb{P}^3_{\mathbb{R}}$. Note that $\dim \mathcal{H}_{4,4} = 35$. So, let's find 35 points $\mathbf{a}_i \in Z_t$ ($1 \le i \le 35$) such that there exists no $g \in \mathcal{H}_{4,4} - \{0\}$ which satisfy $g(\mathbf{a}_i) = 0$ for all $1 \le i \le 35$.

Let $\mathbf{a}_1 := (1{:}\,1{:}\,-1{:}\,-1)$, $\mathbf{a}_2 := (1{:}\,1{:}\,1{:}\,s)$, $\mathbf{a}_3 := (-s{:}\,1{:}\,1{:}\,1)$, $\mathbf{a}_4 := (1{:}\,-s{:}\,1{:}\,1)$, $\mathbf{a}_5 := (1{:}\,1{:}\,-s{:}\,1)$, $\mathbf{a}_6 := (1{:}\,1{:}\,1{:}\,-s)$, $\mathbf{a}_7 := (s{:}\,-1{:}\,1{:}\,1)$, $\mathbf{a}_8 := (s{:}\,1{:}\,-1{:}\,1)$, $\mathbf{a}_9 := (s{:}\,1{:}\,1{:}\,-1)$, $\mathbf{a}_{10} := (-1{:}\,s{:}\,1{:}\,1)$, $\mathbf{a}_{11} := (1{:}\,s{:}\,-1{:}\,1)$, $\mathbf{a}_{12} := (1{:}\,s{:}\,1{:}\,-1)$, $\mathbf{a}_{13} := (-1{:}\,1{:}\,s{:}\,1)$, $\mathbf{a}_{14} := (1{:}\,-1{:}\,s{:}\,1)$, $\mathbf{a}_{15} := (1{:}\,1{:}\,s{:}\,-1)$, $\mathbf{a}_{16} := (-1{:}\,1{:}\,1{:}\,s)$, $\mathbf{a}_{17} := (1{:}\,-1{:}\,1{:}\,s)$, $\mathbf{a}_{18} := (1{:}\,1{:}\,-1{:}\,s)$, $\mathbf{a}_{19} := (s{:}\,1{:}\,-1{:}\,-1)$, $\mathbf{a}_{20} := (s{:}\,-1{:}\,1{:}\,-1)$, $\mathbf{a}_{21} := (s{:}\,-1{:}\,-1{:}\,1)$, $\mathbf{a}_{22} := (-1{:}\,s{:}\,1{:}\,-1)$, $\mathbf{a}_{23} := (-1{:}\,s{:}\,-1{:}\,1)$, $\mathbf{a}_{24} := (1{:}\,s{:}\,-1{:}\,-1)$, $\mathbf{a}_{25} := (-1{:}\,-1{:}\,s{:}\,1)$, $\mathbf{a}_{26} := (1{:}\,-1{:}\,s{:}\,-1)$, $\mathbf{a}_{27} := (-1{:}\,1{:}\,s{:}\,-1)$, $\mathbf{a}_{28} := (1{:}\,-1{:}\,-1{:}\,s)$, $\mathbf{a}_{29} := (-1{:}\,1{:}\,-1{:}\,s)$, $\mathbf{a}_{30} := (-1{:}\,-1{:}\,1{:}\,s)$, $\mathbf{a}_{31} := (v{:}\,1{:}\,0{:}\,0)$, $\mathbf{a}_{32} := (v{:}\,0{:}\,-1{:}\,0)$, $\mathbf{a}_{33} := (v{:}\,0{:}\,0{:}\,-1)$, $\mathbf{a}_{34} := (0{:}\,v{:}\,1{:}\,0)$, $\mathbf{a}_{35} := (0{:}\,v{:}\,0{:}\,1)$. Take 35 monomials $e_1, \dots, e_{35}$ as a basis of $\mathcal{H}_{4,4}$, and denote $g = c_1e_1 + \cdots + c_{35}e_{35} \in \mathcal{H}_{4,4}$. Let $A = (a_{i,j})$ be 35×35-matrix such that $a_{i,j} = e_j(\mathbf{a}_i)$. Then

$$\begin{aligned} \det A = \pm 549755813888\, & t^{13/2}(t-1)^{23}(t+3)^6 \\ & \times u^3(1 + t - 2u)(tu + u - 2)(3u^2 - ut - u - 1). \end{aligned}$$

Note that $3u^2 - ut - u - 1 = (3u^2 - ut - u + 3) - 4 = -4 \ne 0$, $tu + u - 2 = 3u^2 + 1 > 0$ and $u > 0$. There exist no real solutions $1 + t - 2u = 0$, $t = (3u^2 - u + 3)/u$. Thus $\det A \ne 0$ if $t > 0$ and $t \ne 1$. This implies there exists no $g \in \mathcal{H}_{4,4} - \{0\}$ which satisfy $g(\mathbf{a}_i) = 0$ for all $1 \le i \le 35$. □

Proof of Proposition 1.6. Let $t > 5$. We shall show that $\mathfrak{f}_t^c \in \mathcal{E}(\mathcal{P}_{4,4}^+)$. This is equivalent to $\mathfrak{h}_u^c \in \mathcal{E}(\mathcal{P}_{4,4}^+)$ for all $u > 0$.

Let $e_1,\ldots, e_{35}$ be all the monomials in $\mathcal{H}_{4,4}$, and denote $f \in \mathcal{H}_{4,4}$ as $f = \sum_{i=1}^{35} c_i e_i$ ($c_i \in \mathbb{R}$). Let $t := (3u^2 - u + 3)/u$. Let $\mathcal{K}$ be the subspace of all the $f \in \mathcal{H}_{4,4}$ which satisfies the following 34 equalities:

$$
\begin{array}{llll}
f_a(1,1,1,1) = 0, & f_b(1,1,1,1) = 0, & f(t,1,1,1) = 0, & f_a(t,1,1,1) = 0, \\
f_b(t,1,1,1) = 0, & f(1,t,1,1) = 0, & f_a(1,t,1,1) = 0, & f_b(1,t,1,1) = 0, \\
f_c(1,t,1,1) = 0, & f(1,1,t,1) = 0, & f_a(1,1,t,1) = 0, & f_b(1,1,t,1) = 0, \\
f_c(1,1,t,1) = 0, & f(1,1,1,t) = 0, & f_a(1,1,1,t) = 0, & f_b(1,1,1,t) = 0, \\
f_c(1,1,1,t) = 0, & f(0,0,u,1) = 0, & f_c(0,0,u,1) = 0, & f(0,u,0,1) = 0, \\
f_b(0,u,0,1) = 0, & f(0,u,1,0) = 0, & f_b(0,u,1,0) = 0, & f(u,0,0,1) = 0, \\
f_a(u,0,0,1) = 0, & f(u,0,1,0) = 0, & f(u,1,0,0) = 0, & f_a(u,1,0,0) = 0, \\
f(0,0,1,u) = 0, & f(0,1,0,u) = 0, & f(0,1,u,0) = 0, & f(1,0,0,u) = 0, \\
f(1,0,u,0) = 0, & f(1,u,0,0) = 0. & &
\end{array}
$$

The system of these equation can be written as $A\mathbf{c} = \mathbf{0}$ by a certain 34×35-matrix A, i.e. $\mathcal{K} = \operatorname{Ker} A$. Add the vector $(1, 0,\ldots, 0)$ to the bottom of A, and make 35×35-matrix B. Then

$$\det B = \pm t(t+3)(t-1)^{25} u^{12} (u^2 - 1)^{12} (u^2 + 1)^2 (12u^4 + 12u^3 + 21u^2 + 10u + 9) \neq 0.$$

Thus $\dim \operatorname{Ker} A = 1$, and $\operatorname{Ker} A = \mathbb{R} \cdot \mathfrak{h}_u^c$. This implies $\mathfrak{h}_u^c \in \mathcal{E}(\mathcal{P}_{4,4}^+)$. $\square$

It seems that $\mathfrak{f}_t^{ab} \notin \mathcal{E}(\mathcal{P}_{4,4}^+)$ for $t < 5$. But the author does not have proof.

Section 4. Cubic Inequalities of Four Variables

4.1. Structure of $\mathcal{P}_{4,3}^{c0+}$

In this section, we shall study $\mathcal{P}_{4,3}^{c0+} := \mathcal{P}(\mathbb{P}_+^3, \mathcal{H}_{4,3}^{c0})$. We use similar symbols with §3. To state the main theorem of this section we need to fix some symbols. Put

$$S_3 := \sum_{i=0}^{3} a_i^3, \quad S_{2,1,0} := \sum_{i=0}^{3} a_i^2 a_{i+1}, \quad S_{2,0,1} := \sum_{i=0}^{3} a_i^2 a_{i+2},$$
$$S_{1,2,0} := \sum_{i=0}^{3} a_i^2 a_{i+3}, \quad S_{1,1,1} := \sum_{i=0}^{3} a_i a_{i+1} a_{i+2},$$

here we regard $a_{i+4} = a_i$ for all $i \in \mathbb{Z}$. We choose $s_0 := S_3 - S_{1,1,1}$, $s_1 := S_{2,1,0} - S_{1,1,1}$, $s_2 := S_{2,0,1} - S_{1,1,1}$, $s_3 := S_{1,2,0} - S_{1,1,1}$ as a basis of $\mathcal{H}_{4,3}^{c0}$, and define $\Phi_{4,3}^{c0} : \mathbb{P}_+^3 \cdots \to \mathbb{P}_+^3$ by $\Phi_{4,3}^{c0}(\mathbf{a}) = \big(s_0(\mathbf{a}) : s_1(\mathbf{a}) : s_2(\mathbf{a}) : s_3(\mathbf{a})\big)$. The coordinate system of $A = \mathbb{P}_{\mathbb{R}}^3$ is denoted by $(a_0 : a_1 : a_2 : a_3)$ or $(a : b : c : d)$, and the coordinate system of $\mathbb{P}((\mathcal{H}_{4,3}^{c0})^\vee)$ is denoted by $(x_0 : x_1 : x_2 : x_3)$. We represent $f \in \mathcal{H}_{4,3}^{c0}$ as $f = p_0 s_0 + \cdots + p_3 s_3$ ($p_i \in \mathbb{R}$), and the coordinate system of $\mathcal{H}_{4,3}^{c0}$ is denoted by (p_0, p_1, p_2, p_3). If $f \in \mathcal{P}_{4,3}^{c0+}$, then $s_0 \geq 0$. When $p_0 = 1$, we say f is *monic*. When $p_0 = 0$, we say f lies *at infinity*. We denote

$$\check{\mathcal{P}}_{4,3}^{c0+} := \big\{ f \in \mathcal{P}_{4,3}^{c0+} \bigm| f \text{ is monic} \big\}.$$

The characteristic variety is written by $X_{4,3}^{c0+} := \Phi_{4,3}^{c0}(\mathbb{P}_+^3)$. Let

$$A_c^+ := \big\{ (a_0 : a_1 : a_2 : 1) \in \mathbb{P}_+^3 \bigm| 0 \leq a_i \leq 1 \ (i = 0,\, 1,\, 2) \big\},$$
$$E_2 := \big\{ (a : b : a : b) \in \mathbb{P}_+^3 \bigm| a,\, b \in \mathbb{R}_+ \big\},$$
$$E_3 := \big\{ (a_0 : a_1 : a_2 : a_3) \in \mathbb{P}_+^3 \bigm| a_0 + a_2 = a_1 + a_3 \big\},$$

$$
\begin{aligned}
B_0 &:= \left\{(0\colon s\colon t\colon 1) \in \mathbb{P}^3_+ \;\middle|\; s>0,\, t>0,\, (s,t)\neq(1,0),\, t\neq s+1\right\},\\
\overline{B_0} &:= \left\{(0\colon s\colon t\colon u) \in \mathbb{P}^3_+ \;\middle|\; (s\colon t\colon u)\in \mathbb{P}^2_+\right\},\\
S &:= \Phi^{c0}_{4,3}(B_0) \subset X^{c0+}_{4,3},\\
C &:= \left\{\Phi^{c0}_{4,3}(0\colon 0\colon t\colon 1)\in \mathbb{P}^3_+ \;\middle|\; t>0\right\} \subset X^{c0+}_{4,3},\\
L &:= \left\{\Phi^{c0}_{4,3}(0\colon t\colon 0\colon 1)\in \mathbb{P}^3_+ \;\middle|\; t>0\right\} \subset X^{c0+}_{4,3},\\
P_1 &:= (1\colon 0\colon 0\colon 0) = \Phi^{c0}_{4,3}(0\colon 0\colon 0\colon 1) \in X^{c0+}_{4,3},\\
P_2 &:= (1\colon 0\colon 1\colon 0) = \left\{\Phi^{c0}_{4,3}(a\colon b\colon a\colon b)\in \mathbb{P}^3_+ \;\middle|\; a,\, b\in\mathbb{R}_+\right\} \in X^{c0+}_{4,3},\\
P_3 &:= (2\colon 1\colon 0\colon 1) \in X^{c0+}_{4,3}.
\end{aligned}
$$

We denote $\boldsymbol{\mathcal{F}}(P_i)$, $\boldsymbol{\mathcal{F}}(C)$, $\boldsymbol{\mathcal{F}}(S)$ by $\boldsymbol{\mathcal{F}}_{P_i}$, $\boldsymbol{\mathcal{F}}_C$ and $\boldsymbol{\mathcal{F}}_S$. As we will prove in Lemma 4.4, $\boldsymbol{\mathcal{P}}^{c0+}_{4,3} = \boldsymbol{\mathcal{F}}_S \cup \boldsymbol{\mathcal{F}}_C \cup \boldsymbol{\mathcal{F}}_{P_1} \cup \boldsymbol{\mathcal{F}}_{P_2}$, So, we need two discriminants disc_C and disc_S which are defining equations of $\mathrm{Zar}(\boldsymbol{\mathcal{F}}_C)$ and $\mathrm{Zar}(\boldsymbol{\mathcal{F}}_S)$. disc_S is somewhat complicated polynomial.

$$
\begin{aligned}
&\mathrm{disc}_C(p_0,p_1,p_3) := 27p_0^4 + 4p_0p_1^3 + 4p_0p_3^3 - p_1^2p_3^2 - 18p_0^2p_1p_3 = \mathrm{Disc}_3(p_0,p_1,p_3,p_0),\\
&d_S(p_0,p_2,q,r)\\
&:= (p_0-p_2-q)^2(13p_0^2 - 2p_0p_2 + p_2^2 + 2p_0q + 2p_2q)^2\\
&\quad (104p_0^3 + 100p_0^2p_2 - 4p_0p_2^2 + 36p_0^2q + 36p_0p_2q - p_0q^2 - p_2q^2 + 8q^3)\\
&\quad + (17173p_0^7 - 121p_0^6p_2 - 5639p_0^5p_2^2 + 7651p_0^4p_2^3 - 3489p_0^3p_2^4 + 469p_0^2p_2^5\\
&\quad - 45p_0p_2^6 + p_2^7 + 6250p_0^6q + 10028p_0^5p_2q + 3142p_0^4p_2^2q - 1368p_0^3p_2^3q - 746p_0^2p_2^4q\\
&\quad - 20p_0p_2^5q - 6p_2^6q + 898p_0^5q^2 + 7230p_0^4p_2q^2 + 1748p_0^3p_2^2q^2 - 1572p_0^2p_2^3q^2\\
&\quad - 86p_0p_2^4q^2 - 26p_2^5q^2 + 2780p_0^4q^3 - 368p_0^3p_2q^3 + 1448p_0^2p_2^2q^3 - 496p_0p_2^3q^3\\
&\quad + 28p_2^4q^3 + 518p_0^3q^4 + 1018p_0^2p_2q^4 - 190p_0p_2^2q^4 + 78p_2^3q^4 + 164p_0^2q^5\\
&\quad + 168p_0p_2q^5 + 4p_2^2q^5)r^2\\
&\quad + (2495p_0^5 - 317p_0^4p_2 - 1886p_0^3p_2^2 + 842p_0^2p_2^3 - 81p_0p_2^4 + 3p_2^5 + 1768p_0^4q\\
&\quad + 4p_0^3p_2q - 988p_0^2p_2^2q + 380p_0p_2^3q - 12p_2^4q + 291p_0^3q^2 + 897p_0^2p_2q^2 - 463p_0p_2^2q^2\\
&\quad + 83p_2^3q^2 + 226p_0^2q^3 + 92p_0p_2q^3 - 38p_2^2q^3 - p_0q^4 - p_2q^4)r^4\\
&\quad + (95p_0^3 + 65p_0^2p_2 - 43p_0p_2^2 + 3p_2^3 + 98p_0^2q - 20p_0p_2q - 6p_2^2q - 4p_0q^2)r^6\\
&\quad + (-3p_0 + p_2)r^8,\\
&\mathrm{disc}_S(p_0,p_1,p_2,p_3) := \frac{1}{4}d_S(p_0,p_2,p_1+p_3,p_1-p_3).
\end{aligned}
$$

Since $\mathrm{disc}_C(p_0,p_1,p_3)$ has an obstacle branch in the first quadrant $p_1/p_0>0$, $p_3/p_0>0$, we put

$$
d_C(x,z) := \begin{cases} \mathrm{disc}_C(1,x,z) & (\text{if } x<0 \text{ or } z<0)\\ 1 & (\text{if } x\geq 0 \text{ and } z\geq 0)\end{cases}
$$

to avoid complexity. $d_C(x,z)\geq 0$ implies $\mathrm{disc}_C(1,x,z)\geq 0$ or '$x\geq 0$ and $z\geq 0$'. Thus, $d_C(x,z)\geq 0$ defines a convex domain, but $\mathrm{disc}_C(1,x,z)\geq 0$ does not. The following $\eta(x,y)$ is a nice separator whose property is explained in Lemma 4.10.

$$
\begin{aligned}
\eta(x,y) := {}& 61 + 62x + 56y + 32x^2 + 30xy - 6y^2\\
&+ 9x^3 + 4x^2y - 6xy^2 - 16y^3 + x^4 - 4x^2y^2 - 6xy^3 + y^4 - x^3y^2.
\end{aligned}
$$

We also need two constants κ_1, κ_2. Let $\kappa_1 := 0.0129074031\cdots$ be a root of

$$
817808203x^6 - 546807084x^5 + 129155640x^4 - 13342016x^3 + 556080x^2 - 10176x + 64 = 0,
$$

and $\kappa_2 := 0.0318925844\cdots$ be a root of

$$43042537x^6 - 4514514x^5 - 188769x^4 - 38684x^3 + 4119x^2 - 114x + 1 = 0.$$

The aim of this section is to prove the following theorem.

Theorem 4.1. (I) *Take a monic* $f = s_0 + p_1 s_1 + p_2 s_2 + p_3 s_3 \in \breve{\mathfrak{H}}^{c0}_{4,3}$. *Then,* $f(\mathbf{a}) \geq 0$ *for all* $\mathbf{a} \in \mathbb{R}^4_+$, *if and only if one of the following holds:*
(1) $p_2 = -1$ *and* $8(p_1 + p_3) \geq (p_1 - p_3)^2$.
(2) $-1 < p_2 \leq 3$, $\mathrm{disc}_S(1, p_1, p_2, p_3) \geq 0$ *and* $d_C(p_1, p_3) \geq 0$.
(3) $p_2 > 3$, $\kappa_1(p_1 + p_3) + \kappa_2 p_2 \geq 1$, $\mathrm{disc}_S(1, p_1, p_2, p_3) \geq 0$, *and* $d_C(p_1, p_3) \geq 0$.
(4) $p_2 > 3$, $\kappa_1(p_1 + p_3) + \kappa_2 p_2 < 1$, $\eta(p_1 + p_3, p_2) > 0$, $\mathrm{disc}_S(1, p_1, p_2, p_3) \geq 0$, *and* $d_C(p_1, p_3) \geq 0$.
(5) $p_2 > 3$, $\kappa_1(p_1 + p_3) + \kappa_2 p_2 < 1$, $\eta(p_1 + p_3, p_2) \leq 0$, *and* $d_C(p_1, p_3) \geq 0$.

(II) *Let's denote* $f = p_0 s_0 + p_1 s_1 + p_2 s_2 + p_3 s_3$. *Then, all the discriminants of* $\mathfrak{P}^{c0+}_{4,3}$ *are* $\mathrm{disc}_S(p_0, p_1, p_2, p_3)$, $\mathrm{disc}_C(p_0, p_1, p_3)$, $\mathrm{disc}_{P_1} = p_0$, *and* $\mathrm{disc}_{P_2} = p_0 + p_2$.

(III) *If* $f \in \mathfrak{H}^{c0}_{4,3}$ *satisfies* $f(0, s, t, 1) \geq 0$ *for all* $s, t \in \mathbb{R}_+$, *then* $f \in \mathfrak{P}^{c0}_{4,3}$.

This theorem will be proved after Lemma 4.8.

Proposition 4.2.

$$
\begin{aligned}
s_0(a_0, a_1, a_2, a_3) &= \frac{1}{3}\sum_{i=0}^{3}(a_i^3 + a_{i+1}^3 + a_{i+2}^3 - 3a_i a_{i+1} a_{i+2}) \geq 0, \\
s_2(a_0, a_1, a_2, a_3) &= (a_0 - a_1 + a_2 - a_3)(a_0 a_2 - a_1 a_3), \\
s_3(a_0, a_1, a_2, a_3) &= s_1(a_0, a_3, a_2, a_1), \\
s_0 - s_2 &= \frac{1}{3}\sum_{i=0}^{3}(a_i^3 + a_i^3 + a_{i+2}^3 - 3a_i^2 a_{i+2}) \geq 0, \\
s_0 + 2s_2 &= \sum_{i=0}^{3}(a_i^2 a_{i+2} + a_{i+1}^3 + a_i a_{i+2}^2 - 3a_i^2 a_{i+1} a_{i+2}) \geq 0, \\
2s_1 + s_2 &= \sum_{i=0}^{3}(a_i^2 a_{i+1} + a_{i+1}^2 a_{i+2} + a_{i+2}^2 a_i - 3a_i a_{i+1} a_{i+2}) \geq 0, \\
2s_3 + s_2 &= \sum_{i=0}^{3}(a_i a_{i+1}^2 + a_{i+1} a_{i+2}^2 + a_{i+2} a_i^2 - 3a_i a_{i+1} a_{i+2}) \geq 0, \\
s_0 - s_1 &= \frac{1}{3}\sum_{i=0}^{3}(a_i^3 + a_i^3 + a_{i+1}^3 - 3a_i^2 a_{i+1}) \geq 0, \\
s_0 - s_3 &= \frac{1}{3}\sum_{i=0}^{3}(a_i^3 + a_{i+1}^3 + a_{i+1}^3 - 3a_i a_{i+1}^2) \geq 0, \\
s_1 + s_3 &= (a_0 + a_2)(a_1 - a_3)^2 + (a_1 + a_3)(a_0 - a_2)^2 \geq 0.
\end{aligned}
$$

Proof. These follow from direct calculations. ∎

Thus $X_{4,3}^{c0+}$ is a subset of a cube defined by $-1/2 \leq s_1/s_0 \leq 1$, $-1/2 \leq s_2/s_0 \leq 1$, $-1/2 \leq s_3/s_0 \leq 1$. Note that s_1, s_2 and s_3 are not PSD. For example $s_1(1/100, 1/2, 1/10, 1) = -229/20000 < 0$. The rational map $\Phi_{4,3}^{c0} : \mathbb{P}_+^3 \cdots \to X_{4,3}^{c0+}$ splits as

$$\Phi_{4,3}^{c0} : \mathbb{P}_+^3 \xrightarrow{\pi} \mathbb{P}_+^3/(\mathbb{Z}/4\mathbb{Z}) \xrightarrow{\Psi_{4,3}^{c0}} X_{4,3}^{c0+}.$$

It is easy to see that $\Psi_{4,3}^{c0} : \mathbb{P}_+^3/(\mathbb{Z}/4\mathbb{Z}) \cdots \to X_{4,3}^{c0+}$ is a birational map, but is not holomorphic at a singular point $\pi(1\colon 1\colon 1\colon 1)$. We shall provide more precise structure of $X_{4,3}^{c0+}$ at Lemma 4.4. The following $\mathfrak{e}_{s,t}(a_0, a_1, a_2, a_3) \in \boldsymbol{\mathcal{H}}_{4,3}^{c0}$ $(s,\, t \in \mathbb{R})$ has a possibility to be an extremal element. But there exists $(s,\, t)$ such that $\mathfrak{e}_{s,t}$ is not PSD.

Proposition 4.3. *For $(u\colon v\colon w) \in \mathbb{P}_+^2 - \{(1\colon 0\colon 1)\}$, let*

$$g_0^h(u,v,w) := -v\big(uwv^2 - (u+w)(u^2+w^2)v + uw(u-w)^2\big),$$
$$g_1^h(u,v,w) := uv^4 - w(u+2w)v^3 - 2uw(u-w)v^2 - u(2u^3+u^2w-3w^3)v + w(u^2-w^2)^2,$$
$$g_2^h(u,v,w) := v\big(v^4 + (2u^2-3uw+2w^2)v^2 - (u+w)(u^2+w^2)v + (u-w)^2(u^2-uw+w^2)\big),$$
$$g_3^h(u,v,w) := g_1^h(w,v,u),$$
$$\mathfrak{e}_{u,v,w}^h(\mathbf{a}) := \sum_{i=0}^{3} g_i^h(u,v,w)s_i(\mathbf{a}).$$

For simplicity, put $g_i(s,t) := g_i^h(s,t,1)$ and $\mathfrak{e}_{s,t}(\mathbf{a}) := \mathfrak{e}_{s,t,1}^h(\mathbf{a})$. Then the following hold:

(1) $\mathfrak{e}_{w,v,u}^h - \mathfrak{e}_{u,v,w}^h = (u-w)(v^2-(u+w)^2)((u-w)^2+2(u+w)v+v^2)(s_1-s_3).$

(2) $\mathfrak{e}_{t,1,0}^h = t\mathfrak{e}_{0,t,1}^h - (t^2-1)(t^2+1)^2 s_2.$

(3) *Assume that $s > 0$, $t > 0$, $t \neq s+1$, $g_0(s,t) > 0$ and $\mathfrak{e}_{s,t} \in \boldsymbol{\mathcal{P}}_{4,3}^{c0+}$. If $f \in \boldsymbol{\mathcal{P}}_{4,3}^{c0+}$ satisfies $f(0,s,t,1) = 0$, then there exists $\alpha \geq 0$ such that $f = \alpha\mathfrak{e}_{s,t}$. Especially, $\mathfrak{e}_{s,t} \in \boldsymbol{\mathcal{E}}(\boldsymbol{\mathcal{P}}_{4,3}^{c0+})$.*

(4) *Assume that $s = 0$, $t > 0$, $t \neq 1$ and $\mathfrak{e}_{0,t} \in \boldsymbol{\mathcal{P}}_{4,3}^{c0+}$. If $f \in \boldsymbol{\mathcal{P}}_{4,3}^{c0+}$ satisfies $f(0,0,t,1) = 0$ and $\dfrac{\partial}{\partial b}f(0,0,t,1) = 0$, then there exists $\alpha \geq 0$ such that $f = \alpha\mathfrak{e}_{0,t}$. Especially, $\mathfrak{e}_{0,t} \in \boldsymbol{\mathcal{E}}(\boldsymbol{\mathcal{P}}_{4,3}^{c0+})$.*

(5) *Assume that $u > 0$, $v > 0$, and $\mathfrak{e}_{u,v,0}^h \in \boldsymbol{\mathcal{P}}_{4,3}^{c0+}$. If $f \in \boldsymbol{\mathcal{P}}_{4,3}^{c0+}$ satisfies $f(0,u,v,0) = 0$ and $\dfrac{\partial}{\partial d}f(0,u,v,0) = 0$, then there exists $\alpha \geq 0$ such that $f = \alpha\mathfrak{e}_{u,v,0}^h$. Especially, $\mathfrak{e}_{u,v,0}^h \in \boldsymbol{\mathcal{E}}(\boldsymbol{\mathcal{P}}_{4,3}^{c0+})$.*

(6) *If $t = s+1$, then*

$$\begin{aligned}\mathfrak{e}_{s,s+1}(a,b,c,d) &= (s+1)(s^2+1)^2(a-b+c-d)^2(a+b+c+d) \qquad (*)\\ &= (s+1)(s^2+1)^2\mathfrak{e}_{0,1}(a,b,c,d).\end{aligned}$$

If $f \in \boldsymbol{\mathcal{P}}_{4,3}^{c0+}$ satisfies $f(0,0,1,1) = 0$ and $f(0,1,2,1) = 0$, then there exists $\alpha \geq 0$ such that $f = \alpha\mathfrak{e}_{0,1}$. Especially, $\mathfrak{e}_{s,s+1} \in \boldsymbol{\mathcal{E}}(\boldsymbol{\mathcal{P}}_{4,3}^{c0+})$.

(7) *If $g_0(s,\, t) < 0$, then $\mathfrak{e}_{s,t} \notin \boldsymbol{\mathcal{E}}(\boldsymbol{\mathcal{P}}_{4,3}^{c0+})$ and $-\mathfrak{e}_{s,t} \notin \boldsymbol{\mathcal{E}}(\boldsymbol{\mathcal{P}}_{4,3}^{c0+})$.*

(8) *$\mathfrak{e}_{1,0} = \mathfrak{e}_{1,0,1}^h$ is a zero polynomial.*

Proof. Denote $f_a(a,b,c,d) = \dfrac{\partial}{\partial a}f(a,b,c,d)$ and so on.

(1), (2) and (8) follows from direct calculation.

(3) Assume that $f = p_0 s_0 + p_1 s_1 + p_2 s_2 + p_3 s_3 \in \boldsymbol{\mathcal{P}}^{c0+}_{4,3}$ satisfies $f(0,s,t,1) = 0$. Then $f_b(0,s,t,1) = 0$ and $f_c(0,s,t,1) = 0$ hold. Let $a_{0,j} = s_j(0,s,t,1)$, $a_{1,j} = (s_j)_b(0,0,t,1)$, $a_{2,j} = (s_j)_c(0,s,t,1)$, and $A = (a_{i,j})$. Then

$$A = \begin{pmatrix} s^3+t^3-st+1 & t(s^2-s+t) & s(1+s-t) & t(st-s+1) \\ 3s^2-t & (2s-1)t & 2s-t+1 & t(t-1) \\ 3t^2-s & s^2-s+2t & -s & 2st-s+1 \end{pmatrix}.$$

Let B be the square matrix add $(1,0,0,0)$ above A. Then $\det B = (t-s-1)g_0(s,t) \neq 0$. Note that $\mathfrak{e}_{s,t} \in \operatorname{Ker} A$. Thus, $\operatorname{Ker} A = \mathbb{R} \cdot \mathfrak{e}_{s,t}$.

(4), (5) Same with (3).

(6) $(*)$ follows from direct calculation. Assume that $f \in \boldsymbol{\mathcal{P}}^{c0+}_{4,3}$ satisfies $f(0,0,1,1) = 0$ and $f(0,1,2,1) = 0$. Then $f(0,0,1,1) = 0$, $f_a(0,0,1,1) = 0$ and $f_a(0,1,2,1) = 0$. then $f_c(0,0,t,1) = 0$ holds. By the same method as (3), we have the conclusion.

(7) We may assume $t \neq s+1$. If $\mathfrak{e}_{s,t} \in \boldsymbol{\mathcal{E}}(\boldsymbol{\mathcal{P}}^{c0+}_{4,3})$, then $g_0(s,t) = \mathfrak{e}_{s,t}(0,0,0,1) \geq 0$. On the other hand, $\mathfrak{e}_{s,t}(0,0,1,1) = (s+1)(t-s-1)^2((s-1)^2+t^2) > 0$. Thus $-\mathfrak{e}_{s,t} \notin \boldsymbol{\mathcal{E}}(\boldsymbol{\mathcal{P}}^{c0+}_{4,3})$. □

The condition that $\mathfrak{e}_{s,t} \in \boldsymbol{\mathcal{E}}(\boldsymbol{\mathcal{P}}^{c0+}_{4,3})$ will be determined at Theorem 4.13.

Lemma 4.4. *Let* $\mathbf{1} = (1\colon 1\colon 1\colon 1) \in \mathbb{P}^3_+$, $Z := A^+_c - \{\mathbf{1}\} - \bigcup_{\tau \in \mathbb{Z}/4\mathbb{Z}} \tau(E_2 \cup E_3)$ *and*

$$\begin{aligned} & f^{c0}_{4,3}(x_0,x_1,x_2,x_3) \\ &:= (x_1^3 - x_0x_1x_3 + x_3^3)^2 - x_2(x_1^3 - x_0x_1x_3 + x_3^3)(x_0^2 + 3x_1^2 - 4x_1x_3 + 3x_3^2) \\ &\quad + x_2^2\big(x_0^2(x_1^2 - x_1x_3 + x_3^2) + 2x_0x_1x_3(x_1+x_3) \\ &\qquad\qquad + x_1^4 - 7x_1^3x_3 + 9x_1^2x_3^2 - 7x_1x_3^3 + x_3^4\big) \\ &\quad + x_2^3\big(2x_0x_1^2 - x_0(4x_1^2 + x_1x_3 + 2x_3^2) + (x_1+x_3)(x_1^2 - 3x_1x_3 + x_3^2)\big) \\ &\quad + x_2^4(x_1^2 + x_1x_3 + x_3^2). \end{aligned}$$

Then, the following hold:

1. $\Phi^{c0}_{4,3} : A^+_c \cdots \to X^{c0+}_{4,3}$ *is a birational map whose all the exceptional sets are* $\Phi^{c0}_{4,3}(E_2) = P_2$ *and* $\Phi^{c0}_{4,3}(E_3) = P_3$. $\Phi^{c0}_{4,3}\colon Z \longrightarrow \Phi^{c0}_{4,3}(Z)$ *is an isomorphism.* $\operatorname{Bs}\Phi^{c0}_{4,3} = \{\mathbf{1}\}$ *and we can regard* $\Phi^{c0}_{4,3}(\mathbf{1})$ *as the closed line segments* $[P_2P_3]$.
2. $\operatorname{Zar}(\partial X^{c0+}_{4,3}) \subset V_{\mathbb{R}}(f^{c0}_{4,3})$, $\Phi^{c0}_{4,3}(\overline{B_0}) = \partial X^{c0+}_{4,3}$ *and* S *is non-singular.*
3. $\Delta^0(X^{c0}_{4,3}) = \{P_1, P_2\}$, $\Delta^1(X^{c0}_{4,3}) = \{C, (P_1P_2)\}$ *and* $\Delta^2(X^{c0}_{4,3}) = \{S\}$.
4. *Let* $\boldsymbol{\mathcal{L}}^{c0+}_{(0:s:t:1)}$ *be the local cone of* $\boldsymbol{\mathcal{P}}^{c0+}_{4,3}$ *at* $(0\colon s\colon t\colon 1)$. *Take* $(0\colon s\colon t\colon 1) \in B_0$. *If* $\mathfrak{e}_{s,t}$ *is PSD, then* $\mathfrak{e}_{s,t} \in \boldsymbol{\mathcal{E}}(\boldsymbol{\mathcal{P}}^{c0+}_{4,3})$ *and*
$$\boldsymbol{\mathcal{L}}^{c0+}_{(0:s:t:1)} = \mathbb{R}_+ \cdot \mathfrak{e}_{s,t}.$$
If $\mathfrak{e}_{s,t}$ *is not PSD, then* $\boldsymbol{\mathcal{L}}^{c0+}_{(0:s:t:1)} = 0$.

Proof. (1), (2) and (3) $\operatorname{Bs}\Phi^{c0}_{4,3} = \{\mathbf{1}\}$ is trivial. Since $\Phi^{c0}_{4,3}(1, 1+b, 1+c, 1+d) = (2(c^2+(b-d)^2) + (b-c+d)^2 : c^2+(b-d)^2 : (b-c+d)^2, c^2+(b-d)^2) + (\text{higher degree terms})$, we can regard $\Phi^{c0}_{4,3}(1,1,1,1)$ is a line segment $[P_2P_3]$.

$\Phi^{c0}_{4,3}(E_2) = P_2$ and $\Phi^{c0}_{4,3}(E_3) = P_3$ are obtained by the direct calculation.

A_c^+ is a fundamental domain of $\Phi^{c0}_{4,3}$. It is easy to see that $\Phi^{c0}_{4,3}\colon \mathbb{P}^3_+ \to \mathbb{P}^3_+$ is a generically finite map of degree 4. The Jacobian of $\Phi^{c0}_{4,3}$ is equal to

$$J_P := -(a-b+c-d)^3((a-c)^2+(b-d)^2)^2(a+b+c+d).$$

Thus $J_P \neq 0$ on Z. Therefore $\Phi^{c0}_{4,3}\colon Z \longrightarrow \Phi^{c0}_{4,3}(Z)$ is an isomorphism. This also implies $S \subset \partial X^{c0+}_{4,3}$, $C \subset \partial X^{c0+}_{4,3}$, $L \subset \partial X^{c0+}_{4,3}$, and $\{P_1, P_2, P_3\} \subset \partial X^{c0+}_{4,3}$.

We obtain $f^{c0}_{4,3}$ by eliminating a, b, c from $x_i = s_i(a,b,c,0)$ ($i = 0$, 1, 2, 3). Using PC, we fave

$$f^{c0}_{4,3}\big(\Phi^{c0}_{4,3}(a,b,c,d)\big) = abcd(a-b+c-d)^4(a+b+c+d)^2((a-c)^2+(b-d)^2)^4 \geq 0.$$

Thus $\partial X^{c0}_{4,3} \subset V_{\mathbb{R}}(f^{c0}_{4,3})$, and $\Phi^{c0}_{4,3}(\overline{B_0}) = \partial X^{c0+}_{4,3}$. Since $J_P \neq 0$ on B_0, we have $\mathrm{Sing}(S) = \emptyset$.

Since $C = \big\{(t^3+1 : t^2 : 0 : t) \bigm| t > 0\big\}$, C is a cubic curve desined by $x_1^3 + x_2^3 = x_0x_1x_3$, $x_2 = 0$ and $(x_1+x_3)/x_0 > 0$. Note that C has a node at P_1 ($t = 0$ and ∞). But P_3 ($t = 1$) is a non-singular point of C.

Since $L = \big\{(t^3+1 : 0 : t(t+1) : 0) \bigm| t > 0\big\}$, L is a line segment $(P_1P_2]$ desined by $x_1^3 + x_2^3 = x_0x_1x_3$, $x_2 = 0$ and $0 < x_2/x_0 \leq 1$.

Thus $\mathrm{Sing}(\partial X^{c0}_{4,3}) = C \cup (P_1P_2) \cup \{P_1, P_2\}$, This implies (3).

(4) follows from Proposition 4.3(4). ∎

It is easy to draw a graph of $X^{c0+}_{4,3}$ using Mathematica. But it may present incorrect impression. It seems that $X^{c0+}_{4,3}$ is a convex set. But it is not true. The following observation show us that $X^{c0+}_{4,3}$ is not convex near $(1\colon 0\colon 0\colon 0)$. Cut $\partial X^{c0+}_{4,3}$ by the plane $V_{\mathbb{R}}(x_1 - x_3)$. Note that

$$f^{c0}_{4,3}(1,x,y,x) = x^2(2x-3y-1)(2x^3+x^2y-y^3-x^2+2y^2-y).$$

The graph of $V_{\mathbb{R}}(2x^3+x^2y-y^3-x^2+2y^2-y)$ is not convex near $(x,y) = (0,0)$. Thus $X^{c0+}_{4,3}$ is not convex. This also implies that $\mathfrak{e}_{s,t} \notin \boldsymbol{\mathcal{P}}^{c0+}_{4,3}$ for some $(0\colon s\colon t\colon 1) \in B_0$.

It is also possible to obtain $\mathfrak{e}_{s,t}$ by the method explained in Remark 1.28 of [3].

Let $f_i(x_0,x_1,x_2,x_3) := \dfrac{\partial}{\partial x_i} f^{c0}_{4,3}(x_0,x_1,x_2,x_3)$ and

$$h_i(s,t) := f_i\big(\Phi^{c0}_{4,3}(0,s,t,1)\big),$$
$$g_c(s,t) := st(t-s-1)^2(s+t+1)((s-1)^2+t^2)^2.$$

Then $h_i(s,t) = g_c(s,t)g_i(s,t)$ ($i = 0$, 1, 2, 3). Thus we have $\mathfrak{e}_{s,t} = \sum_{i=0}^{3} g_i(s,t)s_i$. We define a rational map $G^S : \overline{B_0} \cdots \to \mathbb{P}(\boldsymbol{\mathcal{H}}^{c0}_{4,3})$ by

$$G^S(0\colon s\colon t\colon 1) := \big(g_0(s,t)\colon g_1(s,t)\colon g_2(s,t)\colon g_3(s,t)\big).$$

Note that $(0\colon 1\colon 0\colon 1) \in \mathrm{Bs}\, G$. If $\mathfrak{e}_{s,t} \in \boldsymbol{\mathcal{P}}^{c0+}_{4,3}$, then $G^S(0,s,t,1) = \mathfrak{e}_{s,t} \in \boldsymbol{\mathcal{F}}_S$. We can extend G to $G^S : \partial\mathbb{P}^3_+ \cdots \to \mathbb{P}(\boldsymbol{\mathcal{H}}^{c0}_{4,3})$ by $G^S(x\colon y\colon 1\colon 0) = G^S(y\colon 1\colon 0\colon x) = G^S(1\colon 0\colon x\colon y) = G^S(0\colon x\colon y\colon 1) := G^S(0,\ x,\ y,\ 1)$.

Lemmma 4.5. (1) $\partial\boldsymbol{\mathcal{P}}^{c0+}_{4,3} = \boldsymbol{\mathcal{F}}_{P_1} \cup \boldsymbol{\mathcal{F}}_{P_2} \cup \boldsymbol{\mathcal{F}}_S \cup \boldsymbol{\mathcal{F}}_C$.

(2) *B_0 is a test set of $\boldsymbol{\mathcal{P}}^{c0+}_{4,3}$. In other words, if $f \in \boldsymbol{\mathcal{H}}^{c0}_{4,3}$ satisfies $f(0,s,t,1) = 0$ for all $s \geq 0$, $t \geq 0$, then $f(\mathbf{a}) \geq 0$ for all $\mathbf{a} \in \mathbb{R}^4_+$.*

Proof. (1) $\partial\mathcal{P}^{c0+}_{4,3} = \bigcup_{D\in\Delta(X^{c0+}_{4,3})} \mathcal{F}(D)$ by Theorem 2.7(1). Let $D_3 := \mathrm{Int}(X^{c0+}_{4,3}) \in \Delta^3(X^{c0+}_{4,3})$. Sice $\mathrm{Zar}(D_3) = \mathbb{P}^3_{\mathbb{R}}$, $\mathcal{F}(D_3)$ is not a face component.

$\mathrm{Zar}\left(\mathcal{F}((P_1P_2))\right)$ is two dimensional plane defined by $p_0 = p_2 = 0$. Thus, $\mathcal{F}((P_1P_2))$ is not a face component. Thus we have the conclusion.

(2) By Lemma 4.4(2) and Theorem 2.10, we have the conclusion. ∎

Note that (III) of Theorem 4.1 follows from the above proposition.

Lemma 4.6. *We regard as* $\mathcal{H}^{c0+}_{4,3} = \mathbb{R}^4$ *by identifying* $(p_0,\ p_1,\ p_2,\ p_3) \in \mathbb{R}^4$ *with* $\sum_{i=0}^{3} p_i s_i \in \mathcal{H}^{c0+}_{4,3}$. *Then,*

(1) $\mathrm{Zar}\left(\mathcal{F}_{P_1}\right) = V_{\mathbb{R}}(p_0)$. *Thus* $\mathcal{F}_{P_1} = \left\{ f \in \mathcal{P}^{c0}_{4,3} \;\middle|\; f \text{ is at infinity} \right\}$.

(2) $\mathrm{Zar}\left(\mathcal{F}_{P_2}\right) = V_{\mathbb{R}}(p_0 + p_2)$.

(3) $\mathrm{disc}_S\left(g_0(s,t),\ g_1(s,t),\ g_2(s,t),\ g_3(s,t)\right) = 0$ *for all* $s,\ t \in \mathbb{R}$.

(4) $\mathrm{disc}_S\left(g_0(s,t),\ g_3(s,t),\ g_2(s,t),\ g_1(s,t)\right) = 0$ *for all* $s,\ t \in \mathbb{R}$.

(5) $\mathrm{Zar}\left(\mathcal{F}_C\right) = V_{\mathbb{R}}(\mathrm{disc}_C)$.

(6) $\mathrm{Zar}\left(\mathcal{F}_S\right) = V_{\mathbb{R}}(\mathrm{disc}_S)$.

Proof. (1) and (2) are trivial.

(3) and (4) follow from direct calculation.

(5) follows from study of $\mathcal{P}^{c+}_{3,3}$. See §3 of [3].

(6) follows from (3). ∎

Now, we shall observe $\mathcal{F}_{P_2}$. In the definition of $\mathfrak{e}_{s,t}$, Remember that $\mathfrak{e}_{1,0} = 0$. In other word, $g_i(1,\ 0) = 0$ $(i = 0,\ 1,\ 2,\ 3)$. This $\mathfrak{e}_{1,0}$ corresponts to $\mathcal{F}_{P_2}$. Put $g_i^{P_2}(c) := \lim_{h\to 0} \dfrac{g_i(ch+1,h)}{4h^2}$. Then $g_0^{P_2}(c) = 1$, $g_1^{P_2}(c) = c(c-2)$, $g_2^{P_2}(c) = -1$, $g_3^{P_2}(c) = c(c+2)$.

Lemma 4.7. *For* $c \in \mathbb{R}$, *let*

$$\mathfrak{e}_c^{P_2} := s_0 + c(c-2)s_1 - s_2 + c(c+2)s_3,$$

and $\mathfrak{e}_\infty^{P_2} := s_1 + s_3$. *Then the following hold:*

(1) $\mathfrak{e}_c^{P_2} \in \mathcal{F}_{P_2} \cap \mathcal{F}_S$ *and* $\mathcal{F}_{P_2} \cap \mathcal{F}_S \cap \mathcal{F}_{P_1} = \mathbb{R}_+ \cdot \mathfrak{e}_\infty^{P_2}$.

(2) $\partial\mathcal{F}_{P_2} \subset \mathcal{F}_S$.

(3) $\mathcal{F}_{P_2} \cap \mathcal{F}_S = V\left(p_0 + p_2,\ 8p_0(p_1+p_3) - (p_1-p_3)^2\right)$.

(4) $\mathfrak{e}_c^{P_2} \in \mathcal{E}(\mathcal{P}^{c0+}_{4,3})$ *for all* $c \in \mathbb{P}^1_{\mathbb{R}}$.

Proof. (0) We shall show that $\mathfrak{e}_c^{P_2} \in \mathcal{P}^{c0+}_{4,3}$ for all $c \in \mathbb{P}^1_{\mathbb{R}}$.

Let $c_2(u,v) := (u-1)^2 + v(u+1)$ and $c_1(u,v) := 2(u-1)v(v-u-1)$. $c_2(u,v) \geq 0$ for $u \geq 0$, $v \geq 0$. Then,

$$\mathfrak{e}_c^{P_2}(0,u,v,1) = vc_2(u,v)\left(c + \frac{c_1(u,v)}{2vc_2(u,v)}\right)^2 + \frac{(u+1)((u-1)^2+v^2)^2}{c_2(u,v)} \geq 0.$$

Thus, $f_c^{P_2}$ is PSD for $c \in \mathbb{R}$. $\mathfrak{e}_\infty^{P_2} = s_1 + s_3$ is PSD by Proposition 4.2.

(1) Since $\mathfrak{e}_c^{P_2} = \lim_{h\to 0} \mathfrak{e}_{ch+1,h}/4h^2$, we have $\mathfrak{e}_c^{P_2} \in \boldsymbol{\mathfrak{F}}_S$. Since $\mathfrak{e}_c^{P_2}(1,0,1,0) = 0$, we have $\mathfrak{e}_c^{P_2} \in \boldsymbol{\mathfrak{F}}_S$. It is easy to see that $\mathfrak{e}_\infty^{P_2} \in \boldsymbol{\mathfrak{F}}_{P_1}$ and $\dim(\boldsymbol{\mathfrak{F}}_{P_2} \cap \boldsymbol{\mathfrak{F}}_C \cap \boldsymbol{\mathfrak{F}}_{P_1}) = 1$. Thus $\boldsymbol{\mathfrak{F}}_{P_2} \cap \boldsymbol{\mathfrak{F}}_S \cap \boldsymbol{\mathfrak{F}}_{P_1} = \mathbb{R}_+ \cdot \mathfrak{e}_\infty^{P_2}$.

(2) We shall determine $(\boldsymbol{\mathfrak{F}}_{P_2} \cap \boldsymbol{\mathfrak{F}}_C) - \boldsymbol{\mathfrak{F}}_{P_1}$. Note that

$$\mathrm{disc}_S(p_0, p_1, -p_0, p_3) = 2p_0\big((p_0 - p_1)^2 + (p_0 - p_3)^2\big)\big(8p_0(p_1 + p_3) - (p_1 - p_3)^2\big)^3.$$

Thus let

$$\check{V}_C := \big\{(1, p_1, -1, p_3) \in \check{\boldsymbol{\mathfrak{H}}}_{4,3}^{c0} \bigm| 8(p_1 + p_3) - (p_1 - p_3)^2 = 0\big\}.$$

Then $\check{V}_C = \big\{(1,\, c(c-2),\, -1,\, c(c+2)) \bigm| c \in \mathbb{R}\big\}$. Each point in $\check{V}_C$ corresponts to $\mathfrak{e}_c^{P_2}$. Since $\mathbb{R}_+ \cdot \check{V}_C \cup \mathbb{R}_+ \cdot \mathfrak{e}_\infty^{P_2}$ is a conic closed convex cone, it must agree with $\boldsymbol{\mathfrak{F}}_C$, and $\partial\boldsymbol{\mathfrak{F}}_C$ is generated by $\mathfrak{e}_c^{P_2}$ $(c \in \mathbb{P}^1_{\mathbb{R}})$.

(3) follows from (1) and (2).

(4) Put $D_{P_2} := \big\{(p_0{:}\, p_1{:}\, p_2{:}\, p_3) \in \mathbb{P}^3_{\mathbb{R}} \bigm| p_0 + p_2 \geq 0,\ 8p_0(p_1 + p_3) \geqq (p_1 - p_3)^2\big\}$. Then $\mathbb{P}(\boldsymbol{\mathfrak{F}}_{P_2}) = D_{P_2}$, and $\mathfrak{e}_c^{P_2} \in \partial\boldsymbol{\mathfrak{F}}_{P_2}$. Any point of ∂D_{P_2} is an extremal point of D_{P_2}. □

To characterize $\mathfrak{e}_c^{P_2}$, we need an infinitesimal local cone. Let $\pi\colon X \to A = \mathbb{P}^2_+$ be the blowing up at $(1{:}\,0{:}\,1)$, and put $\mathfrak{e}_c^b(x,y,z) := \mathfrak{e}_c^{P_2}(xz,\ yz+1,\ z,\ 1)/z^2$. Then $\mathfrak{e}_c^b(x,y,0) = 2(cx + y - t)^2$. This zero locus $V_X(cx + y - t,\ z)$ characterizes $\mathfrak{e}_c^{P_2}$.

Next we shall study $\boldsymbol{\mathfrak{F}}_S \cap \boldsymbol{\mathfrak{F}}_C$. Remember that disc_C is the edge discriminant of $X_{3,3}^{c+}$ and $X_{3,3}^{c0+}$. Let

$$\boldsymbol{\mathfrak{D}}_C := \big\{(1,x,y,z) \in \check{\boldsymbol{\mathfrak{H}}}_{4,3}^{c0+} \bigm| y \geq -1 \text{ and } d_C(x,z) \geq 0\big\}.$$

Then $\boldsymbol{\mathfrak{D}}_C$ is a closed convex set such that $\check{\boldsymbol{\mathfrak{P}}}_{4,3}^{c0+} \subset \boldsymbol{\mathfrak{D}}_C$, and $\big(\partial\check{\boldsymbol{\mathfrak{P}}}_{4,3}^{c0+}\big) \cap \mathrm{Int}(\boldsymbol{\mathfrak{D}}_C) \subset V_{\mathbb{R}}(\mathrm{disc}_S)$ by Lemma 4.6. We need the following polynomial to describe the cusp loci of $V_{\mathbb{R}}(\mathrm{disc}_S)$.

$$\begin{aligned}
f_{Q_0}(x,y) &:= 4(x+1)^2 + (y-3)^2,\\
f_{L_S}(x,y) &:= 2x + y - 1,\\
f_{C^s}(x,y) &:= y^2 + 4x(y+1) - 2y + 13,\\
f_S^{cusp}(x,y) &= 260403739669 + 153581431744x + 102255553008x^2 + 5758906656x^3\\
&\quad + 2375407488x^4 - 2980119168x^5 + 472233216x^6 - 115722240x^7\\
&\quad + 17307648x^8 - 438272x^9 + 4096x^{10} + 89440948796y + 32061417248xy\\
&\quad + 8138124864x^2y - 17528885472x^3y - 2067065472x^4y - 828572544x^5y\\
&\quad + 1188607488x^6y - 112318464x^7y - 15593472x^8y - 126976x^9y + 8192x^{10}y\\
&\quad - 223071977286y^2 - 16231383328xy^2 - 12833341936x^2y^2 + 40377065344x^3y^2\\
&\quad + 5505244544x^4y^2 + 4819181440x^5y^2 - 264563968x^6y^2 + 218927104x^7y^2\\
&\quad + 9482240x^8y^2 + 176128x^9y^2 + 4096x^{10}y^2 + 30713189004y^3 + 8960225536xy^3\\
&\quad + 17703049984x^2y^3 - 2170474624x^3y^3 - 7085133440x^4y^3 - 4728214912x^5y^3\\
&\quad - 1856392192x^6y^3 - 112496640x^7y^3 - 3928064x^8y^3 - 135168x^9y^3\\
&\quad + 61229381323y^4 - 32671427200xy^4 - 16135419808x^2y^4 - 19363454784x^3y^4\\
&\quad + 2347438208x^4y^4 + 668450944x^5y^4 + 1133005568x^6y^4 + 47364096x^7y^4\\
&\quad + 1464320x^8y^4 - 40004520712y^5 + 14114790976xy^5 - 921252992x^2y^5\\
&\quad + 9081775296x^3y^5 + 71177344x^4y^5 + 679918976x^5y^5 - 112298496x^6y^5
\end{aligned}$$

$$
\begin{aligned}
&- 6821888x^7y^5 + 10688483692y^6 - 1398548800xy^6 + 3457102112x^2y^6 \\
&- 1135819904x^3y^6 + 55287936x^4y^6 - 134577536x^5y^6 - 18625280x^6y^6 \\
&- 870429832y^7 + 226903552xy^7 - 733186304x^2y^7 - 48610432x^3y^7 \\
&- 35363712x^4y^7 - 12108928x^5y^7 - 108565637y^8 - 133149760xy^8 \\
&+ 1725104x^2y^8 + 6646560x^3y^8 - 2811392x^4y^8 + 4147404y^9 + 9240992xy^9 \\
&+ 5649472x^2y^9 - 26336x^3y^9 + 2233722y^{10} + 1416544xy^{10} + 84944x^2y^{10} \\
&+ 121340y^{11} + 16896xy^{11} + 517y^{12}.
\end{aligned}
$$

Note that

$$\mathrm{disc}_S(1, x, y, x) = f_{L_S}(x, y)^2 f_{C^s}(x, y)^2(16x^3 - x^2y + 18xy - x^2 - y^2 + 18x + 25y + 26).$$

Lemma 4.8. *Regard* $\breve{\mathfrak{H}}^{c0}_{4,3} \subset \mathbb{P}(\mathfrak{H}^{c0}_{4,3})$*, and consider on* $\breve{\mathfrak{H}}^{c0}_{4,3} : (1, x, y, z) \cong \mathbb{R}^3$*. Then*

$$\{Q_0\} \cup L^s \cup C_1^{cusp} \cup C_2^{cusp} \cup C_3^{cusp} \cup C_4^{cusp} \subset \mathrm{Sing}\left(V_{\mathbb{R}}(\mathrm{disc}_S(1, x, y, z))\right) \cap \breve{\mathfrak{P}}^{c0+}_{4,3} \subset \breve{\mathfrak{H}}^{c0}_{4,3},$$

where Q_0*,* L^s *and* C_i^{cusp} *are defined as follows:*

(1) $Q_0 := V_{\mathbb{R}}(f_{Q_0}) \cap V_{\mathbb{R}}(z + 1) = (1, -1, 3, -1) \in \partial\breve{\mathfrak{P}}^{c0+}_{4,3} \subset \breve{\mathfrak{H}}^{c0}_{4,3}$.

(2) L^s *is the half line defined by* $x = z$*,* $f_L(x, y) = 0$ *and* $y \geq -1$ *in* $\breve{\mathfrak{H}}^{c0}_{4,3}$*. But* $L^s \cap \partial\breve{\mathfrak{P}}^{c0+}_{4,3} = \{Q_0\}$*.*

(3) *Let* C^s *be the hyperbolic curve on a plane defined by* $x = z$ *and* $f_{C^s}(x, y) = 0$ *in* $\breve{\mathfrak{H}}^{c0}_{4,3}$*. But* $C^s \cap \partial\breve{\mathfrak{P}}^{c0+}_{4,3} = \{Q_0\}$*.*

(4) *Let* $x = \alpha_i(y)$ *be all the four real roots of* $f_S^{cusp}(x, y) = 0$ *when we regard* y *to be a constant where* $y \geq 3$ *and* $\alpha_1(y) \leq \alpha_2(y) \leq \alpha_3(y) \leq \alpha_4(y)$*. Note that* $\alpha_1(3) = \alpha_2(3) = \alpha_3(3) = \alpha_4(3) = 1$*. Then, the following four branches are cusps of* S*.*

$$
\begin{aligned}
C_1^{cusp} &= \left\{(1, \alpha_1(y), y, \alpha_4(y)) \in \breve{\mathfrak{H}}^{c0}_{4,3} \;\middle|\; y > 3\right\}, \\
C_2^{cusp} &= \left\{(1, \alpha_2(y), y, \alpha_3(y)) \in \breve{\mathfrak{H}}^{c0}_{4,3} \;\middle|\; y > 3\right\}, \\
C_3^{cusp} &= \left\{(1, \alpha_3(y), y, \alpha_2(y)) \in \breve{\mathfrak{H}}^{c0}_{4,3} \;\middle|\; y > 3\right\}, \\
C_4^{cusp} &= \left\{(1, \alpha_4(y), y, \alpha_1(y)) \in \breve{\mathfrak{H}}^{c0}_{4,3} \;\middle|\; y > 3\right\}.
\end{aligned}
$$

Proof. Let $f(x, y, z) := \mathrm{disc}_S(1, x, y, z)$ and $f_x := \dfrac{\partial f}{\partial x}$ and so on. $\mathrm{Sing}\left(V_{\mathbb{R}}(\mathrm{disc}_S(1, x, y, z))\right)$ can be obtained by solving the system of equations $f(x, y, z) = f_x(x, y, z) = f_y(x, y, z) = f_z(x, y, z) = 0$. But it is next to impossible to proceed this calculation. Instead of it, we eliminate z from $f_x(x, y, z) = 0$, $f_y(x, y, z) = 0$, and $f_z(x, y, z) = 0$. During this elimination process, we obtain $f_{Q_0}(x, y)$, $f_{L_S}(x, y)$, $f_{C^s}(x, y)$ and $f_S^{cusp}(x, y)$. Using PC, we can check $\{Q_0\} \cup L^s \cup C_1^{cusp} \cup C_2^{cusp} \cup C_3^{cusp} \cup C_4^{cusp} \subset \mathrm{Sing}\left(V_{\mathbb{R}}(\mathrm{disc}_S(1, x, y, z))\right)$. ∎

$\mathrm{Sing}\left(V_{\mathbb{R}}(\mathrm{disc}_S(1, x, y, z))\right)$ may has other loci. But we will see that

$$\mathrm{Sing}\left(V_{\mathbb{R}}(\mathrm{disc}_S(1, x, y, z))\right) \cap \partial\breve{\mathfrak{P}}^{c0+}_{4,3} \subset \{Q_0\} \cup C_1^{cusp} \cup C_2^{cusp} \cup C_3^{cusp} \cup C_4^{cusp},$$

during discussion from now.

Proof of Theorem 4.1. We take the section of $\breve{\mathfrak{P}}^{c0+}_{4,3}$ by the hyperplane

$$H_r := \left\{(1, x, y, z) \in \breve{\mathfrak{H}}^{c0}_{4,3} \;\middle|\; y = r\right\}.$$

We regard H_r as $(x,\, z)$-plane. Put

$$
\begin{aligned}
D_r &:= H_r \cap \breve{\mathfrak{P}}^{c0+}_{4,3} = \big\{(x,z) \in H_r \;\big|\; (1,x,r,z) \in \breve{\mathfrak{P}}^{c0+}_{4,3}\big\},\\
D_C &:= \mathfrak{D}_C \cap H_r = \big\{(x,z) \in H_r \;\big|\; d_C(x,z) \geq 0\big\},\\
V_C &:= \partial D_C = \big\{(x,z) \in H_r \;\big|\; d_C(x,z) = 0\big\},\\
V^r_S &:= \big\{(x,z) \in H_r \;\big|\; \mathrm{disc}_S(1,x,r,z) = 0\big\} - (C^s \cup L^s) \cap H_r.
\end{aligned}
$$

(O-1) If $r < -1$, then $D_r = \emptyset$, by Lemma 4.6(2).

(O-2) If $r = -1$, then the condition of (1) of Theorem 4.1 determines the set $\breve{\mathfrak{P}}^{c0+}_{4,3} \cap H_{-1}$, because of Lemma 4.7.

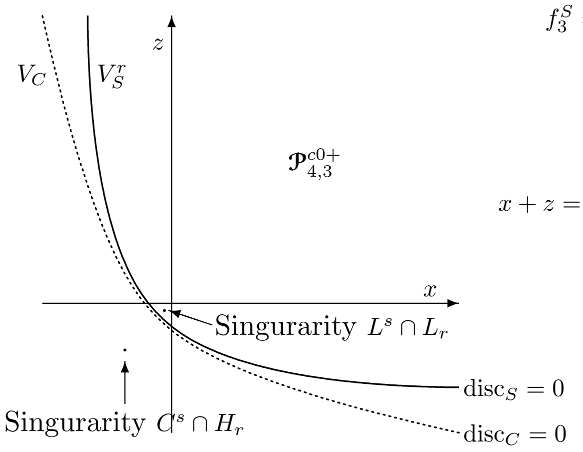


Fig.4.1 : The case $-1 < r < 3$

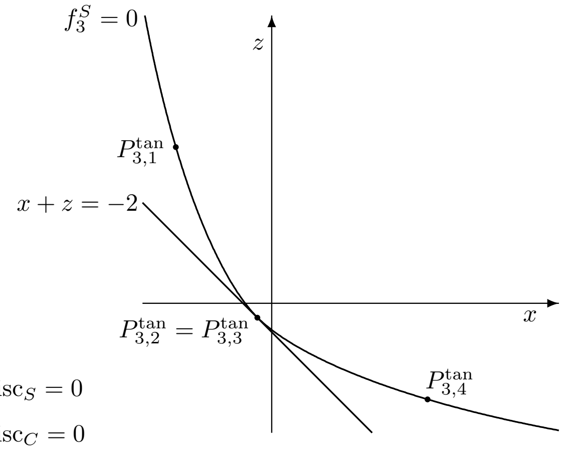


Fig.4.2 : The case $r = 3$

(I) When $-1 < r < 3$, V^r_S is as Fig.4.1. Two points $C^s \cap H_r$ and $L^s \cap H_r$ are all the isolated singularities of $V_{\mathbb{R}}(\mathrm{disc}_S) \cap H_3$. V^r_S is a smooth curve in D_C and enclose a convex set $\breve{\mathfrak{P}}^{c0+}_{4,3} \cap H_r$. Thus,

$$D_r = \big\{(x,z) \in \mathbb{R}^2 \;\big|\; \mathrm{disc}_S(1,x,r,z) \geq 0 \text{ and } d_C(x,z) \geq 0\big\}.$$

Thus, the conditions of (2) of Theorem 4.1 determines $\breve{\mathfrak{P}}^{c0+}_{4,3} \cap H_r$.

(II) Consider the case $r = 3$. Let

$$
\begin{aligned}
f^S_3(x,z) := {}& x^6 - 4x^5z + 7x^4z^2 - 8x^3z^3 + 7x^2z^4 - 4xz^5 + z^6\\
& - 174x^5 - 342x^4z - 508x^3z^2 - 508x^2z^3 - 342xz^4 - 174z^5\\
& - 414x^4 - 712x^3z - 1332x^2z^2 - 712xz^3 - 414z^4\\
& - 800x^3 - 4320x^2z - 4320xz^2 - 800z^3\\
& - 6592x^2 - 16512xz - 6592z^2 - 16384x - 16384z - 11776.
\end{aligned}
$$

Then $\mathrm{disc}_S(1,x,3,z) = -2(x+z+2)^2 f^S_3(x,z)$. As Fig 4.2, $V_{\mathbb{R}}(f^S_3)$ tangents V_C at three points $P^{\tan}_{3,1}$, $P^{\tan}_{3,4}$ and $P^{\tan}_{3,2} = P^{\tan}_{3,3}$ (these symbols will be explained in (III)). Moreover $V_{\mathbb{R}}(f^S_3) \subset D_C$. Thus,

$$D_3 = \big\{(x,z) \in \mathbb{R}^2 \;\big|\; \mathrm{disc}_S(1,x,3,z) \geq 0 \text{ and } d_C(x,z) \geq 0\big\},$$

and the conditions of (2) of Theorem 4.1 determines $\breve{\mathfrak{P}}^{c0+}_{4,3} \cap H_3$.

Note that $V_{\mathbb{R}}(f_3^S) \cap V_{\mathbb{R}}(z + 11.851831\cdots) = \emptyset$, and $V_{\mathbb{R}}(f_3^S) \cap V_{\mathbb{R}}(z - z_0)$ sonsists of two points for $z_0 > -11.851831\cdots$.

(III) Consider the case $r > 3$. Then, V_S^r has just four cusps $P_{r,i}^{cusp} := C_i^{cusp} \cap H_r = \big(\alpha_i(r),\, \alpha_{5-i}(r)\big)$ $(i = 1,\, 2,\, 3,\, 4)$. Since V_S^r is symmetric with respect to the line $V_{\mathbb{R}}(x - z)$, it is enough to consider the part $z \geq x$.

We observe $\boldsymbol{\mathfrak{F}}_S \cap \boldsymbol{\mathfrak{F}}_C \cap \breve{\boldsymbol{\mathcal{H}}}^{c0}_{4,3}$. Let

$$\begin{aligned} L_x &:= \big\{(0\colon 0\colon w\colon 1) \in \mathbb{P}^3_+ \;\big|\; w \in [0, \infty]\big\}, \\ L_y &:= \big\{(0\colon w\colon 0\colon 1) \in \mathbb{P}^3_+ \;\big|\; w \in [0, \infty]\big\}, \\ L_z &:= \big\{(0\colon x\colon y\colon 0) \in \mathbb{P}^3_+ \;\big|\; (x\colon y) \in \mathbb{P}^1_+\big\}. \end{aligned}$$

Note that $\partial\, \mathrm{Cls}_{\mathbb{P}^3_+}(\overline{B_0}) = L_x \cup L_y \cup L_z$.

We define a rational map $G^S : \mathbb{P}^3_+ \cdots \to \mathbb{P}(\boldsymbol{\mathcal{H}}^{c0}_{4,3})$ just before Lemma 4.5. Since $g_0(w, 0) = 0$, $G^S(L_y) \cap \breve{\boldsymbol{\mathcal{H}}}^{c0}_{4,3} = \emptyset$. Since $G^S(0\colon x\colon y\colon 0) = G^S(0\colon 0\colon x/y\colon 1)$, we have $G^S(L_z) = G^S(L_x)$. Since

$$\mathrm{disc}_C\,\big(g_0(0, w),\, g_1(0, w),\, g_3(0, w)\big) = 0,$$

we have $G^S(L_x) \subset V_{\mathbb{R}}(\mathrm{disc}_C) \cap V_{\mathbb{R}}(\mathrm{disc}_S)$. Put $C_x^{\tan} := G^S(L_x)$.

Similarly, we define a rational map $G' : \mathbb{P}^3_+ \cdots \to \mathbb{P}(\boldsymbol{\mathcal{H}}^{c0}_{4,3})$ by

$$G'(0, x, y, 1) := \big(g_0(x, y) : g_3(x, y) : g_2(x, y) : g_1(x, y)\big).$$

Let $C_z^{\tan} := G'(L_x)$. Then $C_x^{\tan} \cup C_z^{\tan} \subset V_{\mathbb{R}}(\mathrm{disc}_C) \cap V_{\mathbb{R}}(\mathrm{disc}_S)$.

Put $H_{\geq 3} := \big\{(1, x, r, z) \in \breve{\boldsymbol{\mathcal{H}}}^{c0}_{4,3} \;\big|\; r \geq 3\big\}$. We regard $H_{\geq 3} \subset \breve{\boldsymbol{\mathcal{H}}}^{c0}_{4,3} \subset \mathbb{P}(\boldsymbol{\mathcal{H}}^{c0}_{4,3})$. We shall determine $C_x^{\tan} \cap H_{\geq 3}$. Let $\delta := 0.2955977425\cdots$ be the real root of $t^3 + t^2 + 3t - 1 = 0$. Then, all the real roots of $g_2(0,\, t)/g_0(0,\, t) = 3$ are $t = 1,\, \delta$. We put

$$\begin{aligned} C_1^{\tan} &:= \big\{G'(0, 0, w, 1) \in \mathbb{P}(\boldsymbol{\mathcal{H}}^{c0}_{4,3}) \;\big|\; 0 < w \leq \delta\big\} \subset C_z^{\tan}, & P_{r,1}^{\tan} &:= C_1^{\tan} \cap H_r \in \breve{\boldsymbol{\mathcal{P}}}^{c0+}_{4,3}, \\ C_2^{\tan} &:= \big\{G^S(0, 0, w, 1) \in \mathbb{P}(\boldsymbol{\mathcal{H}}^{c0}_{4,3}) \;\big|\; w \geq 1\big\} \subset C_x^{\tan}, & P_{r,2}^{\tan} &:= C_2^{\tan} \cap H_r \in \breve{\boldsymbol{\mathcal{P}}}^{c0+}_{4,3}, \\ C_3^{\tan} &:= \big\{G'(0, 0, w, 1) \in \mathbb{P}(\boldsymbol{\mathcal{H}}^{c0}_{4,3}) \;\big|\; w \geq 1\big\} \subset C_z^{\tan}, & P_{r,3}^{\tan} &:= C_3^{\tan} \cap H_r \in \breve{\boldsymbol{\mathcal{P}}}^{c0+}_{4,3}, \\ C_4^{\tan} &:= \big\{G^S(0, 0, w, 1) \in \mathbb{P}(\boldsymbol{\mathcal{H}}^{c0}_{4,3}) \;\big|\; 0 < w \leq \delta\big\} \subset C_x^{\tan}, & P_{r,4}^{\tan} &:= C_4^{\tan} \cap H_r \in \breve{\boldsymbol{\mathcal{P}}}^{c0+}_{4,3}. \end{aligned}$$

Then $C_x^{\tan} \cap H_{\geq 3} = C_1^{\tan} \cup C_3^{\tan}$ and $C_z^{\tan} \cap H_{\geq 3} = C_2^{\tan} \cup C_4^{\tan}$. Note that $\boldsymbol{\mathfrak{F}}_S \cap \boldsymbol{\mathfrak{F}}_C \cap \big\{G^S(0, 0, w, 1) \in \mathbb{P}(\breve{\boldsymbol{\mathcal{H}}}^{c0}_{4,3}) \;\big|\; \delta < w < 1\;\big\} = \emptyset$.

Lemma 4.9. $C_1^{\tan} \cup C_2^{\tan} \cup C_3^{\tan} \cup C_4^{\tan} \subset \mathrm{Zar}(\boldsymbol{\mathfrak{F}}_S \cap \boldsymbol{\mathfrak{F}}_C) \cap H_{\geq 3}$.

Proof. Clear. □

Put $C^{cusp} := \mathrm{Cls}_{\breve{\boldsymbol{\mathcal{H}}}^{c0}_{4,3}}(C_1^{cusp} \cup C_2^{cusp} \cup C_3^{cusp} \cup C_4^{cusp})$. Let's determine $C_x^{\tan} \cap C^{cusp}$. Since $C_x^{\tan} = G^S(L_x) \subset V_{\mathbb{R}}(\mathrm{disc}_C) \cap V_{\mathbb{R}}(\mathrm{disc}_S)$, and

$$G^S(0, 0, w, 1) = \left(1 : \frac{1 - 2w^3}{w^2} : \frac{(w^2 + 1)^2 - w}{w} : \frac{w^3 - 2}{w}\right),$$

we put $G_x^S(w) := (1 - 2w^3)/w^2$, $G_y^S(w) := ((w^2 + 1)^2 - w)/w$ and $G_z^S(w) := (w^3 - 2)/w = G_x^S(1/w)$.

Lemma 4.10. *$\eta(x, y) = 61 + 62x + 56y + 32x^2 + 30xy - 6y^2 + 9x^3 + 4x^2y - 6xy^2 - 16y^3 + x^4 - 4x^2y^2 - 6xy^3 + y^4 - x^3y^2$ has the following properties:*

(1) *If $(1\colon x\colon y\colon z) \in C_x^{\tan} \cup C_z^{\tan}$, then $\eta(x + z,\, y) = 0$.*

(2) *Let $r > 3$. On a plane H_r, the zero locus $\eta(x+z, r) = 0$ is the union of two lines. One is the line $P^{\tan}_{r,1}P^{\tan}_{r,4}$, and the other is the line $P^{\tan}_{r,2}P^{\tan}_{r,3}$. $\eta(x+z, r) < 0$ between these two lines, and $\eta(x+z, r) > 0$ outside.*

Proof. (1) follows from $\eta\big(G^S_x(w) + G^S_z(w),\, G^S_y(w)\big) = 0$.
(2) $\eta(x, r) = 0$ has just two real roots for $r > 3$, and $\eta(y-3, y) < 0$ for $y < 3$. ∎

Note that

$$f^{cusp}_S\big(G^S_x(w),\, G^S_y(w)\big) = \frac{(w-1)^4(w^2+1)^4(w^4-6w^2-8w+1)^2 f_{38}(w)}{w^{22}},$$

here $f_{38}(w)$ is a polynomial of degree 38 whose real roots are two negative numbers $w = -8.590880\cdots$, $-2.4445756\cdots$. Let $\tau_1 := 0.1150\cdots$ and $\tau_2 := 2.9343\cdots$ be the real roots of $w^4 - 6w^2 - 8w + 1 = 0$, and

$$r_1 := \frac{g_2(0,\tau_1)}{g_0(0,\tau_1)} = 7.9207039574\cdots, \quad r_2 := \frac{g_2(0,\tau_2)}{g_0(0,\tau_2)} = 30.474537321\cdots.$$

be the real roots of $r^4 - 28r^3 - 90r^2 - 92r + 16353 = 0$. Then, all positive the roots of $f^{cusp}_S\big(G^S_x(w),\, G^S_y(w)\big) = 0$ are $w = 1,\ \tau_1,\ \tau_2$. In the case $w = 1$, $G^S(0,0,w,1) = (1\colon{-1}\colon 3\colon{-1}) = Q_0$. Thus, $C^{\tan}_x \cap C^{cusp}$ consists of three points Q_0, $P^{\tan}_{r_1,1} = P^{cusp}_{r_1,1} = G^S(0,0,\tau_1,1)$, and $P^{\tan}_{r_2,2} = P^{cusp}_{r_2,2} = G^S(0,0,\tau_2,1)$. Similarly, $C^{\tan}_z \cap C^{cusp}$ consists of three points Q_0, $P^{\tan}_{r_1,4} = P^{cusp}_{r_1,4} = G'(0,0,\tau_1,1)$, and $P^{\tan}_{r_2,3} = P^{cusp}_{r_2,3} = G'(0,0,\tau_2,1)$.

Lemma 4.11. *In $\breve{\mathfrak{H}}^{c0}_{4,3} \cong \mathbb{R}^3 : (x,y,z)$, $\kappa_1(x+z) + \kappa_2 y = 1$ defines the plane which passes through $P^{\tan}_{r_1,1}$, $P^{\tan}_{r_2,2}$, $P^{\tan}_{r_2,3}$ and $P^{\tan}_{r_1,4}$.*

Proof. Note that $P^{\tan}_{r_1,1} = P^{cusp}_{r_1,1} = (\alpha_1(r_1),\, r_1,\, \alpha_4(r_1))$ and so on.

$$\alpha_2(r_1) + \alpha_3(r_1) = \frac{g_1(0,\tau_1) + g_3(0,\tau_1)}{g_0(0,\tau_1)} = G^S_x(\tau_1) + G^S_z(\tau_1),$$
$$\alpha_1(r_2) + \alpha_4(r_2) = \frac{g_1(0,\tau_2) + g_3(0,\tau_2)}{g_0(0,\tau_2)} = G^S_x(\tau_2) + G^S_z(\tau_2).$$

Solve $\kappa_1(G^S_x(w) + G^S_z(w)) + \kappa_2 G^S_y(w) = 1$ for $w = \tau_1$ and τ_2. Then, we obtain

$$\begin{aligned}\kappa_1 &:= \frac{s^2t^2 - t^3 + 2t^2 - t}{s^4 - 2s^3t - 2st^3 + t^4 - 4s^2t + 5st^2 - 2t^3 + 2s^2 - 2st - s + 1} \\ &= 0.0129074031\cdots, \\ \kappa_2 &:= \frac{-st^2 + 2t^2 + s - 2t}{s^4 - 2s^3t - 2st^3 + t^4 - 4s^2t + 5st^2 - 2t^3 + 2s^2 - 2st - s + 1} \\ &= 0.0318925844\cdots,\end{aligned}$$

where $s = \tau_1 + \tau_2$, $t = \tau_1\tau_2$. Let γ, δ be all the imaginal roots $w^4 - 6w^2 - 8w + 1 = 0$, and put $s_2 := \gamma + \delta$, $t_2 := \gamma\delta$. Then $s + s_2 = 0$, $tt_2 = 1$, $t + t_2 + s + s_2 = -6$, $ts_2 + st_2 = 8$. When we eliminate s, t, s_1, t_1 from these relations, we have

$$817808203\kappa_1^6 - 546807084\kappa_1^5 + 129155640\kappa_1^4 - 13342016\kappa_1^3 + 556080\kappa_1^2 - 10176\kappa_1 + 64 = 0,$$
$$43042537\kappa_2^6 - 4514514\kappa_2^5 - 188769\kappa_2^4 - 38684\kappa_2^3 + 4119\kappa_2^2 - 114\kappa_2 + 1 = 0.$$ ∎

Now, we shall complete the proof of Theorem 4.1. To prove (3), (4), (5) of Theorem 4.1, we put

$$D_r^{(3)} := \{(x,z) \in H_r \mid \kappa_1(x+z)+\kappa_2 r \geq 1,\ \mathrm{disc}_S(1,x,r,z) \geq 0,\ d_C(x,z) \geq 0\},$$
$$D_r^{(4)} := \left\{(x,z) \in H_r \;\middle|\; \begin{array}{l} \kappa_1(x+z)+\kappa_2 r < 1,\ \eta(x+z,\, r) > 0, \\ \mathrm{disc}_S(1,x,r,z) \geq 0,\ d_C(x,z) \geq 0 \end{array} \right\},$$
$$D_r^{(5)} := \{(x,z) \in H_r \mid \kappa_1(x+z)+\kappa_2 r < 1,\ \eta(x+z,\, r) \leq 0,\ d_C(x,\, z) \geq 0\}.$$

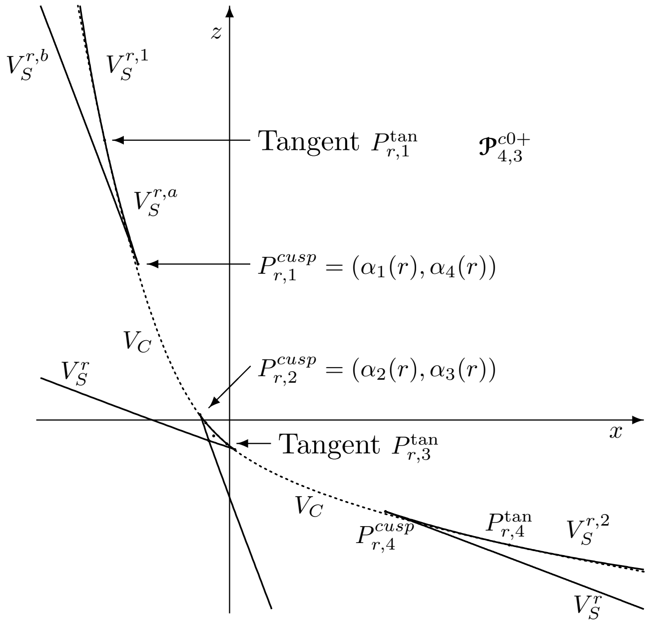


Fig.4.3 : The case $3 < r < r_1$

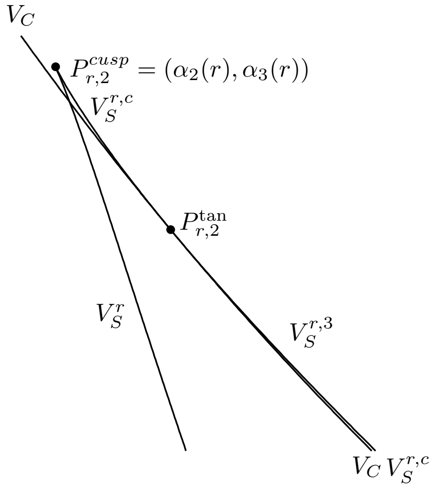


Fig.4.4 : The case $3 < r < r_1$

As Fig. 4.3, we divide the part $z > \alpha_4(y)$ of V_S^r at $P_{r,1}^{cusp}$, and denote the right part by $V_S^{r,a}$ and the left part by $V_S^{r,b}$. We mean $V_S^{r,a} \cap V_S^{r,b} = \{P_{r,1}^{cusp}\}$. Similarly, let $V_S^{r,c}$ be the smooth interval between $P_{r,2}^{cusp}$ and $P_{r,3}^{cusp}$ of V_S^r. We mean $P_{r,2}^{cusp}$, $P_{r,3}^{cusp} \in V_S^r$.

(III-1) If $3 < r < r_1$, then $P_{r,1}^{cusp} = (\alpha_1(r),\, \alpha_4(r)) \in \mathrm{Int}(D_C)$, and $V_S^{r,a}$ tangents to V_C at $P_{r,1}^{\tan}$, as Fig. 4.3. This implies that $P_{r,1}^{\tan} \in (\partial \mathfrak{F}_C) \cap (\partial \mathfrak{F}_S)$. We divide the curve segment $V_S^{r,a}$ at the point $P_{r,1}^{\tan}$, and denote the upper part by

$$V_S^{r,1} := \{(x,z) \in H_r \mid \mathrm{disc}_S(x,r,z) = 0,\ d_C(x,z) \geq 0,\ z \geq z(P_{r,1}^{\tan})\},$$

where $z(P)$ is the z-coordinate of the point $P \in H_r$. Then $V_S^{r,1} = \mathfrak{F}_S \cap V_S^{r,a}$. Every $P \in V_S^{r,a} - V_S^{r,1}$ is obtained as $P = G(0{:}\, s{:}\, t{:}\, 1)$ for a certain $(s,\, t) \in \mathbb{C}^2 - \overline{B_0}$.

Let $V_S^{r,2}$ be the symmetric set of $V_S^{r,1}$ with respect to the line $x = z$ on H_r.

Similarly, $(\alpha_2(r),\, \alpha_3(r)) \in \mathrm{Int}(D_C)$, and $V_S^{r,c}$ tangents to V_C at $P_{r,2}^{\tan}$, as Fig. 4.4. Let

$$V_S^{r,3} := \{(x,z) \in H_r \mid \mathrm{disc}_S(x,r,z) = 0,\ d_C(x,z) \geq 0,\ z(P_{r,3}^{\tan}) \leq z \leq z(P_{r,2}^{\tan})\}$$

be the interval of $V_S^{r,c}$ between $P_{r,2}^{\tan}$ and $P_{r,3}^{\tan}$. Then $V_S^{r,2} = \mathfrak{F}_S \cap V_S^{r,c}$. By Lemma 4.10,

$$V_S^{r,1} \cup V_S^{r,2} \cup V_S^{r,3} = \left\{(1,x,r,z) \in \partial \breve{\mathfrak{P}}_{4,3}^{c0+} \;\middle|\; \begin{array}{l} \mathrm{disc}_S(1,x,r,z) = 0,\ d_C(x,z) \geq 0, \\ \eta(x+z,r) \geq 0 \end{array} \right\}.$$

So, $D_r = D_r^{(3)} \cup D_r^{(4)} \cup D_r^{(5)}$.

(III-2) If $r = r_1$, then $P^{\tan}_{r_1,1} = \big(\alpha_1(r_1),\ \alpha_4(r_1)\big)$, $P^{\tan}_{r_1,4} = \big(\alpha_4(r_1),\ \alpha_1(r_1)\big) \in (\partial\mathfrak{F}_C) \cap (\partial\mathfrak{F}_S)$. The line defined by $\kappa_1(x+z)+\kappa_2 r_1 = 1$ agrees with the line $P^{\tan}_{r_1,1}P^{\tan}_{r_1,4}$. Others are similar as (III)-1.

(III-3) Consider the case $r_1 < r < r_2$. About $V_S^{r,3}$ the situation is same as (III-1).

The situation of $V_S^{r,1}$ and $V_S^{r,2}$ changes. If $r > r_1$, then $\big(\alpha_1(r),\ \alpha_4(r)\big) \notin D_C$. and $P^{\tan}_{r,1} \notin D_C$ as Fig. 4.5. In this case, V_C and $V_S^{r,a}$ intersect at a point Q_r^a transversally. So, $\mathfrak{F}_S \cap V_S^{r,a}$ agrees with the following new $V_S^{r,4}$ in this case.

$$V_S^{r,4} := \big\{(x,z) \in H_r \;\big|\; \mathrm{disc}_S(x,r,z) = 0,\ d_C(x,z) \geq 0,\ z \geq z(Q_r^a)\big\},$$

be the interval of $V_S^{r,a}$ upper than Q_r^a. Let $V_S^{r,5}$ be the symmetric set of $V_S^{r,4}$ with respect to $V_{\mathbb{R}}(x-z)$. Then,

$$V_S^{r,4} \cup V_S^{r,5} = \left\{(1,x,r,z) \in \partial\check{\mathfrak{P}}^{c0+}_{4,3} \;\middle|\; \begin{array}{l} \mathrm{disc}_S(1,x,r,z) = 0,\ d_C(x,z) \geq 0, \\ \kappa_1(x+z)+\kappa_2 r \geq 1 \end{array}\right\},$$

$$V_S^{r,3} = \left\{(1,x,r,z) \in \partial\check{\mathfrak{P}}^{c0+}_{4,3} \;\middle|\; \begin{array}{l} \mathrm{disc}_S(1,x,r,z) = 0,\ d_C(x,z) \geq 0, \\ \eta(x+z,r) \geq 0,\ \kappa_1(x+z)+\kappa_2 r < 1 \end{array}\right\}.$$

So, $D_r = D_r^{(3)} \cup D_r^{(4)} \cup D_r^{(5)}$.

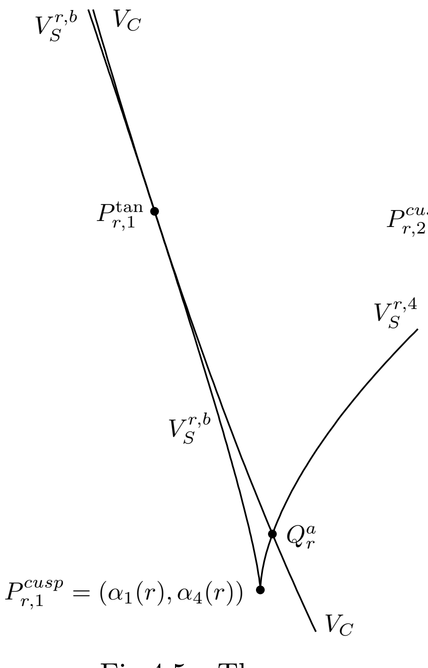


Fig.4.5 : The case $r > r_1$

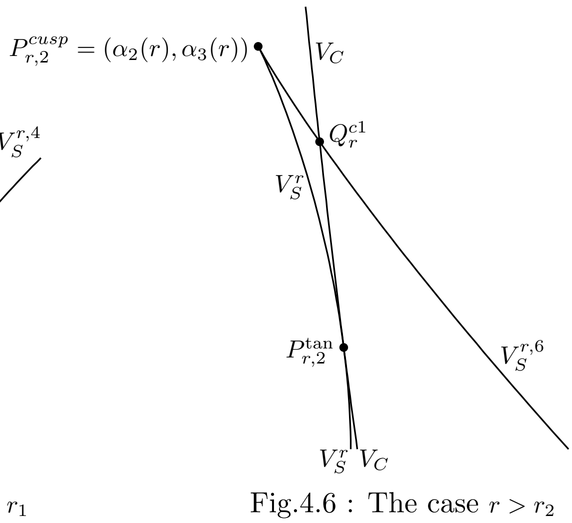


Fig.4.6 : The case $r > r_2$

(III-4) If $r = r_2$, then $P^{\tan}_{r_2,2} = \big(\alpha_2(r_2),\ \alpha_3(r_2)\big)$, $P^{\tan}_{r_2,3} = \big(\alpha_2(r_3),\ \alpha_3(r_2)\big) \in (\partial\mathfrak{F}_C) \cap (\partial\mathfrak{F}_S)$. Others are similar as (III-3).

(III-5) If $r > r_2$, then $\big(\alpha_2(r),\ \alpha_3(r)\big) \notin D_C$, and $P^{\tan}_{r,2}$, $P^{\tan}_{r,3} \notin D_C$ as Fig.4.6. In this case, V_C and $V_S^{r,c}$ intersect at two points Q_r^{c1}, Q_r^{c2} transversally. So, $\mathfrak{F}_S \cap V_S^{r,c}$ agrees with the following new $V_S^{r,6}$ in this case.

$$V_S^{r,6} := \big\{(x,z) \in H_r \;\big|\; \mathrm{disc}_S(x,r,z) = 0,\ d_C(x,z) \geq 0,\ z(Q_r^{c2}) \leq z \leq z(Q_r^{c3})\big\}$$

be the interval of $V_S^{r,c}$ between Q_r^{c1} and Q_r^{c2}. Then

$$V_S^{r,4} \cup V_S^{r,5} \cup V_S^{r,6} = \{(1,x,r,z) \in \partial\breve{\mathfrak{P}}_{4,3}^{c0+} \mid \mathrm{disc}_S(1,x,r,z) = 0,\ d_C(x,z) \geq 0\}.$$

If $r > r_2$, then $\kappa_1(x+z) + \kappa_2 r \geq 1$ holds for any $(x,\ z) \in D_C$. Thus $D_r = D_r^{(3)}$ in this case.

By (III-1)—(III-5) and Lemma 4.11, we conclude that the conditions of (3), (4), (5) of Theorem 4.1 determine $\breve{\mathfrak{P}}_{4,3}^{c0+}$ when $r > 3$. ∎

Next we observe $\partial\mathfrak{F}_{P_1}$. Note that $\mathfrak{e}_{s,t} \in \mathfrak{F}_{P_1}$ when $g_0(s,t) = 0$.

Proposition 4.12. *Let*

$$\begin{aligned} h_\xi(t) &:= t^4 - 3t^3 - 27t^2 - 64t + 2, \\ h_\mu(t) &:= t^4 + t^3 - 2t^2 - 3t + 1, \\ h_{\nu,a}(t) &:= t^4 - 7t^3 + 13t^2 - 20t + 2, \\ h_{\nu,b}(t) &:= t^4 - 4t^3 + 3t^2 - 6t + 2. \end{aligned}$$

Take the real roots of these polynomials as follows:

$$\begin{aligned} V_{\mathbb{R}}(h_\xi) &= \{\xi_1 := 0.0308472031\cdots,\ \xi_2 := 7.631998798\cdots\}, \\ V_{\mathbb{R}}(h_\mu) &= \{\mu_1 := 0.2882309962\cdots,\ \mu_4 := 1.4587325322\cdots\}, \\ V_{\mathbb{R}}(h_{\nu,a}) &= \{\nu_1 := 0.1070225045\cdots,\ \nu_4 := 5.2319384324\cdots\}, \\ V_{\mathbb{R}}(h_{\nu,b}) &= \{\nu_2 := 0.3713081034\cdots,\ \nu_3 := 3.586633132\cdots\}. \end{aligned}$$

Moreover, put $\mu_2 := 1/\mu_4$ and $\mu_3 := 1/\mu_1$. Then the following hold:

(1) $s_1 + s_3 + cs_2 \in \mathfrak{P}_{4,3}^{c0+}$, *if and only if* $0 \leq c \leq 16$. *Moreover* $s_1 + s_3 + 16s_2 = (1/64)\mathfrak{e}_{1,4}$ *and* $s_1 + s_2 = \mathfrak{e}_\infty^{P_2}$.

(2) $s_1 + cs_2$ *and* $s_3 + cs_2$ *are PSD, if and only if* $\xi_1 \leq c \leq \xi_2$.

(3) *There exists* $\alpha_i > 0$ $(i = 1,\ 2,\ 3,\ 4)$ *such that*

$$\begin{aligned} \mathfrak{e}_{\mu_1,\nu_1} &= \alpha_1(s_1 + \xi_1 s_2), \quad & \mathfrak{e}_{\mu_2,\nu_2} &= \alpha_2(s_3 + \xi_1 s_2), \\ \mathfrak{e}_{\mu_3,\nu_3} &= \alpha_3(s_3 + \xi_2 s_2), \quad & \mathfrak{e}_{\mu_4,\nu_4} &= \alpha_4(s_1 + \xi_2 s_2). \end{aligned}$$

(4) $\mathfrak{F}_{P_1}$ *is given as the following. Normalize* $f \in \mathfrak{F}_{P_1}$ *as* $f = xs_1 + ys_2 + (1-x)s_3$, *and correspond this* f *to the point* $(x,\ y) \in \mathbb{R}^2$. *Let*

$$\begin{aligned} D(P_1) &:= \{(x,y) \in \mathbb{R}^2 \mid xs_1 + ys_2 + (1-x)s_3 \in \mathfrak{F}_{P_1}\}, \\ V_S^u &:= \{(x,y) \in \mathbb{R}^2 \mid 0 \leq x \leq 1,\ 4 \leq y \leq 8,\ \mathrm{disc}_S(0,x,y,1-x) = 0\}, \\ V_S^l &:= \{(x,y) \in \mathbb{R}^2 \mid 0 \leq x \leq 1,\ 0 < y \leq 4,\ \mathrm{disc}_S(0,x,y,1-x) = 0\} \cup \{(1/2,0)\}. \end{aligned}$$

Then, $D(P_1)$ *is a convex domain enclosed by* V_S^u, V_S^l *and lines* $x = 0$, $x = 1$. *We can identify* $D(P_1)$ *with* $\mathbb{P}(\mathfrak{F}_{P_1}) \subset \mathbb{P}(\mathfrak{H}_{4,3}^{c0})$.

Proof. (1) Let $f_t := s_1 + s_3 + ts_2$, $w_f(u) := u + 1/u$, $v_f(t,u) := \dfrac{u}{2(u+1)}\bigl(t + 2 - w_f(u)\bigr)$, and $r_f(t,u) := -w_f(u)^2 + 2(3t+2)w_f(u) - (t-2)^2$. Then

$$f_t(0,u,v,1) = (u+1)(v - v_f(t,u))^2 + \frac{u^2 r_f(t,u)}{4(u+1)}.$$

Note that $w_f(u) \geq 2$.

Consider the case $w_f(u) > t+2$. Then $v_f(t,u) < 0$ and $f_t(0,u,v,1)$ is monotonically increasing with respect to v in $v \geq 0$. Thus $f_t(0,u,v,1) \geq f_t(0,u,0,1) = tu(u+1) \geq 0$.

Consider the case $2 \leq w_f(u) \leq t+2$. Then $r_f(t,u) \geq r_f(t,1) = t(16-t)$. Thus, $f_t(0,u,v,1) \geq 0$ if $0 \leq t \leq 16$. If $t < 0$ or $t > 16$, then $v_f(t,2) > 0$ and $f_t(0,1,v_f(t,1),1) = r_f(t,1)/8 = t(16-t)/8 < 0$.

Thus $f_t \in \boldsymbol{\mathfrak{P}}^{c0+}_{4,3}$ if and only if $0 \leq t \leq 16$.

Since $f_{16}(0,1,v,1) = 2(v-4)^2$ and $g_1(1,4) = 64$, we have $f_{16} = \mathfrak{e}_{1,4}/64$.

(2) Let $g_t := s_1 + ts_2$, $v_g(t,u) := u(t+1-u)/2$, and $r_g(t,u) := -u^3 + (2t+2)u^2 - (t-1)^2u + 4t$. Then

$$g_t(0,u,v,1) = (v - v_g(t,u))^2 + \frac{u}{4} r_g(t,u).$$

If $u > t+1$, then $g_t(0,u,v,1) \geq g_t(0,u,0,1) = tu(u+1) \geq 0$.

Assume that $0 \leq u \leq t+1$. Observe the cubic function $r_g(t,u)$. The roots of $(\partial/\partial u) r_g(t,u) = 0$ are $u_\pm := (2(t+1) \pm \sqrt{t^4 + 14t + 1})/3$. Note that $0 \leq u_- < t+1 < u_+$. Thus $\min g_t(0,u,v,1) = \min r_g(t,u) = r_g(t,u_-)$. If $g_y \in \boldsymbol{\mathcal{E}}(\boldsymbol{\mathfrak{P}}^{c0+}_{4,3})$ then $r_g(t,u_-) = 0$ and the cubic equation $r_g(t,u) = 0$ has a double root at $u = u_-$. Then $\mathrm{Disc}_3(-1,\ 2t+2,\ -(t-1)^2,\ 4) = 0$. Note that $\mathrm{Disc}_3(-1,\ 2t+2,\ -(t-1)^2,\ 4t) = 16t \cdot h_\xi(t)$. Thus $t = \xi_1$ or ξ_2 if $g_t \in \boldsymbol{\mathcal{E}}(\boldsymbol{\mathfrak{P}}^{c0+}_{4,3})$.

We can also see that g_t is PSD if and only if $\xi_1 \leq t \leq \xi_2$. Since $\mathrm{disc}_S(0,\ 1,\ t,\ 0) = -t^2 h_\xi(t)$, $g_t \in \boldsymbol{\mathfrak{F}}_S$ if and only if $t = \xi_1$ and ξ_2.

Since $\mathrm{disc}_S(0,x,y,z) = \mathrm{disc}_S(0,z,y,x)$ and $s_3(a,b,c,d) + ts_2(a,b,c,d) = s_1(b,a,d,c) + ts_2(b,a,d,c)$, $s_3 + ts_2$ is PSD if and only if $\xi_1 \leq t \leq \xi_2$.

(3) Assume that $t = \xi_1$ or ξ_2. Then $r_g(\xi_i, u_0) = 0$ for $\exists u_0 \in \mathbb{R}$ and $h_\xi(\xi_i) = 0$. Eliminate t from $r_g(t,u) = 0$ and $h_\xi(t) = 0$, we obtain

$$h_\mu(u)^2(u^4 - 16u^3 + 48u^2 - 384u + 512) = 0.$$

$u = u_0$ must be a multiple root of the above equation. Thus $h_\mu(u_0) = 0$, and $u_0 = \mu_1$ or μ_4. Let $\nu_1 := v_g(\xi_1, \mu_1)$ and $\nu_4 := v_g(\xi_2, \mu_4)$. Then $g_{\xi_1}(0,\mu_1,\nu_1,1) = 0$ and $g_{\xi_2}(0,\mu_4,\nu_4,1) = 0$. Thus $g_{\xi_1} \in \mathbb{R}_+ \cdot \mathfrak{e}_{\mu_1,\nu_1}$ and $g_{\xi_2} \in \mathbb{R}_+ \cdot \mathfrak{e}_{\mu_4,\nu_4}$ by Proposition 4.3(5). Eliminate t and u from $v = v_g(t,u)$, $h_\xi(t) = 0$ and $h_\mu(u) = 0$, we obtain $h_{\nu,a}(\nu_1) = h_{\nu,a}(\nu_4) = 0$.

Let $h_t := s_1 + ts_2$. Then $h_t(0,u,v,1) = u^3 g_t(0,1/u,v/u,1)$. $\mu_2 = 1/\mu_4$ and $\mu_3 = 1/\mu_2$ are roots of $u^4 h_\mu(1/u)$. Let $\nu_3 := \nu_1/\mu_1$ and $\nu_2 := \nu_4/\mu_4$. Then $h_{\xi_2}(0,\mu_2,\nu_2,1) = 0$ and $h_{\xi_1}(0,\mu_4,\nu_4,1) = 0$. Thus $h_{\xi_2} \in \mathbb{R}_+ \cdot \mathfrak{e}_{\mu_2,\nu_2}$ and $h_{\xi_2} \in \mathbb{R}_+ \cdot \mathfrak{e}_{\mu_3,\nu_3}$. Eliminate u from $v = uv_g(t,1/u)$, $h_\xi(t) = 0$ and $h_\mu(1/u) = 0$, we obtain $h_{\nu,b}(\nu_2) = h_{\nu,b}(\nu_3) = 0$.

(4) For $f = p_0s_0 + p_1s_1 + p_2s_2 + p_3s_3 \in \boldsymbol{\mathfrak{H}}^{c0}_{4,3}$, $\mathrm{disc}(P_1) = p_0$ and $\mathrm{disc}(P_2) = p_0 + p_2$. By Lemma 4.5, $\partial \boldsymbol{\mathfrak{F}}_{P_1} \subset \boldsymbol{\mathfrak{F}}_{P_2} \cup \boldsymbol{\mathfrak{F}}_S \cup \boldsymbol{\mathfrak{F}}_C$. $\mathrm{disc}(P_2) = 0$ corresponds to $y = 0$. Thus, $D(P_1)$ must be included in the upper half space $y \geq 0$. Since $\mathrm{disc}_C(0,\ x,\ (1-x)) = -x^2(1-x)^2$ and $(1/2,\ 1) \in D(P_1)$, $D(P_1)$ is included in the stlipe $0 \leq x \leq 1$. V^u_S and V^l_S are curves as Fig 4.7. Thus, we have the conclusion. □

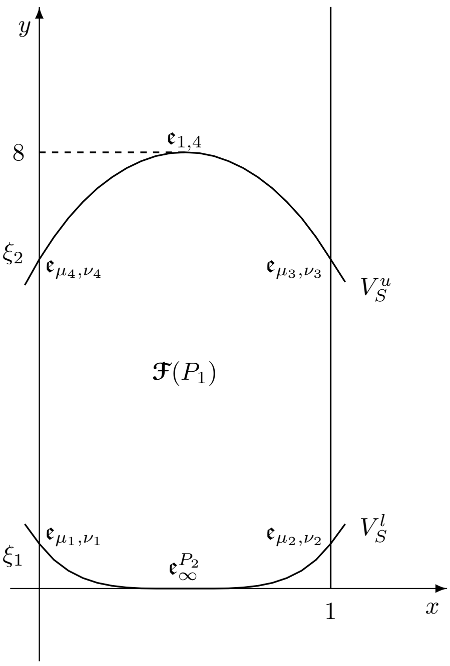


Fig. 4.7 $\mathfrak{F}_{P_1}$

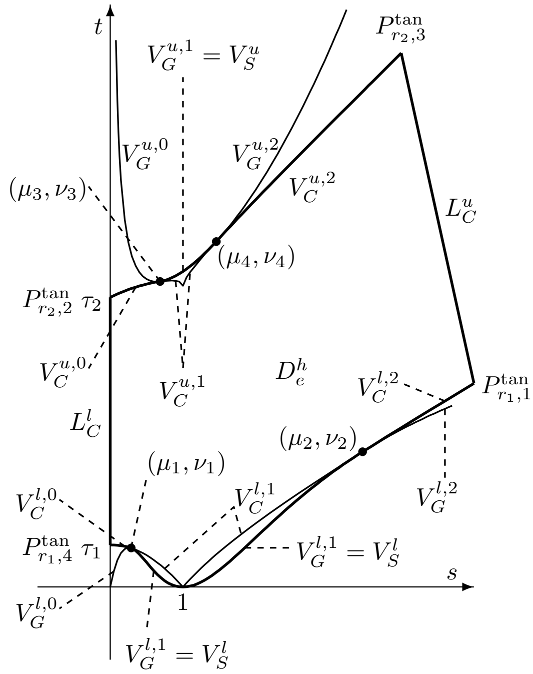


Fig. 4.8 D_e^h

Let

$$D_e^h := \left\{ (u\colon v\colon w) \in \mathbb{P}_+^2 \;\middle|\; \mathfrak{e}_{u,v,w}^h \in \boldsymbol{\mathcal{E}}(\boldsymbol{\mathcal{P}}_{4,3}^{c0+}) \right\} = \left\{ (u\colon v\colon w) \in \mathbb{P}_+^2 \;\middle|\; \mathfrak{e}_{u,v,w}^h \in \boldsymbol{\mathcal{P}}_{4,3}^{c0+} \right\},$$

$$d_e^{Ch}(u,v,w) := \frac{\operatorname{disc}_C\left(g_0^h(u,v,v),\, g_1^h(u,v,w),\, g_3^h(u,v,w)\right)}{u^2 w^2 (u+w-v)^2((u-w)^2+v^2)^2},$$

$$d_e^C(s,t) := d_e^{Ch}(s,t,1).$$

$d_e^{Ch}(u,v,w)$ is a homogeneous polynomial of degree 10. Let $L_w := V_+(w) \subset \mathbb{P}_+^2$ be the line segment at infinity. For $(u\colon v\colon w) \in \mathbb{P}_+^2 - L_w$, let $s := u/w$, $t := v/w$ and regard $\mathbb{P}_+^2 - L_w$ to be the the first quadrant of the $(s,\, t)$-plane $\mathbb{R}_+^2$. The point $(s,t) = (1,0) \notin D_e^h$ because $\mathfrak{e}_{1,0} = 0$. For completion of D_G^h, it is better to put $\mathfrak{e}_\infty^{P_2} = s_1 + s_3$ at $(s,\, t) = (1,\, 0)$. In the quadrant $s \geq 0$ and $t \geq 0$, the curve $V_C := V_{\mathbb{R}}(d_e^C(s,t))$ has two connected components V_C^l and V_C^u. Similarly, $V_G := V_{\mathbb{R}}(g_0(s,t))$ has two connected components V_G^l and V_G^u. V_C^l and V_G^l are included in $t < s+1$, and V_C^u, V_G^u are included in $t > s+1$.

$V_C^l \cap V_G^l = \left\{ (\mu_1,\nu_1),\, (\mu_2,\nu_2) \right\}$, and $V_C^u \cap V_G^u = \left\{ (\mu_3,\nu_3),\, (\mu_4,\nu_4) \right\}$. Divide V_C^l and V_G^l by the points (μ_1,ν_1) and (μ_2,ν_2), and define $V_C^{l,i}$ and $V_G^{l,i}$ $(i = 0,\, 1,\, 2)$ as Fig. 4.8. Similarly, we divide V_C^u and V_G^u by the points (μ_3,ν_3) and (μ_4,ν_4), and we define $V_C^{u,i}$ and $V_G^{u,i}$ $(i = 0,\, 1,\, 2)$ as Fig. 4.8. The segment $V_G^{l,1}$ corresponds to V_S^l, and $V_G^{u,1}$ corresponds to V_S^u.

Theorem 4.13.

$D_e^h = \left\{ (u\colon v\colon w) \in \mathbb{P}_+^2 \;\middle|\; g_0^h(u,v,w) \geq 0,\ v > 0 \text{ and one of the following (1) or (2) holds.} \right\}$.

(1) $d_e^{Ch}(u,v,w) \geq 0$.

(2) $g_1^h(u,v,w) \geq 0$ *and* $g_3^h(u,v,w) \geq 0$.

Proof. We already proved that if $\mathfrak{e}_{u,v,w}^h$ is PSD then $\mathfrak{e}_{u,v,w}^h \in \boldsymbol{\mathcal{E}}(\boldsymbol{\mathcal{P}}_{4,3}^{c0+})$. By Proposition 4.3, $g_0^h(u,v,w) \geq 0$ is required.

(i) Consider the case $g_0^h(u,v,w) > 0$.

Let $p_i = g_i^h(u,v,w)/g_0^h(u,v,w)$ (i = 1, 2, 3). $\mathfrak{e}^h_{u,v,w}$ is PSD, if and only if (p_1, p_2, p_3) satisfy the condition of Theorem 4.1. If $\mathfrak{e}^h_{u,v,w}$ is PSD, then $\mathfrak{e}^h_{u,v,w} \in \boldsymbol{\mathfrak{F}}_S$ and $\mathrm{disc}_S(1, p_1, p_2, p_3) = 0$. Conditions about $\eta(p_1 + p_3, p_2)$ and $\kappa_1(p_1 + p_3) + \kappa_2 p_2 - 1$ do not have special sence in this case. Thus, $\mathfrak{e}^h_{u,v,w}$ is PSD, if and only if $d_C(p_1, p_3) \geq 0$. That is, $\mathrm{disc}_C(1, p_1, p_3) \geq 0$ or '$p_1 \geq 0$ and $p_3 \geq 0$'. $\mathrm{disc}_C(1, p_1, p_3) \geq 0$ is equivalent to $u \geq 0$, $w \geq 0$ and $d_e^{Ch}(u,v,w) \geq 0$. Thus, we have the conclusion.

(ii) Consider the case $g_0^h(u,v,w) = 0$.

In this case, V_S^l and V_S^u of Proposition 4.12 appears in ∂D_e^h. By Proposition 4.12, $V_G^l \cup V_G^u$ is determined by $g_0(s,t) = 0$, $g_1(s,t) \geq 0$ and $g_3(s,t) \geq 0$. □

By the avobe theorem, $\mathfrak{e}_{0,t}$ is PSD, if and only if $\tau_1 \leq t \leq \tau_2$. Similarly, $\mathfrak{e}^h_{t,1,0} = t\mathfrak{e}^h_{0,t,1} - (t^2-1)(t^2+1)^2 s_2$ is PSD, if and only if $1/\tau_2 \leq t \leq 1/\tau_1$.

We shall observe ∂D_e^h precisely. $\boldsymbol{\mathfrak{F}}_S \cap \boldsymbol{\mathfrak{F}}_{P_1}$ and $\boldsymbol{\mathfrak{F}}_S \cap \boldsymbol{\mathfrak{F}}_{P_2}$ are determined already. We observe the part of ∂D_e^h corresponding to $\boldsymbol{\mathfrak{F}}_S \cap \boldsymbol{\mathfrak{F}}_C$.

Let L_C^l be the line segment defined by $s = 0$ and $\tau_1 \leq t \leq \tau_2$, and put $V_{SC}^1 := V_C^{l,0} \cup L_C^l \cup V_C^{u,0}$. Since $V_{SC}^1 \subset V(\mathrm{disc}_C) \cap \partial D_e^h$, if $(s,t) \in V_{SC}^1$, there exists $\rho \in \mathbb{P}^1_{\mathbb{R}}$ such that $\mathfrak{e}_{s,t}(0,0,\rho,1) = 0$. We denote this ρ by $\rho(s,t) = \rho(s{:}\,t{:}\,1)$. Note that $\rho(0,\tau_1) = \tau_1$, $\rho(\mu_1,\, \nu_1) = 0$. If $(s,t) \in V_C^{l,0}$, $\rho(s,t)$ is monotonically decreasing from τ_1 to 0 with respect to s. Similarly, $\rho(0,\tau_2) = \tau_2$, $\rho(\mu_3,\, \nu_3) = +\infty$, and of $(s,t) \in V_C^{u,0}$, $\rho(s,t)$ is monotonically increasing from τ_2 to $+\infty$ with respect to s. If $(s,t) \in L_C^l$, then $\rho(s,t) = t$. So, each $\rho \in [0, +\infty]$, there exists unique $(s,\, t) \in V_{SC}^1$ such that $\rho(s,t) = \rho$. That is, $\mathfrak{e}_{s,t}(0,0,u,1) = 0$. Note that $(s,t) = (0,\tau_1)$ corresponds to $P^{\tan}_{r_1,4} = P^{cusp}_{r_1,4}$, and $(s,t) = (0,\tau_2)$ corresponds to $P^{\tan}_{r_2,2} = P^{cusp}_{r_2,2}$.

When $w = 0$, let L_C^u be the interval of $L_w = V_+(w)$ between $(1{:}\,\tau_1{:}\,0)$ and $(1{:}\,\tau_2{:}\,0)$. Note that $V_C^{l,2} \cap L_w = (1{:}\,\tau_1{:}\,0)$ and $V_C^{u,2} \cap L_w = (1{:}\,\tau_2{:}\,0)$. Put $V_{SC}^2 := V_C^{u,2} \cup L_C^u \cup V_C^{l,2}$. Note that $\rho(\mu_4,\nu_4) = 0$, $\rho(1{:}\,t{:}\,0) = 1/t$, and $\rho(\mu_3,\nu_3) = +\infty$. So, each $\rho \in [0, +\infty]$, there exists unique $(u{:}\,v{:}\,w) \in V_{SC}^2$ such that $\rho(u{:}\,v{:}\,w) = \rho$.

L_C^l corresponts to $C_x^{\tan}$, and L_C^r corresponts to $C_z^{\tan}$. $P_{r,1}^{\tan}$ moves on the interval of L_C^u defined by $1/\tau_2 \leq v/u \leq 1$. Q_r^a moves on $V_C^{l,2}$. $P_{r,2}^{\tan}$ moves on the interval of L_C^l defined by $1 \leq t \leq \tau_2$. Q_r^{c1} moves on $V_C^{u,0}$.

If $(s,t) \in V_C^{l,0} \cup V_C^{l,2} \cup V_C^{u,0} \cup V_C^{2,2}$ and $\rho = \rho(s,t)$, then s and ρ satisfy the following relation:

$$\begin{aligned}
&(\rho^3+1)^2(\rho^4 - 8\rho^3 - 6\rho^2 + 1)s^4 \\
&+ (3\rho+1)(-\rho^9 - 3\rho^8 - 2\rho^7 - 6\rho^6 - 14\rho^5 + 6\rho^4 - 2\rho^3 - 6\rho^2 - 5\rho + 1)s^3 \\
&- 2(\rho^{10} + 12\rho^8 + 26\rho^7 - \rho^6 + 4\rho^5 - \rho^4 + 26\rho^3 + 12\rho^2 + 1)s^2 \\
&+ (\rho+3)(\rho^9 - 5\rho^8 - 6\rho^7 - 2\rho^6 + 6\rho^5 - 14\rho^4 - 6\rho^3 - 2\rho^2 - 3\rho - 1)s \\
&+ (\rho^3+1)^2(\rho^4 - 6\rho^2 - 8\rho + 1) = 0.
\end{aligned}$$

Especially, we have the following:

Proposition 4.14. *For $t \in [0, +\infty]$. let $\boldsymbol{\mathcal{L}}_t^C \subset \boldsymbol{\mathfrak{F}}_C$ be the local cone of $\boldsymbol{\mathfrak{P}}_{4,3}^{c0+}$ at $(0{:}\,0{:}\,t{:}\,1) \in \mathbb{P}^3_+$. Take $(u_i{:}\,v_i{:}\,w_i) \in V_{SC}^i$ such that $\rho(u_i{:}\,v_i{:}\,w_i) = 1$ (i = 1, 2). Then*

$$\boldsymbol{\mathcal{L}}_t^C = \mathbb{R}_+ \cdot \mathfrak{e}^h_{u_1,v_1,w_1} + \mathbb{R}_+ \cdot \mathfrak{e}^h_{u_2,v_2,w_2}.$$

Theorem 4.15. *All the elements of* $\mathfrak{E}(\mathcal{P}^{c0+}_{4,3})$ *is the positive multiple of* $\mathfrak{e}^h_{u,v,w}$ $((u\colon v\colon w) \in D^h_e)$ *or* $\mathfrak{e}^{P_2}_t$ $(t \in \mathbb{P}^1_{\mathbb{R}})$.

Proof of Proposition 1.11. Let $e_1,\dots, e_{20}$ be all the monomials in $\mathcal{H}_{4,3}$. Assume that $(s\colon t\colon 1) \in D^h_e$, $s > 0$, $t > 0$ and $t \neq s+1$. Put $u := \sqrt{s}$, $v := \sqrt{t}$ and $E_{s,t}(a, b, c, d) := \mathfrak{e}_{s,t}(a^2, b^2, c^2, d^2)$. $V_{\mathbb{R}}(E_{s,t})$ contains at least 27 isolated points. Among $V_{\mathbb{R}}(E_{s,t})$, we choose the following 20 points: $\mathbf{a}_1 = (1\colon 1\colon 1\colon 1)$, $\mathbf{a}_2 = (-1\colon 1\colon 1\colon 1)$, $\mathbf{a}_3 = (1\colon -1\colon 1\colon 1)$, $\mathbf{a}_4 = (1\colon 1\colon -1\colon 1)$, $\mathbf{a}_5 = (1\colon 1\colon 1\colon -1)$, $\mathbf{a}_6 = (1\colon 1\colon -1\colon -1)$, $\mathbf{a}_7 = (1\colon -1\colon 1\colon -1)$, $\mathbf{a}_8 = (1\colon -1\colon -1\colon 1)$, $\mathbf{a}_9 = (0\colon u\colon v\colon 1)$, $\mathbf{a}_{10} = (1\colon 0\colon u\colon v)$, $\mathbf{a}_{11} = (v\colon 1\colon 0\colon u)$, $\mathbf{a}_{12} = (u\colon v\colon 1\colon 0)$, $\mathbf{a}_{13} = (0\colon u\colon v\colon -1)$, $\mathbf{a}_{14} = (-1\colon 0\colon u\colon v)$, $\mathbf{a}_{15} = (v\colon -1\colon 0\colon u)$, $\mathbf{a}_{16} = (u\colon v\colon -1\colon 0)$, $\mathbf{a}_{17} = (0\colon u\colon -v\colon 1)$, $\mathbf{a}_{18} = (1\colon 0\colon u\colon -v)$, $\mathbf{a}_{19} = (-v\colon 1\colon 0\colon u)$, $\mathbf{a}_{20} = (u\colon -v\colon 1\colon 0)$. Let $a_{i,j} := e_j(\mathbf{a}_i)$ and $A := (a_{i,j})$. Then

$$\det A = \pm 1048576 s^2 t^2 (t-s-1)^4 ((s-1)^2 + t^2)^4 \neq 0.$$

Thus, there exists no $g \in \mathcal{H}_{4,3} - \{0\}$ such that $g(\mathbf{a}_i) = 0$ for all $1 \le i \le 20$. Thus $E_{s,t} \notin \Sigma_{4,6}$. □

It seems that if $(s,t) \in V^{l,0}_C \cup V^{l,1}_G \cup V^{l,2}_C \cup V^{u,0}_C \cup V^{u,1}_G \cup V^{u,2}_C - (L^l_C \cup L^u_C)$, then $\mathfrak{e}_{s,t} \in \mathfrak{E}(\mathcal{P}^+_{4,3})$. If $(s,t) \in \mathrm{Int}(D^h_e) \cup L^l_C \cup L^u_C$, then $\mathfrak{e}_{s,t} \notin \mathfrak{E}(\mathcal{P}^+_{4,3})$. This suggests that $\mathfrak{E}(\mathcal{P}^+_{4,3})$ is not so simple.

If $(s,t) \in V^{l,0}_C \cup V^{l,2}_C \cup V^{u,0}_C \cup V^{u,2}_C - (L^l_C \cup L^u_C) - \{(\mu_i, \nu_i) \mid i = 1,\ 2,\ 3,\ 4\}$, then $\mathfrak{e}_{s,t}(a^2, b^2, c^2, d^2)$ has 35 isolated zeros, because $\mathfrak{e}_{s,t}(0,0,r,1) = 0$ by $r = \rho(s,t) > 0$, $t \neq 1$. So, $\mathfrak{e}_{s,t}(a^2, b^2, c^2, d^2)$ will be an extremal element of $\mathcal{P}_{4,6}$ which is irreducible.

4.2. Structure of $\mathcal{P}^{c+}_{4,3}$

We have not complete any of (I1), (I2), (I3) for $\mathcal{P}^{c+}_{4,3}$. But, we shall give (I4) and some information about $X^{c+}_{4,3}$.

We choose $s_0 := S_3 - S_{1,1,1}$, $s_1 := S_{2,1,0} - S_{1,1,1}$, $s_2 := S_{2,0,1} - S_{1,1,1}$, $s_3 := S_{1,2,0} - S_{1,1,1}$, $s_4 := S_{1,1,1}$ as a basis of $\mathcal{H}^c_{4,3}$, and define $\Phi^c_{4,3} : \mathbb{P}^3_+ \longrightarrow \mathbb{P}^4_+$ by $\Phi^c_{4,3}(a) = \big(s_0(a) : s_1(a) : s_2(a) : s_3(a) : s_4(a)\big)$. Put $X^{c+}_{4,3} := \Phi^c_{4,3}(\mathbb{P}^3_+)$. $\Psi^{c0}_{4,3} : \mathbb{P}^3_+/(\mathbb{Z}/4\mathbb{Z}) \cdots \to X^{c0+}_{4,3}$ split as

$$\Psi^{c0}_{4,3} : \mathbb{P}^3_+/(\mathbb{Z}/4\mathbb{Z}) \xrightarrow{\Psi^c_{4,3}} X^{c+}_{4,3} \xrightarrow{\mathrm{pr}} X^{c0+}_{4,3}.$$

Proposition 4.16. *Let*

$$\begin{aligned} &f^c_{4,3}(x_0, x_1, x_2, x_3, x_4) \\ &:= x_1^3 - x_0x_1x_3 + x_3^3 + x_1^2x_2 + x_1x_2^2 + x_2^2x_3 + x_2x_3^2 - x_0x_1x_2 - x_0x_2x_3 - x_1x_2x_3 \\ &\quad + x_4\Big(x_0^2 + 5x_1^2 + x_2^2 + 5x_3^2 - 2x_0x_1 - 2x_0x_2 - 2x_0x_3 + 2x_1x_2 - 6x_1x_3 + 2x_2x_3\Big). \end{aligned}$$

Then $\mathrm{Zar}(X^{c+}_{4,3}) = \{\mathbf{x} \in \mathbb{P}^4_{\mathbb{R}} \mid f^c_{4,3}(\mathbf{x}) = 0,\ f^{c0}_{4,3}(\mathbf{x}) \ge 0\}$ *with the coordinate system* $x_i = s_i(a_0\colon \cdots\colon a_3)$ $(i = 0,\dots, 4)$. *This cubic hypersurface* $V_{\mathbb{R}}(f^c_{4,3})$ *has an isolated singularity at* $\Phi^c_{4,3}(1\colon 1\colon 1\colon 1) = (0\colon 0\colon 0\colon 0\colon 1)$.

Proof. Using PC, we have $f^c_{4,3}(s_0,\ s_1,\ s_2,\ s_3,\ s_4) = 0$. Define $\mathrm{pr} : X^{c+}_{4,3} \cdots \to X^{c0+}_{4,3}$ by $\mathrm{pr}(x_0\colon \cdots\colon x_4) = (x_0\colon \cdots\colon x_3)$. This is a birational map. By Lemma 4.4, we have the conclusion. □

Proposition 4.17. $X^{c+}_{4,3}$ *does not have the main component.*

Proof. Assume that $X_{4,3}^{c+}$ has the main component. Note that $\mathcal{E}(\mathcal{P}_{4,3}^{c0+}) = \mathcal{E}(\mathcal{P}_{4,3}^{c+}) \cap \mathcal{H}_{4,3}^{c0+}$. Let f be an element of the main component such that $f \in \mathcal{E}(\mathcal{P}_{4,3}^{c+}) - \mathcal{E}(\mathcal{P}_{4,3}^{c0+})$. Then, there exists $\mathbf{a} = (a\colon b\colon c\colon 1) \in \mathrm{Int}(\mathbb{P}_{+}^{3})$ such that $f(\mathbf{a}) = 0$. $(a, b, c) \neq (1, 1, 1)$, since $f \notin \mathcal{P}_{4,3}^{c0+}$. Put $\mathbf{b} := (b\colon c\colon 1\colon a) \in \mathrm{Int}(\mathbb{P}_{+}^{3})$. Note that $\mathbf{a} \neq \mathbf{b}$. Then the line $\mathbf{ab}$ is a bitangent line of the cubic surface $V_{\mathbb{C}}(f) \in \mathbb{P}_{\mathbb{C}}^{3}$. But a cubic surface has no bitangent line. A contradiction. □

Proof of Theorem 1.10. Let $\overline{B_0} := \{(0\colon s\colon t\colon 1) \in \mathbb{P}_{+}^{3} \mid s, t \in \mathbb{R}_{+}\}$, and $\Omega := \{(1\colon 1\colon 1\colon 1)\} \cup \overline{B_0}$. By Theorem 2.10, it is enough to show $\mathcal{E}(X_{4,3}^{c+}) \subset \Phi_{4,3}^{c}(\Omega)$. Take any $\mathbf{x} \in \mathcal{E}(X_{4,3}^{c+})$. Then, there exists $D \in \Delta(X_{4,3}^{c+})$ such that $\mathbf{x} \in D$ and that $\mathcal{F}_D$ is a face component. By the above proposition, $D \subset \partial X_{4,3}^{c+} \cup \mathrm{Sing}(X_{4,3}^{c+})$. If $\mathbf{x} \in \partial X_{4,3}^{c+}$, then $\mathbf{x} \in \Phi_{4,3}^{c}(\overline{B_0})$. If $\mathbf{x} \in \mathrm{Sing}(X_{4,3}^{c+})$, then $\mathbf{x} = \Phi_{4,3}^{c}(1\colon 1\colon 1\colon 1)$ by Proposition 4.16. □

For test set, we can prove the following by the same idea.

Proposition 4.18. *Assume that* $f(x_1,\ldots, x_n) \in \mathcal{H}_{n,3}$, *and there exists* $\mathbf{a} \in \mathrm{Int}(\mathbb{P}_{+}^{n-1})$ *such that* $f(\mathbf{a}) = 0$ *and* $\dfrac{\partial}{\partial x_i} f(\mathbf{a}) = 0$ *for all* $i = 1,\ldots, n$. *Then* $f \in \mathcal{P}_{n,3}^{+}$ *if and only if* $f(\mathbf{b}) \geq 0$ *for all* $\mathbf{b} \in \partial\mathbb{P}_{+}^{n-1}$.

Proof. Assume that $f(\mathbf{c}) < 0$ for a certain $\mathbf{c} \in \mathrm{Int}(\mathbb{P}_{+}^{n-1})$. We may assume that f take a minimal value at $\mathbf{c}$. Put $g(t) := f((1 - t)\mathbf{a} + t\mathbf{c})$. Then, a cubic polynomial $g(t)$ takes minimal values at $t = 0$ and $t = 1$. A contradiction. □

Section 5. Philosophy of Semialgebraic Variety.

5.1. Real algebraic quasi-variety.

Till §4, we used the notion of (quasi-) semialgebraic varieties without exact definition. In this section, we shall discuss how its definition should be, at least for theory of PDS cones. Before to give it, we must discuss what a real algebraic variety is.

Usually, we say $(X, \mathcal{O}_X)$ is an *algebraic variety over* $\mathbb{R}$ when $(X, \mathcal{O}_X)$ is an integral separated scheme of finite type over $\mathbb{R}$. $X(\mathbb{R})$ denotes the set of $\mathbb{R}$-rational points, and $X_{\mathbb{C}} := X \times_{\mathrm{Spec}\,\mathbb{R}} \mathrm{Spec}\,\mathbb{C}$. By this definition, X and $X_{\mathbb{C}}$ are irreducible and reduced. To treat possibly reducible or non-reduced varieties, we shall call a separated scheme of finite type over $\mathrm{Spec}(\mathbb{R})$ to be an *algebraic quasi-variety.* This notion is not convenient for algebraic inequalities. For example, there exists infinitely many algebraic varieties X over $\mathbb{R}$ such that $X(\mathbb{R}) = \mathbb{R}^2$. X may not be affine even if $X(\mathbb{R}) = \mathbb{R}^2$.

The definition of a real algebraic variety is given in §3.2 in [8]. According to this definition, every real algebraic variety is reduced but may be reducible (i.e. not irreducible). To keep consistency with complex algebraic geometry, we shall add a restriction that real algebraic varieties must be irreducible and separated. To treat possibly non-reduced varieties, we shall give alternative definition of real algebraic quasi-varieties as the following:

Definition 5.1.(Real algebraic quasi-variety) (I) A locally ringed space $(X, \mathcal{R}_X)$ is called a *real algebraic quasi-variety*, if there exists a separated scheme $(Y, \mathcal{O}_Y)$ of finite type over $\mathrm{Spec}\,\mathbb{R}$ which satisfies the following:

(1) There exists an injective morphism $\iota\colon(X,\,\mathcal{R}_X)\longrightarrow(Y,\,\mathcal{O}_Y)$ as locally ringed spaces, and ι induces a homeomorphism $X\to Y(\mathbb{R})$ as topological spaces with respect to Zariski topology and Euclidean topology.
(2) Take any affine open subset $V\subset Y$. Let $\mathfrak{n}_P$ be the maximal ideal of $\mathcal{O}_Y(V)$ corresponding to a closed point $P\in Y$. For an arbitral non-empty subset $U\subset V\cap\iota(X)$, we put

$$S_U:=\bigcap_{P\in U}\big(\mathcal{O}_Y(V)-\mathfrak{n}_P\big).$$

If U is an Euclidean open set, then $\iota^*:S_U^{-1}\mathcal{O}_Y(V)\longrightarrow\mathcal{R}_X(\iota^{-1}(U))$ is an isomorphism of $\mathbb{R}$-algebra. Thus, each maximal ideal $\mathfrak{m}\subset\mathcal{R}_X(\iota^{-1}(V))$ corresponds to a point $P\in\iota^{-1}(V)\subset X$.
(3) Take an arbitral affine open subset $V\subset Y$. Then

$$\big\{f\in\mathcal{O}_Y(V)\;\big|\;f(P)=0\text{ for all }P\in V(\mathbb{R})\big\}$$

is a nilpotent ideal of $\mathcal{O}_Y(V)$.

In this case, Y is said to be a $\mathbb{R}$-*scheme which represents* X. If we can choose Y such that $Y_{\mathbb{C}}$ is irreducible and reduced, then we shall call X to be a *real algebraic variety* (See Notation 0.1 of [18]).

$U\subset X$ is called an *affine open subset* of X, if there exists an affine open subset $U_Y\subset Y$ such that $U=\iota^{-1}(U_Y(\mathbb{R}))$. Zariski open (resp. closed) subsets are defied similarly. The *Euclidean topology* of X is the topology induced from the analytic topology of $Y_{\mathbb{C}}$. $Y(\mathbb{R})$ is also denoted as $Y_{\mathbb{C}}(\mathbb{R})$. When $V\subset Y$ is an affine open subset and $B\subset V(\mathbb{R})$ is a subset such that $\mathrm{Cls}_{Y(\mathbb{R})}(\mathrm{Int}(B))=\mathrm{Cls}_{Y(\mathbb{R})}(B)$, we put

$$S_B:=\bigcap_{P\in B}\big(\mathcal{O}_Y(V)-\mathfrak{n}_P\big),$$

and $\mathcal{R}_X(\iota^{-1}(B)):=\iota^*\big(S_B^{-1}\mathcal{O}_Y(V)\big)$. By this definition, $(X,\,\mathcal{R}_X)$ can be also regarded as a locally ringed space with respect to the Zariski topology and the Euclidean topology. We usually omit to write ι. For example, we write $X=Y(\mathbb{R})$.

Note that if $(X,\,\mathcal{R}_X)$ is a (possibly reducible) separated real algebraic variety in the sense of [8], there exists a reduced scheme $(Y,\,\mathcal{O}_Y)$ which satisfies the above conditions. Contrary, if $(X,\,\mathcal{R}_X)$ is a reduced real algebraic quasi-variety as Definition 5.1, then $(X,\,\mathcal{R}_X)$ is a real algebraic variety in the sense of [8]. Definition 5.1 may not be so clear, the author wishes someone will give more nice definition.

5.2. Semialgebraic quasi-variety.

Definition 5.2.(Semialgebraic quasi-variety) A locally ringed space $(A,\,\mathcal{R}_A)$ is called *semialgebraic quasi-variety*, if there exists a real algebraic quasi-variety $(X,\,\mathcal{R}_X)$ and a finite affine open covering $\{V_i\}_{i=1}^r$ of X which satisfies the following:
(1) There exists an injective morphism $\iota\colon(A,\,\mathcal{R}_A)\longrightarrow(X,\,\mathcal{R}_X)$ as locally ringed spaces, and ι induces a homeomorphism $A\to\iota(A)$ as Euclidean spaces. Moreover, $\iota(A)$ is a semialgebraic subset of X, i.e. $\iota(A)\cap V_i$ is a semialgebraic subset of V_i for each $i=1,\ldots,r$.
(2) $\mathrm{Zar}_X(A)=X$.

(3) Take an arbitral $i \in \{1, 2, \ldots, r\}$, and take any Euclidean open subset $U \subset \iota^{-1}(V_i)$. Put $R_i := \mathfrak{R}_{V_i}(V_i)$. For a point $P \in \iota(U)$, let $\mathfrak{m}_P$ be the maximal ideal of R_i corresponding to P, and let

$$S_U := \bigcap_{P \in U} \big(R_i - \mathfrak{m}_P\big) \subset R_i.$$

Then $\iota^* : S_U^{-1} R_i \longrightarrow \mathfrak{R}_A(U)$ is an isomorphism of $\mathbb{R}$-algebra.

Moreover, if X is a real algebraic variety, then A is said to be an *semialgebraic variety.* In this case, the field of fractions $Q(\mathfrak{R}_A(U_i))$ is called the *field of rational functions*, and is denoted by $\mathrm{Rat}(A) := Q\big(\mathfrak{R}_A(U_i)\big)$.

The Zariski topology and the Euclidean topology on A are defined naturally. A semialgebraic quasi-variety A is called *irreducible* if it is irreducible with respect to the Zariski topology. A is said to be *reduced* if $\mathfrak{R}_{A,P}$ has no nilpotent elements except 0 for every $P \in A$. $\dim A$ is defined by $\dim A = \max_{P \in A} \operatorname{Krull} \dim \mathfrak{R}_{A,P}$. A is called *connected* if it is connected with respect to Euclidean topology. Note that A may not be connected even if A is irreducible. A is called *affine*, if we can choose X to be isomorphic to a closed Zariski subset of $\mathbb{R}^n$ for a certain n.

Notions about singularities of A are defined using $\mathfrak{R}_{A,P}$. Note that if Y is a $\mathbb{R}$-scheme which represents X, then $\mathfrak{R}_{A,P} \cong \mathfrak{O}_{Y,P}$. We denote

$$\mathrm{Sing}(A) := \big\{P \in A \bigm| \mathfrak{R}_{A,P} \text{ is not a regular local ring}\big\},$$
$$\mathrm{Reg}(A) := \mathrm{Int}(A) - \mathrm{Sing}(A).$$

A *regular map* or *holomorphic map* (resp. *isomorphism*) between semialgebraic quasi-varieties is defined as a morphism (resp. isomorphism) of locally ringed space.

We can choose a real algebraic quasi-variety X and a separated scheme Y of finite type over $\mathbb{R}$ so that $Y_{\mathbb{C}}$ is complete and Y represents X. Then, we say X is a *real envelope* of A, and $Y_{\mathbb{C}}$ is a *complex envelope* of A.

X and $Y_{\mathbb{C}}$ are not unique for A, but it is easy to see that:

Proposition 5.3. *Let A be a semialgebraic quasi-variety, $Y_{\mathbb{C}}$ and $Y'_{\mathbb{C}}$ be complex envelopes of A. Then $Y_{\mathbb{C}}$ and $Y'_{\mathbb{C}}$ are birational equivalent. If A is a semialgebraic variety, then* $\mathrm{Rat}(A) \otimes_{\mathbb{R}} \mathbb{C} = \mathrm{Rat}(Y_{\mathbb{C}})$.

This follows from Proposition 5.10 given later.

By this proposition, if $\nu(Y_{\mathbb{C}})$ is a certain birational invariant of complex algebraic varieties, then we can define $\nu(A) := \nu(Y_{\mathbb{C}})$ to be an invariant of A. Especially, when A is non-singular semialgebraic variety, we can choose Y to be non-singular projective, and we can define $h^i(A) := \dim_{\mathbb{C}} H^i(Y_{\mathbb{C}}, \mathfrak{O}_{Y_{\mathbb{C}}})$ and $P_m(A) := \dim_{\mathbb{C}} H^0(Y_{\mathbb{C}}, \mathfrak{O}_{Y_{\mathbb{C}}}(mK_{Y_{\mathbb{C}}}))$ for $m \geq 0$. Using $P_m(A)$, we can define the *Kodaira dimension* $\kappa(A)$,

Remark 5.4. (1) $\mathrm{Reg}(A) \neq \emptyset$ if A is reduced.

(2) $\mathrm{Reg}(A)$ is not always dense in A with respect to the Euclidean topology. For example, consider the case that A has an isolated singularity as a connected component.

(3) If $P \in \mathrm{Reg}(A) \cap \mathrm{Int}(A)$ and $\dim A = n$, then there exists an Euclidean open neighborhood $P \in U \subset A$ such that U is homeomorphic to an open subset of $\mathbb{R}^n$.

(4) By our definition, an isolated singular locus of A is included in $\mathrm{Int}(A)$. But $\mathrm{Sing}(A)$ sometimes acts as if it is a boundary. So it will be safe to discuss $\mathrm{Int}(A) \cap \mathrm{Reg}(A)$.

In complex algebraic geometry, a subscheme is a closed subscheme of an open subscheme. But to define semialgebraic subvarieties, we must be careful. For example, any semialgebraic subset B of a real algebraic variety A, must be able to be treated as semialgebraic quasi-subvariety of A.

Definition 5.5.(Image of a regular map) Let A, B be semialgebraic quasi-varieties, and $\varphi: A \to B$ be a regular map. Let $C := \varphi(B)$. By Tarski-Seidenberg theorem, C is a semialgebraic subset of B. We define $\mathfrak{R}_C$ as the following:

We may assume A and B are affine, since definition of $\mathfrak{R}_C$ is local. Let $R_A := \mathfrak{R}_A(A)$, $R_B := \mathfrak{R}_B(B)$, and $\varphi^*: R_B \to R_A$ be the homomorphism induced by φ. We put $R := R_B / \operatorname{Ker} \varphi^*$. Note that R defines $\operatorname{Zar}_B(C)$. For a point $P \in C$, there exists the unique maximal ideal $\mathfrak{m}_P \subset R$ corresponding to P. Put $S := \bigcap_{P \in C} (R - \mathfrak{m}_P)$, and $R_C := S^{-1}R$. Note that R_C is a R_B-module. The structure sheaf of C is defined by $\mathfrak{R}_C := \widetilde{R_C}$ which is the coherent $\mathfrak{R}_B$-module defined by R_C.

$(C, \mathfrak{R}_C)$ is called the *image* of φ, and simply denoted by $C = \varphi(A)$.

Definition 5.6.(Semialgebraic quasi-subvariety) Let A, B be semialgebraic quasi-varieties. A morphism $\varphi : (B, \mathfrak{R}_B) \longrightarrow (A, \mathfrak{R}_A)$ is called an *immersion*, if φ induces an isomorphism $B \longrightarrow \varphi(B)$.

If B is a semialgebraic subset of A, and the inclusion map $B \to A$ is an immersion, then B is called a semialgebraic *quasi-subvariety* of A.

If A is a semialgebraic quasi-variety, and $B \subset A$ be a semialgebraic subset. Then, there exists a unique sheaf of rings $\mathfrak{R}_B$ such that $(B, \mathfrak{R}_B)$ is a semialgebraic quasi-subvariety of $(A, \mathfrak{R}_A)$ and $(B, \mathfrak{R}_B)$ is reduced. $\mathfrak{R}_B$ is called the *reduced structure* of $B \subset A$.

Assume that A, B, C are non-singular semialgebraic varieties such that $A = B \cup C$, and $P \in B \cap C$. It may happen that $\mathfrak{R}_{B,P} \ncong \mathfrak{R}_{C,P}$. It is easy to see that $\mathfrak{R}_{A,P}$ agree with one of $\mathfrak{R}_{B,P}$ and $\mathfrak{R}_{C,P}$.

Definition 5.7.(Fibre product) Let A, B, C be semialgebraic quasi-varieties, and $f: A \to C$, $g: B \to C$ be regular maps. The *fiber product* $A \times_C B$ is a semialgebraic set

$$A \times_C B = \{(a, b) \in A \times B \mid f(a) = g(b)\}$$

with a structure sheaf $\mathfrak{R}_A \otimes_{\mathfrak{R}_C} \mathfrak{R}_B$.

Definition 5.8.(Inverse image) Let A, B be semialgebraic quasi-varieties, and $\varphi: A \to B$ be a regular map. Let $C \subset B$ be a semialgebraic quasi-subvariety. The *inverse image* $\varphi^{-1}(C)$ is defined as the fiber product $\varphi^{-1}(C) := A \times_B C$.

Definition 5.9.(Birational map) Let A, B be semialgebraic quasi-varieties. If there exists Zariski open subsets $U \subset A$ and $W \subset B$ such that $\operatorname{Zar}_A(U) = A$, $\operatorname{Zar}_B(W) = B$ and there exists a regular map $\varphi: U \to W$, then we say that there exists a *rational map* $\varphi: A \cdots \to B$. Moreover, if $\varphi: U \to W$ is an isomorphism, we say that $\varphi: A \cdots \to B$ is a *birational map*, and A and B are *birational equivalent*.

Proposition 5.10. *Let A, B be semialgebraic quasi-varieties, and let X, Y be complex envelopes of A, B.*

(1) *If there exists a rational map $\varphi: A \cdots \to B$, then there exists a rational map $\Phi: X_{\mathbb{C}} \cdots \to Y_{\mathbb{C}}$ such that $\Phi|_A = \varphi$.*
(2) *In (1), if φ is a birational map, then Φ is a birational map.*

Proof. (1) We may assume φ is a regular map. Take a point $P \in \mathrm{Int}(A)$ such that $Q := \varphi(P) \in \mathrm{Int}(B)$, and take an affine open subset $W \subset Y$ such that $Q \subset W$.

We can choose $f_1, \ldots, f_r \in \mathcal{R}_{Y,Q}$ such that we can regard $f_i \in \mathcal{O}_Y(W)$ and $\mathcal{O}_Y(W) = \mathbb{C}[f_1, \ldots, f_r]$. Put $g_j := \varphi^*(f_j) \in \mathcal{R}_{A,P}$. We can find an affine open subset $U \subset X_{\mathbb{C}}$ such that $g_1, \ldots, g_r$ are holomorphic (regular) on U, and that $U \cap X$ is dense in X and $U \cap A$ is dense in A. Then, $\psi^*: \mathcal{R}_B \to \mathcal{R}_A$ induces $\Psi^*: \mathcal{O}_Y(W) \longrightarrow \mathcal{O}_X(U)$. Ψ^* induces a rational map $\Phi: X \cdots \to Y$.

(2) is easy. □

5.3. Some properties of semialgebraic quasi-varieties.

A notion of semialgebraic quasi-varieties brings some merits to Real Algebraic Geometry.

Theorem 5.11. *Every semialgebraic quasi-variety is affine. In other words, if A is a semialgebraic quasi-variety, then there exists $n \in \mathbb{N}$ and an immersion $\iota: A \to \mathbb{R}^n$.*

Proof. Let A be a semialgebraic quasi-variety. We can take a real envelope X of A. Take an affine open covering $\{V_1, \ldots, V_r\}$ of X. Fix a $1 \le j \le r$. We may assume V_j is a closed subset of $\mathbb{R}^n$. Let $(x_1, \ldots, x_n)$ be the coordinate system of $\mathbb{R}^n$, and $s_i := 1/(x_i^2 + 1)$, $t_i := x_i/(x_i^2 + 1)$. For $P \in X - V_j$, we put $s_i(P) = 0$ and $t_i(P) = 0$. Then s_i and t_i are regular functions on X. The set of functions $F_j := \{s_i, t_i \mid 1 \le i \le n\}$ defines a map $\Phi_j: X \longrightarrow \mathbb{R}^{2n}$. This Φ_j is a regular map as semialgebraic quasi-varieties, and $\Phi_j|_{V_j}: V_j \longrightarrow \mathbb{R}^{2n}$ is an immersion. Note that $\Phi_j(X)$ is a semialgebraic quasi-variety but is not always algebraic quasi-variety. Put $F := F_1 \cup \cdots \cup F_r$ and $N := \#F$. F defines a regular map $\Phi: X \to \mathbb{R}^N$, and F is an immersion as semialgebraic quasi-varieties. □

Remark 5.12. A real algebraic variety is an affine semialgebraic variety, but is not always a real affine variety. For example, $\mathbb{R}^2 - \{(0,0)\}$ is not a real affine variety.

Corollary 5.13. *Let A be a semialgebraic quasi-variety (or a real algebraic quasi-variety) and put $R_A := \mathcal{R}_A(A)$. Then, $\mathcal{R}_A$ is the sheaf obtained as $\widetilde{R_A}$.*

Note that R_A is a Noetherian ring, but is not finitely generated over $\mathbb{R}$ if $\dim A \ge 1$. Each maximal ideal of R_A corresponds to a certain point of A.

Corollary 5.14. *Let A be a semialgebraic quasi-variety (or a real algebraic quasi-variety) and $\mathcal{F}$ be a quasi-coherent $\mathcal{R}_A$-module. Then, $H^i(A, \mathcal{F}) = 0$ for all $i > 0$.*

Proof. There exists an immersion $\iota: A \to \mathbb{R}^n$. As Definition 5.5, there exists a closed real algebraic quasi-subvariety $X \subset \mathbb{R}^n$ such that X is real envelope of A. Let $R_X := \mathcal{R}_X(X)$ and $R_A := \mathcal{R}_A(A)$. We can present as $R_A = S_A^{-1} R_X$ by a certain multiplicatively closed set S_A. Since R_A is an R_X-module, $\mathcal{F}$ is a quasi-coherent $\mathcal{R}_X$-module. Thus, $\mathcal{F}$ is a quasi-coherent $\mathcal{R}_{\mathbb{R}^m}$-module. Thus we have

$$H^i(A, \mathcal{F}) \cong H^i\big(\mathbb{R}^m, \mathcal{F}\big) = 0$$

(see [16] Chap.III, Theorem 3.5). □

By the way, birational geometries of complex and real algebraic varieties are very different. In a complete complex algebraic variety, exceptional subsets are special subsets. This is not true for complete real algebraic varieties.

Theorem 5.15. *Let A be a semialgebraic quasi-variety, $E \subset A$ be a closed semialgebraic subset such that $E = \mathrm{Zar}_A(E) \subsetneqq A$. Then there exists a semialgebraic quasi-variety B and a regular surjective morphism $\varphi: A \to B$ such that $P := \varphi(E)$ is a point and that $\varphi|_{A-E} : (A - E) \longrightarrow (B - P)$ is an isomorphism, i.e. φ is a contraction of E to a point P.*

Proof. We may assume $A \subset \mathbb{R}^n$. Let $f_1, \ldots, f_r$ be defining polynomials of $\mathrm{Zar}_{\mathbb{R}^n}(E)$ in $\mathbb{R}[x_1, \ldots, x_n]$. Consider a map $\Phi: \mathbb{R}^n \to \mathbb{R}^{rn}$ defined by linear system with the base $\{x_i f_j \mid 1 \leq i \leq n, 1 \leq j \leq r\}$. Φ is a regular map. Put $B := \Phi(A)$ and $\varphi := \Phi|_A: A \to B$. Then, B and φ satisfy the conclusion of the Proposition. □

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