SOME EXTREMAL SYMMETRIC INEQUALITIES

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Abstract. Let $\mathcal{H}_{n,d} := \mathbb{R}[x_1,\ldots, x_n]_d$ be the set of all the homogeneous polynomials of degree d, and let $\mathcal{H}^s_{n,d} := \mathcal{H}^{\mathfrak{S}_n}_{n,d}$ be the subset of all the symmetric polynomials. For a semialgebraic subset of $A \subset \mathbb{R}^n$ and a vector subspace $\mathcal{H} \subset \mathcal{H}_{n,d}$, we define a PSD cone $\mathcal{P}(A, \mathcal{H})$ by $\mathcal{P}(A, \mathcal{H}) := \{f \in \mathcal{H} \mid f(a) \geq 0\ (\forall a \in A)\}$. In this article, we study a family of extremal symmetric polynomials of $\mathcal{P}_{3,6} := \mathcal{P}(\mathbb{R}^3, \mathcal{H}_{3,6})$ and that of $\mathcal{P}_{4,4} := \mathcal{P}(\mathbb{R}^4, \mathcal{H}_{4,4})$. We also determine all the extremal polynomials of $\mathcal{P}^{s+}_{3,5} := \mathcal{P}(\mathbb{R}^3_+, \mathcal{H}^s_{3,5})$ where $\mathbb{R}_+ := \{x \in \mathbb{R},\ x \geq 0\}$. Some of them provide extremal polynomials of $\mathcal{P}_{3,10}$.

1. Introduction

First, we should explain what an extremal inequality is. Let $\mathcal{H}_{n,d} := \mathbb{R}[x_1,\ldots, x_n]_d$ (the part of degree d), and $\mathcal{H} \subset \mathcal{H}_{n,d}$ be a vector subspace. For a semialgebraic subset A of $\mathbb{R}^n$ or $\mathbb{P}^{n-1}_{\mathbb{R}}$, the closed convex cone

$$\mathcal{P}(A, \mathcal{H}) := \{f \in \mathcal{H} \mid f(a) \geq 0 \text{ for all } a \in A\}$$

is called the PSD cone on A in $\mathcal{H}$. PSD means Positive Semi-Definite. This is a semialgebraic set whose boundary is a finite union of irreducible semialgebraic sets (see [2, Theorem 2.7]). An element of $\mathcal{P}(A, \mathcal{H})$ can be regarded as an inequality on A. In general, for a closed convex cone $\mathcal{P}$, a half line $\mathbb{R}_+ \cdot f$ ($f \in \mathcal{P} - \{0\}$) or $\mathbb{R}_{++} \cdot f$ is called an *extremal ray*, if $g, h \in \mathcal{P}$ satisfy $g + h = f$ then $g, h \in \mathbb{R}_+ \cdot f$, where $\mathbb{R}_+ := \{x \in \mathbb{R} \mid x \geq 0\}$ and $\mathbb{R}_{++} := \mathbb{R}_+ - \{0\}$. In this case, f is called an *extremal* element of $\mathcal{P}$. The set of all the extremal elements of $\mathcal{P}$ is denoted by $\mathcal{E}(\mathcal{P})$.

The notion of 'extremal' is relative. When $\mathcal{H}' \subset \mathcal{H}$ is a vector subspace, $\mathcal{E}(\mathcal{P}(A, \mathcal{H}')) \not\subset \mathcal{E}(\mathcal{P}(A, \mathcal{H}))$ may occur. But it is useful to study $\mathcal{E}(\mathcal{P}(A, \mathcal{H}'))$ to find relations of $\mathcal{P}(A, \mathcal{H}')$ and $\mathcal{P}(A, \mathcal{H})$. The set $\mathcal{E}(\mathcal{P}(A, \mathcal{H}))$ also depends on A. When $B \subset A$ is a semialgebraic subset, $\mathcal{E}(\mathcal{P}(A, \mathcal{H})) \not\subset \mathcal{E}(\mathcal{P}(B, \mathcal{H}))$ may also occur.

When $A = \mathbb{R}^n$ or $A = \mathbb{R}^n_+$, there are many cases that we have better to study $\mathcal{P}(\mathbb{P}^{n-1}_{\mathbb{R}}, \mathcal{H})$ or $\mathcal{P}(\mathbb{P}^{n-1}_{+}, \mathcal{H})$ instead of $\mathcal{P}(\mathbb{R}^n, \mathcal{H})$ or $\mathcal{P}(\mathbb{R}^n_+, \mathcal{H})$. One of reasons is as follows. For $f \in \mathcal{H}_{n,d}$ and $K = \mathbb{R}$ or $\mathbb{C}$, we denote

$$V_K(f) := \{\mathbf{x} \in \mathbb{P}^{n-1}_K \mid f(\mathbf{x}) = 0\}.$$

In the theory of inequalities, elements of $V_{\mathbb{R}}(f)$ are treated as 'equality conditions'. In many cases some points of $V_{\mathbb{R}}(f)$ are singular points of $V_{\mathbb{C}}(f)$, if $f \in \mathcal{P}(A, \mathcal{H})$. Especially, when f is an irreducible polynomial, the structure of an algebraic variety $V_{\mathbb{C}}(f)$, or the structure of singular points in $V_{\mathbb{R}}(f)$ plays an important role for study of inequality $f \geq 0$. When f is extremal, $V_{\mathbb{R}}(f)$ usually contains many points. The set $V_{\mathbb{R}}(f)$ often determines f itself. This fact is recognized at least from [10]. For more details, please see [6] and [3, §2].

There is an another reason. Consider the case that a finite group G (for example, the symmetric group $G = \mathfrak{S}_n$) acts on A, and $\mathcal{H}$ is a G-invariant set $\mathcal{H}^G = \mathcal{H}$.

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Then $\mathcal{P}(A, \mathcal{H})$ can be identified with $\mathcal{P}(A/G, \mathcal{H})$. In many cases, the structure of A/G plays an important role to study $\mathcal{E}(\mathcal{P}(A, \mathcal{H}))$. In our cases, the structure of $\mathbb{P}^{n-1}_{\mathbb{R}}/\mathfrak{S}_n$ or $\mathbb{P}^{n-1}_{+}/\mathfrak{S}_n$ is more essential than that of $\mathbb{R}^n/\mathfrak{S}_n$ or $\mathbb{R}^n_+/\mathfrak{S}_n$ (see §4.2).

By the way, PSD cones $\mathcal{P}_{n,2d} := \mathcal{P}(\mathbb{R}^n, \mathcal{H}_{n,2d})$ are studied in many articles with interest for SOS problem. An element $f \in \mathcal{P}_{n,2d}$ is called *SOS* (Sum Of Squares), if there exists $r \in \mathbb{N}$ and $g_1,\ldots, g_r \in \mathcal{H}_{n,d}$ such that $f = g_1^2 + \cdots + g_r^2$. The set of all the SOS elements of $\mathcal{P}_{n,2d}$ is denoted by $\Sigma_{n,2d}$, and is called a *SOS cone*. Hilbert proved that $\mathcal{P}_{n,2d} = \Sigma_{n,2d}$ if and only if $(n, d) = (3, 2)$ or $d = 1$ or $n \le 2$ ([13]). In many articles, $\mathcal{P}_{n,2d} - \Sigma_{n,2d}$ are studied, but I feel that not so many elements of $\mathcal{P}_{n,2d} - \Sigma_{n,2d}$ are known yet. One of reasons will be that $\dim \mathcal{H}_{n,2d}$ is too large to proceed precise analysis. Studies on $\mathcal{P}(A, \mathcal{H})$ for some small $\mathcal{H}$ often bring new results.

The set of symmetric polynomials $\mathcal{H}^s_{n,d} := \mathcal{H}^{\mathfrak{S}_n}_{n,d}$ is one of nice vector subspace which is easy to treat. For example, a nice condition to distinguish PSD is provided in [15]. By our experience, the equality condition $f(a,\ldots, a) = 0$ (i.e. $f(1,\ldots, 1) = 0$) also often makes situation simple. Now, we fix some symbols. Let

$$\mathcal{H}^s_{n,d} := \{f \in \mathcal{H}_{n,d} \mid f(x_{\sigma(1)}, \ldots, x_{\sigma(n)}) = f(x_1, \ldots, x_n) \text{ for all } \sigma \in \mathfrak{S}_n\},$$
$$\mathcal{H}^0_{n,d} := \{f \in \mathcal{H}_{n,d} \mid f(a, a, \ldots, a) = 0 \text{ for all } a \in \mathbb{R}\},$$

and $\mathcal{H}^{s0}_{n,d} := \mathcal{H}^s_{n,d} \cap \mathcal{H}^0_{n,d}$. We denote $\mathcal{P}_{n,d} := \mathcal{P}(\mathbb{R}^n, \mathcal{H}_{n,d})$, $\mathcal{P}^+_{n,d} := \mathcal{P}(\mathbb{R}^n_+, \mathcal{H}_{n,d})$, $\mathcal{P}^s_{n,d} := \mathcal{P}(\mathbb{R}^n, \mathcal{H}^s_{n,d})$, $\mathcal{P}^{s+}_{n,d} := \mathcal{P}(\mathbb{R}^n_+, \mathcal{H}^s_{n,d})$, $\mathcal{P}^{s0}_{n,d} := \mathcal{P}(\mathbb{R}^n, \mathcal{H}^{s0}_{n,d})$, and $\mathcal{P}^{s0+}_{n,d} := \mathcal{P}(\mathbb{R}^n_+, \mathcal{H}^{s0}_{n,d})$. The rule of indexing will be clear. "s" means symmetric, "0" means an equality condition $f(a,\ldots, a) = 0$, and "+" means $A = \mathbb{R}^n_+$. These symbols are used in [1, 2]. In [7], $\mathcal{P}^s_{n,2d}$ is denoted by $\mathcal{P}^S_{n,2d}$. The symbols $\mathcal{H}^e_{n,2d} := \mathcal{H}_{n,2d} \cap \mathbb{R}[x_1^2,\ldots, x_n^2]$ and $\mathcal{P}^e_{n,2d} := \mathcal{P}_{n,2d} \cap \mathcal{H}^e_{n.2d}$ (even PSD cone) are also often used.

Note that if $f \in \mathcal{E}(\mathcal{P}_{n,2d})$, then there exists $\mathbf{a} \in \mathbb{R}^n$ such that $f(\mathbf{a}) = 0$. By a linear bijective map $\varphi\colon \mathbb{R}^n \to \mathbb{R}^n$ such that $\varphi(1,\ldots, 1) = \mathbf{a}$, we have $f\circ\varphi \in \mathcal{E}(\mathcal{P}^0_{n,2d})$. Moreover, $\mathcal{E}(\mathcal{P}^0_{n,2d}) \subset \mathcal{E}(\mathcal{P}_{n,2d})$ holds. Thus, a study of $\mathcal{E}(\mathcal{P}^0_{n,2d})$ is useful to study $\mathcal{E}(\mathcal{P}_{n,2d})$.

About the cone $\mathcal{P}^s_{n,2d}$ of PSD symmetric forms, there are many studies relating $\Sigma_{n,2d}$. Many famous elements of $\mathcal{P}_{n,2d} - \Sigma_{n,2d}$ are found out from $\mathcal{P}^{s0}_{n,2d}$ or $\mathcal{P}^s_{n,2d}$. So, symmetric inequalities are studied in many artcles with special interests (for example [12, 15, 16, 17, 18]).

When d is odd, there are a few studies about $\mathcal{P}^{s+}_{n,d}$. But the cone $\mathcal{P}^{s+}_{n,d}$ is also useful, since $\mathcal{P}^+_{n,d} \cong \mathcal{P}^e_{n,2d}$ and $\mathcal{P}^{s+}_{n,d} \cong \mathcal{P}^{es}_{n,2d} := \mathcal{P}^e_{n,2d} \cap \mathcal{P}^s_{n,2d}$, by the corresponding $f(x_1,\ldots, x_n) \longrightarrow f(x_1^2,\ldots, x_n^2)$.

As is already commented, $\mathcal{E}(\mathcal{P}^s_{n,2d}) \subset \mathcal{E}(\mathcal{P}_{n,2d})$ is not always correct. But $\mathcal{E}(\mathcal{P}^{s0}_{n,2d}) \subset \mathcal{E}(\mathcal{P}^s_{n,2d})$ and $\mathcal{E}(\mathcal{P}^{s0+}_{n,d}) \subset \mathcal{E}(\mathcal{P}^{s+}_{n,d})$ always hold. This is one of the reason why we study $\mathcal{P}^{s0}_{n,2d}$ and $\mathcal{P}^{s0+}_{n,d}$.

We review easy cases that $n = 3$ and d is small. Let

$$S_i := x^i + y^i + z^i, \quad S_{i,j} := x^i y^j + y^i z^j + z^i x^j, \quad T_{i,j} := S_{i,j} + S_{j,i},$$

and $U := xyz$. The following proposition will be well known.

Proposition 1.1. *The three dimensional PSD cone $\mathcal{P}^{s+}_{3,3}$ is a triangular cone which has three extremal rays. Each edge of $\mathcal{E}(\mathcal{P}^{s+}_{3,3})$ is generated by one of $f_1^{3,s} := T_{2,1} - 6U$, $f_2^{3,s} := S_3 + 3U - T_{2,1}$ or $f_3^{3,s} := U$. The polynomials $f_i^{3,s}$ are characterized in $\mathcal{P}^{s+}_{3,3}$ by the equality conditions $f_1^{3,s}(1,0,0) = f_1^{3,s}(1,1,1) = 0$, $f_2^{3,s}(1,1,0) = f_2^{3,s}(1,1,1) = 0$ and $f_3^{3,s}(1,0,0) = f_3^{3,s}(1,1,0) = 0$.*

The sentence '$f \in \mathcal{P}$ is characterized by the condition $(*)$' means that if $g \in \mathcal{P}$ satisfies the condition $(*)$ then there exists $\alpha \geq 0$ such that $g = \alpha f$.

The inequality $f_2^{3,s} \geq 0$ is called Schur's inequality of degree 3. Note that $f_1^{3,s}$, $f_2^{3,s}$, $f_3^{3,s} \in \mathcal{E}(\mathcal{P}_{3,3}^{+})$. Thus $\mathcal{E}(\mathcal{P}_{3,3}^{s+}) \subset \mathcal{E}(\mathcal{P}_{3,3}^{+})$. Note that all the elements of $\mathcal{E}(\mathcal{P}_{3,3}^{+})$ are determined in [3]. It is also proved that if $f(x,y,z) \in \mathcal{E}(\mathcal{P}_{3,3}^{+})$, then $f(x^2,y^2,z^2) \in \mathcal{E}(\mathcal{P}_{3,6})$ ([3, Theorem 1.7]). If $f(x,y,z) \in \mathcal{E}(\mathcal{P}_{3,3}^{+})$ is irreducible, then $f(x^2,y^2,z^2) \notin \Sigma_{3,6}$. So, the study of $\mathcal{E}(\mathcal{P}_{n,d}^{+})$ may bring us new aspects.

The following proposition follows from [1, Proposition 4.13].

Proposition 1.2. *Each extremal ray of the four dimensional PSD cone $\mathcal{P}_{3,4}^{s}$ is generated by one of the following polynomials:*
(1) $f_t^{4,s} := S_4 - (t+1)T_{3,1} + (t^2+2t)S_{2,2} - (t^2-1)US_1$ *where* $t \in \mathbb{R}$.
(2) $f_\infty^{4,s} := S_{2,2} - US_1$.
(3) $\mathfrak{e}_k^X := (kS_2 - S_{1,1})^2$ *where* $-1/2 \leq k \leq 1$.

The polynomial $f_t^{4,s}$ is characterized in $\mathcal{P}_{3,4}^{s}$ by the equality condition $f_t^{4,s}(t,1,1) = f_t^{4,s}(1,1,1) = 0$.

Note that the inequality $f_0^{4,s} \geq 0$ is the Schur's inequality of degree 4. We should mention that $\mathfrak{e}_k^X \in \mathcal{E}(\mathcal{P}_{4,4})$ but $f_t^{4,s}$, $f_\infty^{4,s} \notin \mathcal{E}(\mathcal{P}_{4,4})$. For example, $f_t^{4,s} \notin \mathcal{E}(\mathcal{P}_{4,4})$ since

$$6f_t^{4,s}(x_1,x_2,x_3) = \sum_{i=1}^{3} \left(2x_i^2 - x_{i+1}^2 - x_{i+2}^2 - (t+1)(x_ix_{i+1} + x_ix_{i+2} - 2x_{i+1}x_{i+2})\right)^2,$$

where $x_{i+3} := x_i$. Moreover, $f_t^{4,s}(x,y,z)$ is a product of two imaginal quadratic polynomials. The following proposition follows from [1, Therem 4.10].

Proposition 1.3. *Each extremal ray of the four dimensional PSD cone $\mathcal{P}_{3,4}^{s+}$ is generated by one of the following polynomials: $f_t^{4,s}$ $(t \geq 0)$, $f_\infty^{4,s}$, $\mathfrak{e}_k^X$ $(0 \leq k \leq 1)$, $T_{3,1} - 2S_{2,2}$ or US_1.*

In §4 of this article, we determine all the elements of $\mathcal{E}(\mathcal{P}_{3,5}^{s+})$. Since the definitions of extremal polynomials $\mathfrak{e}_{t,u}^A$, $\mathfrak{e}_{t,u}^B$, $\mathfrak{e}_t^C$, $\mathfrak{e}_t^D$ and $\mathfrak{e}_t^E$ are long, we give them in §4.1.

Theorem 1.4. *Each extremal ray of the five dimensional PSD cone $\mathcal{P}_{3,5}^{s+}$ is generated by one of the following polynomials: $\mathfrak{e}_{t,u}^A$ $(0 \leq t \leq 7,\ 0 \leq u \leq \mu_A(t))$, $\mathfrak{e}_{t,u}^B$ $(t \geq 2,\ \mu_B(t) \leq u \leq 1)$, $\mathfrak{e}_t^C$ $(0 \leq t \leq 2)$, $\mathfrak{e}_t^D$ $(t \in [0,\infty])$, $\mathfrak{e}_t^E$ $(t \in [7,\infty])$ or $U(S_2 - S_{1,1})$. Polynomials $\mathfrak{e}_{t,u}^A$, $\mathfrak{e}_{t,u}^B$, $\mathfrak{e}_t^C$, $\mathfrak{e}_t^D$ and $\mathfrak{e}_t^E$ are characterized in $\mathcal{P}_{3,5}^{s+}$ by the following conditions for general t and u:*

$$\begin{aligned}
\mathfrak{e}_{t,u}^A(t,1,1) &= \mathfrak{e}_{t,u}^A\left(\frac{(t+2)(7-t)-u}{(t+2)(5t+1)},\ 1,\ 1\right) = 0,\\
\mathfrak{e}_{t,u}^B(t,1,1) &= \mathfrak{e}_{t,u}^B(0,u,1) = \frac{\partial \mathfrak{e}_{t,u}^B}{\partial y}(0,u,1) = 0,\\
\mathfrak{e}_t^C(t,1,1) &= \mathfrak{e}_t^C(1,1,1) = \mathfrak{e}_t^C(0,1,1) = 0,\\
\mathfrak{e}_t^D(t,1,1) &= \mathfrak{e}_t^D(1,1,1) = \mathfrak{e}_t^D(0,0,1) = 0,\\
\mathfrak{e}_t^E(t,1,1) &= \mathfrak{e}_t^E(0,1,1) = \mathfrak{e}_t^E(0,0,1) = 0.
\end{aligned}$$

We also prove that if $(u,\, t)$ satisfies certain conditions, then $\mathfrak{e}_{t,u}^B \in \mathcal{E}(\mathcal{P}_{3,5}^{+})$, and

$$\mathfrak{e}_{t,u}^B(x^2,\, y^2,\, z^2) \in \mathcal{E}(\mathcal{P}_{3,10}) - \Sigma_{3,10},$$

in Theorem 4.26 and 4.28. Note that the condition $(\partial \mathfrak{e}_{t,u}^B/\partial y)(0,u,1) = 0$ can be described using the notion of 'infinitely near zero' introduced [3, §2].

The cones $\mathcal{P}_{3,6}$ and $\mathcal{P}_{4,4}$ are studied with special interests (for example [11, 5]). The cones $\mathcal{P}^s_{3,6}$ and $\mathcal{P}^s_{4,4}$ are also studied in many articles (for example [9, 12]). Let $\mathcal{P}$ be a closed convex cone which contains no line. An element $f \in \mathcal{E}(\mathcal{P})$ is called an *exposed*, if there exists a hyperplane H of $\mathcal{H}$ such that $H \cap \mathcal{P} = \mathbb{R}_+ \cdot f$. For example, if $\mathcal{P}$ is an polyhedral convex cone, then all $f \in \mathcal{E}(\mathcal{P})$ are exposed. In [4], it is proved that if $f \in \mathcal{E}(\mathcal{P}_{3,6}) - \Sigma_{3,6}$ is exposed, then $V_{\mathbb{C}}(f)$ is an irreducible rational curve with 10 acnodes. All the extremal even sextics are determined in [3]. It provides many elements of $\mathcal{E}(\mathcal{P}_{3,6}) - \Sigma_{3,6}$. Some important symmetric elements of $\mathcal{E}(\mathcal{P}_{3,6})$ are also provided in [8, 12]. But, all the symmetric elements of $\mathcal{E}(\mathcal{P}_{3,6})$ are not determined yet. In §3 of this article, we prove the following theorem about the six dimensional PSD cone $\mathcal{P}^{s0}_{3,6}$.

Theorem 1.5. *There exists a non-empty open subset $\mathcal{U} \subset \mathbb{R}^3$ such that for every $(u,\ v,\ w) \in \mathcal{U}$ there exists $\mathfrak{f}_{u,v,w} \in \mathcal{P}^{s0}_{3,6}$ which satisfies the following (1), (2), (3) and (4):*

(1) $\mathfrak{f}_{u,v,w}(u,v,1) = \mathfrak{f}_{u,v,w}(w,1,1) = \mathfrak{f}_{u,v,w}(1,1,1) = 0$.
(2) $\mathfrak{f}_{u,v,w}$ *is irreducible in* $\mathbb{C}[x,y,z]$.
(3) $\mathfrak{f}_{u,v,w} \in \mathcal{E}(\mathcal{P}_{3,6}) - \Sigma_{3,6}$.
(4) $V_{\mathbb{C}}(\mathfrak{f}_{u,v,w})$ *is an irreducible rational curve which has 10 acnodes.*

The structure of $\mathcal{U}$ is very complicated to describe it. So, it will not be easy to determine all the symmetric elements of $\mathcal{E}(\mathcal{P}_{3,6})$.

Next, we consider the cases n=4. Let

$$
\begin{aligned}
&S^4_d := \sum_{i=1}^4 x_i^d,\\
&T^4_{p,q} := \sum_{i=1}^4 x_i^p(x_{i+1}^q + x_{i+2}^q + x_{i+3}^q),\\
&S^4_{p,p} := \sum_{1\le i<j\le 4} x_i^p x_j^p,\\
&T^4_{p,q,q} := \sum_{i=1}^4 x_i^p(x_{i+1}^q x_{i+2}^q + x_{i+1}^q x_{i+3}^q + x_{i+2}^q x_{i+3}^q),\\
&S^4_{p,p,p} := \sum_{i=1}^4 x_i^p x_{i+1}^p x_{i+2}^p,\\
&U^4 := x_1x_2x_3x_4
\end{aligned}
$$

wheher $x_{i\pm4} = x_i$. We also use (a,b,c,d) instead of (x_1,x_2,x_3,x_4). The following proposition is easy to prove but may not be well known. A proof will be given in §2.2.

Proposition 1.6. *The three dimensional PSD cone $\mathcal{P}^{s+}_{4,3}$ is a quadrangular cone which has four extremal rays. Each edge of $\mathcal{E}(\mathcal{P}^{s+}_{4,3})$ is generated by one of $g^{3,s}_1 := T^4_{2,1} - 3S^4_{1,1,1}$, $g^{3,s}_2 := 3S^4_3 + 3S^4_{1,1,1} - 2T^4_{2,1}$, $g^{3,s}_3 := S^4_{1,1,1}$, or $g^{3,s}_4 := S^4_3 + 3S^4_{1,1,1} - T^4_{2,1}$. These $g^{3,s}_i$ are characterized in $\mathcal{P}^{s+}_{4,3}$ by the equality conditions*

$$
\begin{aligned}
g^{3,s}_1(1,0,0,0) &= g^{3,s}_1(1,1,1,1) = 0,\\
g^{3,s}_2(1,1,1,0) &= g^{3,s}_2(1,1,1,1) = 0,\\
g^{3,s}_3(1,0,0,0) &= g^{3,s}_3(1,1,0,0) = 0,\\
g^{3,s}_4(1,1,0,0) &= g^{3,s}_4(1,1,1,0) = 0.
\end{aligned}
$$

Note that $g_1^{3,s}$, $g_2^{3,s}$, $g_3^{3,s} \notin \mathcal{E}(\mathcal{P}_{4,3}^+)$. But $g_4^{3,s} \in \mathcal{E}(\mathcal{P}_{4,3}^+)$, and

$$g_4^{3,s}(a^2, b^2, c^2, d^2) \in \mathcal{E}(\mathcal{P}_{4,6}) - \Sigma_{4,6}.$$

All the elements of $\mathcal{E}(\mathcal{P}_{4,4}^{s0})$ and $\mathcal{E}(\mathcal{P}_{4,4}^{s0+})$ are completely determined in [2].

Theorem 1.7. ([2, Theorem 1.2]) *Each extremal ray of the four dimensional PSD cone $\mathcal{P}_{4,4}^{s0}$ is generated by one of the following polynomials:*

$$\begin{aligned}
3\mathfrak{g}_t(a,b,c,d) &:= \big(a^2+b^2-c^2-d^2+(t+1)(cd-ab)\big)^2 \\
&\quad + \big(a^2-b^2+c^2-d^2+(t+1)(bd-ac)\big)^2 \\
&\quad + \big(a^2-b^2-c^2+d^2+(t+1)(bc-ad)\big)^2, \\
\mathfrak{g}_\infty(a,b,c,d) &:= (ab-cd)^2+(ac-bd)^2+(ad-bc)^2, \\
\mathfrak{p} &:= (a-b)^2(c-d)^2+(a-c)^2(b-d)^2+(a-d)^2(b-c)^2.
\end{aligned}$$

Here, $t \in \mathbb{R}$. Conversely, these are extremal elements of $\mathcal{P}_{4,4}^{s0}$.

$\mathfrak{g}_t$ *($t \neq 1, -3$) is characterized by the equality conditions $\mathfrak{g}_t(t, 1, 1, 1) = \mathfrak{g}_t(-1, -1, 1, 1) = 0$. $\mathfrak{g}_1$ is characterized by the equality conditions $\mathfrak{g}_1(x, x, 1, 1) = 0$ for all $x \in \mathbb{P}_{\mathbb{R}}^1$. $\mathfrak{g}_{-3}$ is characterized by the equality conditions $\mathfrak{g}_{-3}(a, b, c, -a-b-c) = 0$ for all a, b, $c \in \mathbb{R}$. $\mathfrak{g}_\infty$ is characterized by the equality conditions $\mathfrak{g}_\infty(0, 0, 0, 1) = \mathfrak{g}_\infty(-1, -1, 1, 1) = 0$.*

$\mathfrak{p}$ is characterized by the equality conditions $\mathfrak{p}(0, 0, 0, 1) = 1$ and $\mathfrak{p}(s, 1, 1, 1) = 0$ for all $s \in \mathbb{R}$.

Using this, we have $\mathcal{P}_{4,4}^{s0} \subset \Sigma_{4,4}$ and $\mathcal{E}(\mathcal{P}_{4,4}^{s0}) \cap \mathcal{E}(\mathcal{P}_{4,4}) = \emptyset$.

Theorem 1.8. ([2, Theorem 1.4]) *Each extremal ray of the four dimensional PSD cone $\mathcal{P}_{4,4}^{s0+}$ is generated by one of the following polynomials:*

$$\begin{aligned}
3\mathfrak{f}_t^{ab}(a,b,c,d) &:= 3S_4^4 - 2(t+1)T_{3,1}^4 + 2(2t-1)S_{2,2}^4 \\
&\quad + (t^2+3)T_{2,1,1}^4 - 12(t^2+1)U^4 \quad (0 \le t \le 5), \\
9\mathfrak{f}_t^{c}(a,b,c,d) &:= 9S_4^4 - 6(t+1)T_{3,1}^4 + (t^2+2t+19)S_{2,2}^4 \\
&\quad + 2(t^2+5t08)T_{2,1,1}^4 - 6(5t^2+10t-19)U^4 \quad (t \ge 5), \\
\mathfrak{p}(a,b,c,d) &:= S_{2,2}^4 - T_{2,1,1}^4 + 6U^4. \\
\mathfrak{q}_1(a,b,c,d) &:= T_{3,1}^4 - 2S_{2,2}^4, \\
\mathfrak{q}_2(a,b,c,d) &:= T_{2,1,1}^4 - 12U^4.
\end{aligned}$$

Conversely, these are extremal elements of $\mathcal{P}_{4,4}^{s0+}$.

$\mathfrak{f}_t^{ab}$ *($0 \le t < 1$ or $1 < t \le 5$) is characterized by the equality conditions*

$$\mathfrak{f}_t^{ab}(t,1,1,1) = \mathfrak{f}_t^{ab}(0,0,1,1) = 0.$$

$\mathfrak{f}_t^{c}$ *($t > 5$) is characterized by the equality conditions*

$$\mathfrak{f}_t^{c}(t,1,1,1) = \mathfrak{f}_t^{c}(0,0,u,1) = 0,$$

where $u \in \mathbb{R}_+$ is any root of $3u^2 - (t+1)u + 3 = 0$. $\mathfrak{f}_1^{ab}$ is characterized by the equality conditions $\mathfrak{f}_1^{ab}(t,t,1,1) = 0$ for all $t \ge 0$ and $\dfrac{\partial^2}{\partial a^2}\mathfrak{f}_1^{ab}(1,1,1,1) = 0$. $\mathfrak{q}_1$ is characterized by the equality conditions

$$\mathfrak{q}_1(1,1,1,0) = \mathfrak{q}_1(1,1,0,0) = \mathfrak{q}_1(1,0,0,0) = 0.$$

$\mathfrak{q}_2$ is characterized by the equality conditions $\mathfrak{q}_2(s,1,0,0) = 0$ for all $s \ge 0$.

By the above representation, we have $\mathfrak{p}(a^2,\, b^2,\, c^2,\, d^2),\, \mathfrak{q}_i(a^2,\, b^2,\, c^2,\, d^2) \in \Sigma_{4,8}$ $(i = 1,\, 2)$. But if $0 < t \leq 5$ and $t \neq 1$ then $\mathfrak{f}_t^{ab}(a^2,\, b^2,\, c^2,\, d^2) \notin \Sigma_{4,8}$, and if $t > 5$ then $\mathfrak{f}_t^{c}(a^2,\, b^2,\, c^2,\, d^2) \notin \Sigma_{4,8}$.

The set $\mathcal{P}_{4,4} - \Sigma_{4,4}$ is studied in many articles . Extremal elements of $\mathcal{P}_{4,4}$ have similar properties with that of $\mathcal{E}(\mathcal{P}_{3,6})$. If $f \in \mathcal{E}(\mathcal{P}_{4,4}) - \Sigma_{4,4}$ is irreducible, then $V_{\mathbb{C}}(f)$ is a K3-surface with 10 real rational double points of A_1-type (see [4]). In §2 of this article, we prove the following theorem about the five dimensional PSD cone $\mathcal{P}^s_{4,4}$.

Theorem 1.9. *There exists a non-empty open subset $\mathcal{U} \subset \mathbb{R}^2$ (this $\mathcal{U}$ is described in Theorem 2.5 and Theorem 2.6) and polynomials $\mathfrak{g}_{t,u}(a,b,c,d) \in \mathcal{P}^s_{4,4}$ for $(t,\, u) \in \mathcal{U}$ (this $\mathfrak{g}_{t,u}$ will be defined in Definition 2.1) which satisfy the following properties:*
(1) $\mathfrak{g}_{t,u}(t,1,1,1) = 0$ *and* $\mathfrak{g}_{t,u}(u,u,1,1) = 0$.
(2) $\mathfrak{g}_{t,u} \in \mathcal{E}(\mathcal{P}_{4,4}) - \Sigma_{4,4}$.
(3) $\mathfrak{g}_{t,u}$ *is irreducible in* $\mathbb{C}[a,b,c,d]$.
(4) $V_{\mathbb{R}}(\mathfrak{g}_{t,u})$ *is a set of 10 isolated points.*

As we have already mentioned, $\mathcal{E}(\mathcal{P}^{s0}_{4,4}) \subset \mathcal{E}(\mathcal{P}^s_{4,4}) \cap \Sigma_{4,4}$ and $\mathcal{E}(\mathcal{P}^{s0}_{4,4}) \cap \mathcal{E}(\mathcal{P}_{4,4}) = \emptyset$. But $\mathcal{E}(\mathcal{P}^s_{4,4}) \cap \big(\mathcal{E}(\mathcal{P}_{4,4}) - \Sigma_{4,4}\big) \neq \emptyset$ by the above theprem. This fact suggests that $\mathcal{E}(\mathcal{P}^s_{4,4})$ is very complicated.

2. Some extremal elements of $\mathcal{P}^s_{4,4}$

Among Theorem 1.4, 1.5 and 1.9, Theorem 1.9 is most easy to prove. So, we start from this.

2.1. Quartic polynomial $\mathfrak{g}_{t,u}$.

In this subsection, we prove Theorem 1.9. We have studied the structure of $\mathcal{P}^{s0}_{4,4}$ in [2]. It is fairly simple. But the structure of $\mathcal{P}^s_{4,4}$ is very complicated. We only provide here a family of extremal elements of $\mathcal{P}^s_{4,4}$. But these extremal elements will be interesting with a view of theory of K3 surfaces.

In this section, we use the following symbols. We denote the standard coordinate system of $\mathbb{P}^3_{\mathbb{R}}$ by $(a_0 \colon a_1 \colon a_2 \colon a_3)$. We also denote $a := a_0$, $b := a_1$, $c := a_2$, $d := a_3$. We choose the following $s_0, \ldots, s_4$ as a basis of $\mathcal{H}^s_{4,4}$:

$$\begin{aligned}
s_0(a,b,c,d) &:= S_4^4 - 4U^4 = a^4 + b^4 + c^4 + d^4 - 4abcd,\\
s_1(a,b,c,d) &:= T_{3,1}^4 - 12U^4 = \frac{1}{2}\sum_{\sigma \in \mathfrak{S}_4} a^2_{\sigma(0)} a_{\sigma(1)} - 12abcd,\\
s_2(a,b,c,d) &:= S_{2,2}^4 - 6U^4 = \sum_{0 \leq i < j \leq 3} a_i^2 a_j^2 - 6abcd,\\
s_3(a,b,c,d) &:= T_{2,1,1}^4 - 12U^4 = \frac{1}{2}\sum_{\sigma \in \mathfrak{S}_4} a^2_{\sigma(0)} a_{\sigma(1)} a_{\sigma(2)} - 12abcd,\\
s_4(a,b,c,d) &:= U^4 = abcd.
\end{aligned}$$

Note that $\{s_0,\, s_1,\, s_2,\, s_3\}$ is a base of $\mathcal{H}^{s0}_{4,4}$. Let $\mathbf{s}(a,b,c,d)$ be the vector $(s_0,\, s_1,\, s_2,\, s_3,\, s_4)$. We denote $\mathbf{s}_a := \left(\dfrac{\partial s_0}{\partial a}, \ldots, \dfrac{\partial s_4}{\partial a}\right)$ and so on.

Definition 2.1. For $t,\ w,\ u,\ a,\ b,\ c,\ d \in \mathbb{R}$, we put

$$\begin{aligned}
&\omega(u) := u + \frac{1}{u} - 2 = \frac{(u-1)^2}{u},\\
&p_0^G(t,w) := (4t+2)w^2 - 3(t-1)^2 w,\\
&p_1^G(t,w) := -2(t+1)^2 w^2 + 2(t+1)(t-1)^2 w,\\
&p_2^G(t,w) := 4t^2w^2 - 2(t-1)^2(2t-1)w + 2(t-1)^4,\\
&p_3^G(t,w) := 2(t+1)^2w^2 - (t-1)^2(t^2+3)w - 2(t-1)^4,\\
&p_4^G(t,w) := 2(t-1)^4 w^2,\\
&\mathfrak{g}_{t,u}(a,b,c,d) := u^2 \sum_{i=0}^{4} p_i^G(t,\ \omega(u)) s_i(a,b,c,d).
\end{aligned}$$

Note that if $(t,\ u) = (1,\ 1)$, then $\mathfrak{g}_{1,1} = 0$.

Theorem 2.2. *Let $t,\ u \in \mathbb{R}$. Aline 4 vectors $\mathbf{s}(t,1,1,1)$, $\mathbf{s}_a(t,1,1,1)$, $\mathbf{s}(u,u,1,1)$, $\mathbf{s}_a(u,u,1,1)$ and make a 4×5 matrix $A(t,u)$. Moreover, put $\mathbf{e}_1 = (1,0,0,0,0)$ at the top of $A(t,u)$, and make 5×5 matrix $B(t,u)$. Then*

$$\det B(t,u) = 3(t-1)^2(u^2-1)u^2 p_0^G(t,\ \omega(u)).$$

If $\det B(t,u) \neq 0$, then $\operatorname{Ker} A(t,u)$ is generated by $\mathfrak{g}_{t,u}$.

Proof. This follows from a direct calculation using Mathematica. □

By the above theorem, if $\mathfrak{g}_{t,u} \in \mathcal{P}_{4,4}^s$ and $\det B(t,u) \neq 0$, then $\mathfrak{g}_{t,u} \in \mathcal{E}(\mathcal{P}_{4,4}^s)$. As a special case of [15, Corollary 1.3] or [16, Corollary 2.1], we have the following Lemma. See also [17] and [18].

Lemma 2.3. *Let $f \in \mathcal{H}_{4,4}^s$. Then, $f \in \mathcal{P}_{4,4}^s$ if and only if the following (1) and (2) hold:*
(1) *$f(x,\ x,\ 1,\ 1) \geq 0$ for all $x \in \mathbb{R}$.*
(2) *$f(x,\ 1,\ 1,\ 1) \geq 0$ for all $x \in \mathbb{R}$.*

Theorem 2.4. *Let $V_F(t,w) := (3+6t-t^2)w^2 - 6(t-1)^2 w$. Assume that $u \neq 0$, 1. Then $\mathfrak{g}_{t,u} \in \mathcal{P}_{4,4}^s$ if and only if $V_F(t,\ \omega(u)) \geq 0$. Moreover, $-\mathfrak{g}_{t,u} \notin \mathcal{P}_{4,4}^s$ for any $t,\ u \in \mathbb{R}$.*

Proof. If $u \neq 0$, 1, then $w := \omega(u) \neq 0$. Let

$$\mathfrak{g}_{t,w}^*(a,b,c,d) := \sum_{i=0}^{4} p_i^G(t,w) s_i(a,b,c,d).$$

Then $\mathfrak{g}_{t,u}(a,b,c,d) = u^2 \mathfrak{g}_{t,\omega(u)}^*(a,b,c,d)$. Thus, $\mathfrak{g}_{t,u} \in \mathcal{P}_{4,4}^s$ if and only if $\mathfrak{g}_{t,\omega(u)}^* \in \mathcal{P}_{4,4}^s$. Since $\mathfrak{g}_{t,w}^*(1,0,0,0) = p_0^G(t,w)$, if $f \in \mathcal{P}_{4,4}^s$ then $p_0^G(t,w) \geq 0$. Since

$$2p_0^G(t,w) - V_F(t,w) = (t+1)^2 w^2 > 0,$$

if $V_F(t,w) \geq 0$ then $p_0^G(t,w) > 0$.

Since

$$\mathfrak{g}_{t,w}^*(x,x,1,1) = 2(t-1)^4\big(xw - (x-1)^2\big)^2 \geq 0,$$

(1) of Lemma 2.3 holds. This also proves that $-\mathfrak{g}_{t,u} \notin \mathcal{P}_{4,4}^s$ if $t \neq 1$. $-\mathfrak{g}_{1,u} \notin \mathcal{P}_{4,4}^s$ follows from $p_0^G(1,w) = 6w^2 > 0$.

On the other hand,

$$\mathfrak{g}^*_{t,w}(x,1,1,1) = (x-t)^2\left(p_0^G(t,w)\left(x-\frac{a_1(t,w)}{p_0^G(t,w)}\right)^2+\frac{(t-1)^2w^2V_F(t,w)}{p_0^G(t,w)}\right),$$

where $a_1(t,\ w) := (3+4t-t^2)w^2 - 3(t-1)^2w$. Thus, if $V_F(t,\ w)\geq 0$, then (2) of Lemma 2.3 holds. Conversely, consider the case $x = a_1(t,w)/p_0^G(t,w)$, $V_F(t,\ w)\geq 0$ must hold for (2). □

Theorem 2.5. *Let*

$$\begin{aligned} D_F(t,u) := (3+6t-t^2) &- (13+6t+5t^2)u \\ &+ (10+9t+4t^2+t^3)u^2 - (1+6t+t^2)u^3. \end{aligned}$$

Assume that $t\notin\{0,\ 1\}$, $u\notin\{0,\ \pm1\}$, $2u\neq t+1$, $tu+u\neq 2$, $D_F(t,\ u)\neq 0$ *and* $V_F(t,\ \omega(u))>0$. *Then* $\mathfrak{g}_{t,u}\in\mathcal{E}(\mathcal{P}_{4,4})$.

Proof. Let $e_i(a,b,c,d)$ $(i=1,\ldots,\ 35)$ be all the monic monomials of $\mathcal{H}_{4,4}$. Every $f\in\mathcal{H}_{4,4}$ can be written as $f = c_1e_1+\cdots+c_{35}e_{35}$ $(\exists c_i\in\mathbb{R})$.

Let $\mathbf{a}_1 := (t,1,1,1)$, $\mathbf{a}_2 := (1,t,1,1)$, $\mathbf{a}_3 := (1,1,t,1)$, $\mathbf{a}_4 := (1,1,1,t)$, $\mathbf{a}_5 := (u,u,1,1)$, $\mathbf{a}_6 := (u,1,u,1)$, $\mathbf{a}_7 := (u,1,1,u)$, $\mathbf{a}_8 := (1,u,u,1)$, $\mathbf{a}_9 := (1,u,1,u)$, $\mathbf{a}_{10} := (1,1,u,u)$. Consider the following 34 equations for f.

$$\begin{aligned} &f(\mathbf{a}_i)=0, \quad f_a(\mathbf{a}_i)=0, \quad f_b(\mathbf{a}_i)=0, \quad f_c(\mathbf{a}_i)=0 \quad (i=1,\ldots,\ 7),\\ &f(\mathbf{a}_8)=0, \quad f_a(\mathbf{a}_8)=0, \quad f_b(\mathbf{a}_8)=0,\\ &f(\mathbf{a}_9)=0, \quad f_a(\mathbf{a}_9)=0,\\ &f(\mathbf{a}_{10})=0. \end{aligned}$$

This system of equalities can be written using a 34×35 matrix $A_{t,u}$ and a vector $\mathbf{c}_f := {}^t(c_1,\ldots,\ c_{35})$ as $A_{t,u}\mathbf{c}_f=\mathbf{0}$. Note that if $f=\mathfrak{g}_{t,u}$, the condition $A\mathbf{c}_f=\mathbf{0}$ is satisfied. Thus, $\mathfrak{g}_{t,u}\in\operatorname{Ker}A_{t,u}$.

Let $B_{t,u}$ be the 35×35 matrix obtained by putting $\mathbf{e}_1=(1,\ 0,\ldots,\ 0)$ at the top of A. Then

$$\begin{aligned} \det B_{t,u} = \pm t(t-1)^{29}u^5(u-1)^{27}(u+1)^9(t-2u+1)^4(tu+u-2)^3 \\ \times p_0^G(t,\ \omega(u))V_F(t,\ \omega(u))D_F(t,u)^2. \end{aligned}$$

Rememver that if $V_F(t,\ w)>0$, then $p_0^G(t,\ w)>0$. Thus, under the given condition, we have $\det B_{t,u}\neq 0$. Therefore, $\dim\operatorname{Ker}A_{t,u}=1$ and $\operatorname{Ker}A_{t,u}=\mathbb{R}\cdot\mathfrak{g}_{t,u}$. We have $\mathfrak{g}_{t,u}\in\mathcal{P}_{4,4}$ by the previous theorem.

Assume that $\mathfrak{g}_{t,u}=f+g$ $(f,\ g\mathcal{P}_{4,4})$. Then $f,\ g\in\operatorname{Ker}A_{t,u}=\mathbb{R}\cdot\mathfrak{g}_{t,u}$. Thus we have $\mathfrak{g}_{t,u}\in\mathcal{E}(\mathcal{P}_{4,4})$. □

Theorem 2.6. *Assume that* $t\neq 1$, $u\notin\{0,\ \pm1\}$ *and* $V_F(t,\ \omega(u))>0$. *Moreover, we assume* $\mathfrak{g}_{t,u}\in\mathcal{E}(\mathcal{P}_{4,4})$. *then* $\mathfrak{g}_{t,u}(a,b,c,d)$ *is irreducible in* $\mathbb{C}[a,b,c,d]$ *and* $\mathfrak{g}_{t,u}\notin\Sigma_{4,4}$.

Proof. Let $\mathbf{a}_1 := (t,1,1,1),\ldots,\ \mathbf{a}_{10} := (1,1,u,u)$ be the same as in the proof of Theorem 2.5.

(1) We prove that if $t\neq 1$, $u\notin\{0,\ \pm1\}$ and $V_F(t,\ \omega(u))>0$, then there exists no quadric $g\in\mathbb{C}[a,b,c,d]$ such that $g(\mathbf{a}_i)=0$ for all $i=1,\ldots,\ 10$.

Let $q_1,\ldots,\ q_{10}$ be all the monic monomials of $\mathcal{H}_{4,2}$. Every $g\in\mathcal{H}_{4,2}$ can be written as $g=c_1q_1+\cdots+c_{10}q_{10}$ $(\exists c_i\in\mathbb{R})$. Consider the 10 equations $g(\mathbf{a}_i)=0$ for all $i=1,\ldots,\ 10$. This system of equalities can be written by a 10×10 matrix $B_{t,u}=\big(e_i(\mathbf{a}_j)\big)$, and a vector $\mathbf{c}_g := {}^t(c_1,\ldots,\ c_{10})$ as $B_{t,u}\mathbf{c}_g=\mathbf{0}$. Using PC, we have

$$\det B_{t,u} = \pm(t-1)^6(u-1)^5(u+1)^3u^2V_F(t,\ \omega(u))\neq 0.$$

Thus, $\operatorname{Ker} B_{t,u} = 0$ and we have (1).

(2) Note that if $f \in \mathcal{E}(\mathcal{P}_{4,4}) \cap \Sigma_{4,4}$, then there exists a quadric $g \in \mathcal{H}_{4,2}$ such that $f = g^2$. If $\mathfrak{g}_{t,u} = g^2$, then $\big(g(\mathbf{a}_i)\big)^2 = \mathfrak{g}_{t,u}(\mathbf{a}_i) = 0$. But this is impossible by (1). Thus, $\mathfrak{g} \notin \Sigma_{4,4}$.

(3) We shall show that $\mathfrak{g}_{t,u}$ is irreducible if $\mathfrak{g}_{t,u} \in \mathcal{E}(\mathcal{P}_{4,4})$. Assume that $\mathfrak{g}_{t,u} = gh$ ($\exists g,\ h \in \mathbb{C}[a,b,c,d] - \mathbb{C}$) with $\deg g \le \deg h$. Then $\deg g \le 2$. As is well known, g and h are homogeneous.

(3-1) Consider the case $\deg g = 2$ and $\alpha g \notin \mathbb{R}[a,b,c,d]$ for any $\alpha \in \mathbb{C}^\times$.

Then $\mathfrak{g}_{t,u}$ can be divided by the complex conjugate $\overline{g}$. We may assume that $\mathfrak{g}_{t,u} = g\overline{g}$. Then $g(\mathbf{a}_i) = 0$ for all $i = 1, \ldots, 10$. This is impossible by (1).

(3-2) Consider the case $\deg g = 2$ and $g \in \mathbb{R}[a,b,c,d]$.

Note that $V_{\mathbb{C}}(\mathfrak{g}_{t,u}) = V_{\mathbb{C}}(g) \cup V_{\mathbb{C}}(h)$. If $V_{\mathbb{C}}(g) = V_{\mathbb{C}}(h)$, then there exists $\alpha \in \mathbb{R}$ such that $h = \alpha g$. Thus, $\mathfrak{g}_{t,u} = \alpha g^2$. Then $g(\mathbf{a}_i) = 0$ for all $i = 1, \ldots, 10$. This is impossible by (1). So $V_{\mathbb{C}}(g) \ne V_{\mathbb{C}}(h)$. It is easy to see that $g,\ h \in \mathcal{E}(\mathcal{P}_{4,2})$, otherwise $f \notin \mathcal{E}(\mathcal{P}_{4,4})$. Since $\mathcal{P}_{4,2} = \Sigma_{4,2}$, there exists $g_1,\ h_1 \in \mathcal{H}_{4,1}$ such that $g = g_1^2$, $h = h_1^2$. So, at least 5 points among $\mathbf{a}_1, \ldots, \mathbf{a}_{10}$ lie on the line $V_{\mathbb{R}}(g_1)$ or $V_{\mathbb{R}}(h_1)$. This is impossible.

(3-3) Consider the case $\deg g = 1$ and $\alpha g \in \mathbb{R}[a,b,c,d]$ for any $\alpha \in \mathbb{C}^\times$.

Then $\mathfrak{g}_{t,u}$ change the signature across $V_{\mathbb{R}}(g)$ unless $\mathfrak{g}_{t,u}$ is divisible by g^2. This is impossible by (3-2).

(3-4) Consider the case $\deg g = 1$ and $g \notin \mathbb{R}[a,b,c,d]$.

Then $\mathfrak{g}_{t,u}$ can be divided by the complex conjugate $\overline{g}$. So, we can write $\mathfrak{g}_{t,u} = g\overline{g}h$. This is impossible by (3-2). □

Thus, we obtain Theorem 1.9.

2.2. Proof of Proposition 1.6.

There are many ways to prove Proposition 1.6. We give a short direct proof which use theory of PSD cone. Note that by [15, Corollary 1.3], the following lemma holds.

Lemma 2.7. *Let $f \in \mathcal{H}^s_{4,3}$. Then $f \in \mathcal{P}^{s+}_{4,3}$ if and only if*

$$f(0,0,x,1) \ge 0, \quad f(0,x,1,1) \ge 0, \quad f(x,x,1,1) \ge 0, \quad f(x,1,1,1) \ge 0$$

for all $x \ge 0$.

Proof of Proposition 1.6. Choose $s_0 := S^4_3 - S^4_{1,1,1}$, $s_1 := T^4_{2,1} - 3S^4_{1,1,1}$, $s_2 := S^4_{1,1,1}$ as a base of $\mathcal{H}^s_{4,3}$, where $S^4_3 = a^3 + b^3 + c^3 + d^3$, $S^4_{1,1,1} := bcd + acd + abd + abc$, and $T^4_{2,1} := a^2(b+c+d) + b^2(a+c+d) + c^2(a+b+d) + d^2(a+b+c)$. Remember that $g^{3,s}_1 = s_1$, $g^{3,s}_2 = 3s_0 - 2s_1$, $g^{3,s}_3 = s_2$ and $g^{3,s}_4 = s_0 - s_1 + s_2$.

Define $\Phi^s_{4,3} \colon \mathbb{P}^3_+ \to \mathbb{P}^2_{\mathbb{R}}$ by $\Phi^s_{4,3}(\mathbf{a}) = \big(s_0(\mathbf{a}) \colon s_1(\mathbf{a}) \colon s_2(\mathbf{a})\big)$. Let $X^{s+}_{4,3} := \Phi^s_{4,3}(\mathbb{P}^3_+)$. As [2, Example 3.2(4)],

$$A^+_s := \big\{(a\colon b\colon c\colon 1) \in \mathbb{P}^3_{\mathbb{R}} \bigm| 0 \le a \le b \le c \le 1\big\}$$

is a fundamental domain of $\Phi^s_{4,3}$. Let $\Phi\colon A^+_s \to X^{s+}_{4,3}$ be the restriction of $\Phi^s_{4,3}$. The above lemma implies that $\partial X^{s+}_{4,3}$ is included in the image of 6 edges of the tetrahedron A^+_s by Φ. Using this, it is easy to see that the convex closure of $X^{s+}_{4,3}$ is a quadrilateral $P_0P_1P_2P_3$, where

$$P_0 = \Phi(0\colon 0\colon 0\colon 1) = (1\colon 0\colon 0), \quad P_1 = \Phi(0\colon 0\colon 1\colon 1) = (1\colon 1\colon 0),$$
$$P_2 = \Phi(0\colon 1\colon 1\colon 1) = (2\colon 3\colon 1), \quad P_3 = \Phi(1\colon 1\colon 1\colon 1) = (0\colon 0\colon 1).$$

By [1, Proposition 1.14(2)], $\mathbb{P}(\mathcal{P}^{s+}_{4,3})$ is the dual of the convex closure of $X^{s+}_{4,3}$. Thus, $\mathbb{P}(\mathcal{P}^{s+}_{4,3})$ is a quadrilateral whose vertices are $g^{3,s}_1 = s_1$ (dual of P_3P_0), $g^{3,s}_2 = 3s_0 - 2s_1$ (dual of P_2P_3), $g^{3,s}_3 = s_2$ (dual of P_0P_1), and $g^{3,s}_4 = s_0 - s_1 + s_2$ (dual of P_1P_2). □

Corollary 2.8. *Let* $f \in \mathcal{H}^s_{4,3}$. *Then* $f \in \mathcal{P}^{s+}_{4,3}$ *if and only if*

$$f(0,0,0,1) \geq 0, \quad f(0,0,1,1) \geq 0, \quad f(0,1,1,1) \geq 0, \text{ and } f(1,1,1,1) \geq 0.$$

We define the map $\varphi_n \colon \mathcal{H}^s_{n,d} \longrightarrow \mathcal{H}^s_{n+1,d}$ by

$$\varphi_n\big(f(a_1, \ldots, a_n)\big) := \sum_{i=1}^{n} f(a_1, \ldots, a_{i-1}, a_{i+1}, \ldots, a_{n+1}).$$

If $n \geq d$, then φ_n is an isomorphism (see [7, Proposition 2.3]). In general, $\varphi_n(\mathcal{P}^{s+}_{n,d}) \subset \mathcal{P}^{s+}_{n+1,d}$. Especially $\varphi_3 \colon \mathcal{H}^s_{3,3} \longrightarrow \mathcal{H}^s_{4,3}$ is an isomorphism. Note that $\varphi_3(f^{3,s}_1) = 2g^{3,s}_1$, $\varphi_3(f^{3,s}_2) = g^{3,s}_2$ and $\varphi_3(f^{3,s}_3) = g^{4,s}_3$, where $f^{3,s}_i$ are defined in Proposition 1.1. This implies $g^{3,s}_1$, $g^{3,s}_2$, $g^{3,s}_3 \notin \mathcal{E}(\mathcal{P}_{4,3})$. Thus, we have:

Corollary 2.9. $\varphi_3\big(\mathcal{E}(\mathcal{P}^{s+}_{3,3})\big) \subset \mathcal{E}(\mathcal{P}^{s+}_{4,3}) \not\subset \mathcal{E}(\mathcal{P}^{+}_{4,3})$.

But, it seems that $\varphi_n(\mathcal{E}(\mathcal{P}^{s+}_{n,d})) \subset \mathcal{E}(\mathcal{P}^{s+}_{n+1,d})$ and $\varphi_n(\mathcal{E}(\mathcal{P}^{s}_{n,2d})) \subset \mathcal{E}(\mathcal{P}^{s}_{n+1,2d})$ don't hold in general.

Theorem 2.10. $g^{3,s}_4(a^2, b^2, c^2, d^2) \in \mathcal{E}(\mathcal{P}_{4,6}) - \Sigma_{4,6}$ *and* $g^{3,s}_4(a,b,c,d) \in \mathcal{E}(\mathcal{P}^{+}_{4,3})$.

Proof. (1) We prove $g^{3,s}_4(a^2, b^2, c^2, d^2) \in \mathcal{E}(\mathcal{P}_{4,6})$.

Put $g(a,b,c,d) := g^{3,s}_4(a^2, b^2, c^2, d^2) \in \mathcal{P}_{4,6}$. Let $e_1, \ldots, e_{84}$ be all the monic monomials in $\mathcal{H}_{4,6}$. We define $\tau \in \mathrm{Aut}(\mathcal{H}_{4,6})$ by $\tau(a,b,c,d) = (-a,b,c,d)$. Let $G \subset \mathrm{Aut}(\mathcal{H}_{4,6})$ be the subgroup generated by τ and the symmetric group $\mathfrak{S}_4$, and let

$$\mathcal{Z}_1 := \big\{\sigma(1,1,1,0) \;\big|\; \sigma \in G\big\}, \quad \mathcal{Z}_2 := \big\{\sigma(1,1,0,0) \;\big|\; \sigma \in G\big\}.$$

The set $\mathcal{Z}_1$ consists of $4 \times 4 = 16$ points and $\mathcal{Z}_2$ consists of $6 \times 2 = 12$ points. Let $\mathcal{Z}_1 \cup \mathcal{Z}_2 = \{\mathbf{z}_1, \ldots, \mathbf{z}_{28}\}$, and

$$a_{5i-4,j} := e_j(\mathbf{z}_i), \quad a_{5i-3,j} := \frac{\partial e_j}{\partial a}(\mathbf{z}_i), \quad a_{5i-2,j} := \frac{\partial e_j}{\partial b}(\mathbf{z}_i),$$
$$a_{5i-1,j} := \frac{\partial e_j}{\partial c}(\mathbf{z}_i), \quad a_{5i,j} := \frac{\partial e_j}{\partial d}(\mathbf{z}_i),$$

for $1 \leqq i \leqq 28$ and $1 \leqq j \leqq 84$. Construct a 140×84 matrix $A = (a_{i,j})$.

Note that $g(a,b,c,d) \in \operatorname{Ker} A$. Using Mathematica, we have $\operatorname{rank} A = 83$. Thus $\operatorname{Ker} A = \mathbb{R} \cdot g(a,b,c,d)$. This implies $g(a,b,c,d) \in \mathcal{E}(\mathcal{P}_{4,6})$. Therefore, $g^{3,s}_4(a,b,c,d) \in \mathcal{E}(\mathcal{P}^{+}_{4,3})$.

(2) We prove $g(a,b,c,d) \notin \Sigma_{4,6}$.

Assume that $g(a,b,c,d) \in \Sigma_{4,6}$. Then, there exists $h(a,b,c,d) \in \mathcal{H}_{4,3}$ such that $g(a,b,c,d) = h(a,b,c,d)^2$. We have $h(\mathbf{z}_i) = 0$ for $i = 1, \ldots, 28$, because $g(\mathbf{z}_i) = 0$. Using Mathematica, we can check that there exists no such cubic $h(a,b,c,d) \in \mathcal{H}_{4,3}$. In fact, we can check this as the following. Let $e'_1, \ldots, e'_{20}$ be all the monic monomials in $\mathcal{H}_{4,3}$, and construct 28×20 matrix $B := \big(e'_i(\mathbf{z}_j)\big)$. Then $\operatorname{rank} B = 20$ and $\operatorname{Ker} B = 0$. □

Conjecture 2.11. Let $n \geq 3$ and $\mathbf{b}_i := (\underbrace{0, \ldots, 0}_{n-i}, \underbrace{1, \ldots, 1}_{i})$. The three dimensional PSD cone $\mathcal{P}^{s+}_{n,3}$ will be a polyhedral cone whose base is a n-gon. The extremal rays $\mathbb{R}_+ \cdot f_i$ of $\mathcal{P}^{s+}_{n,3}$ will satisfy the following: $f_i(\mathbf{b}_i) = f_i(\mathbf{b}_{i+1}) = 0$ for $i = 1, \ldots, n-1$, and $f_n(\mathbf{b}_n) = f_n(\mathbf{b}_1) = 0$.

3. Some extremal elements of $\mathcal{P}^{s0}_{3,6}$

In this section, we prove Theorem 1.5. Our idea of proof is similar to that of Theorem 1.9. But it is more difficult to judge whether $f \in \mathcal{H}^{s0}_{3,6}$ is PSD or not.

In this section, we use the following symbols. We denote the coordinate system of $\mathbb{P}^2_{\mathbb{R}}$ by $(a\colon b\colon c)$, and put

$$\begin{aligned} S_{m,n} &= S_{m,n}(a,b,c) := a^m b^n + b^m c^n + c^m a^n, \\ S_n &:= S_n(a,b,c) = S_{n,0}(a,b,c) = a^n + b^n + c^n, \\ T_{m,n} &:= S_{m,n}(a,b,c) + S_{n,m}(a,b,c), \\ U &= U(a,b,c) := abc. \end{aligned}$$

3.1. Preparation.

We use the following theorem.

Theorem 3.1. *If $f \in \mathcal{P}_{3,6}$ is an exposed extremal element and $f \notin \Sigma_{3,6}$, then $V_{\mathbb{C}}(f)$ is an irreducible rational curve which has 10 acnodes $P_1, \ldots, P_{10}$, and $V_{\mathbb{R}}(f) = \{P_1, \ldots, P_{10}\}$. On the other hand, if $f \in \mathcal{P}_{3,6}$ and $V_{\mathbb{C}}(f)$ is an irreducible curve which has 10 nodes in $\mathbb{P}^2_{\mathbb{R}}$, then $f \in \mathcal{E}(\mathcal{P}_{3,6})$.*

The latter half of the above theorem was proved in [14, Theorem 7.2] and the first half was proved in [4, Remark 8]. See also [3, Theorem 2.17].

Assume that $u \neq 1$, $v \neq 1$, $w \neq 1$ and $u \neq v$. If $f \in \mathcal{P}^{s0}_{3,6}$ satisfies $f(u,v,1) = 0$ and $f(w,1,1) = 0$, then $V_{\mathbb{R}}(f)$ contains 10 points $(1\colon 1\colon 1)$, $(u\colon v\colon 1)$, $(u\colon 1\colon v)$, $(v\colon u\colon 1)$, $(v\colon 1\colon u)$, $(1\colon u\colon v)$, $(1\colon v\colon u)$, $(w\colon 1\colon 1)$, $(1\colon w\colon 1)$, $(1\colon w\colon 1)$. Moreover, if f is irreducible in $\mathbb{C}[a,b,c]$, then $f \in \mathcal{E}(\mathcal{P}_{3,6})$. In this case $f \notin \Sigma_{3,6}$. Because, if $f \in \Sigma_{3,6}$, then f is a square of a cubic polynomial. This is impossible, because f is irreducible.

Definition 3.2. In this section, we say 10 points $P_1, \ldots, P_{10} \in \mathbb{P}^2_{\mathbb{R}}$ are in *general position*, if the following (1) and (2) hold.

(1) No three points are colinear.
(2) There exists no cubic homogeneous polynomial $g \in \mathbb{C}[a,b,c] - \{0\}$ such that $g(P_i) = 0$ for all $i = 1, \ldots, 10$,

In the above definition, we don't assume that 'no 6 points are on a same quadric curve'. So, this is an unusual definition.

Lemma 3.3. *Let $f \in \mathcal{P}_{3,6}$. Assume that $\{\mathbf{c}_1, \ldots, \mathbf{c}_{10}\} \subset V_{\mathbb{R}}(f)$ and $\mathbf{c}_1, \ldots, \mathbf{c}_{10}$ are in general position. Then f is irreducible in $\mathbb{C}[a,b,c]$ and $f \notin \Sigma_{3,6}$.*

Proof. (1) Assume that $f = g_1^2 + \cdots + g_r^2 \in \Sigma_{3,6}$. Then cubic polynomials g_i satisfy $g_i(\mathbf{c}_i) = 0$ for all $i = 1, \ldots, 10$. This contradicts to our assumption that $\mathbf{c}_1, \ldots, \mathbf{c}_{10}$ are in general position.

(2) Assume that f is reducible in $\mathbb{C}[a,b,c]$. Reznick has proved in [14, Lemma 7.1] that if $f \in \mathcal{P}_{3,6}$ is reducible, then $f \in \Sigma_{3,6}$. □

Proposition 3.4. *For a, b, c, $w \in \mathbb{R}$ and $\mathbf{u} = (u\colon v\colon 1) \in \mathbb{P}^2_{\mathbb{R}}$, let*

$$\delta_1(a,b,c,w) := 2S_3(a,b,c) - (w+2)T_{2,1}(a,b,c) + (6w+6)U(a,b,c),$$
$$\delta_2(a,b,c,w) := (2w+1)S_2(a,b,c) - (w^2+2)S_{1,1}(a,b,c),$$
$$V_{u,v,w} := \begin{Bmatrix} (1\colon 1\colon 1),\ (u\colon v\colon 1),\ (u\colon 1\colon v),\ (v\colon u\colon 1),\ (v\colon 1\colon u), \\ (1\colon u\colon v),\ (1\colon v\colon u),\ (w\colon 1\colon 1),\ (1\colon w\colon 1),\ (1\colon 1\colon w) \end{Bmatrix} \subset \mathbb{P}^2_{\mathbb{R}}.$$

Then 10 points of $V_{u,v,w}$ are in general position if and only if the following (1), (2) and (3) hold:
(1) $\delta_1(u,v,1,w) \neq 0$ *and* $\delta_2(u,v,1,w) \neq 0$.
(2) $u \neq 1$, $v \neq 1$, $w \neq 1$, $u \neq v$ *and* $u+v+1 \neq 0$.
(3) $u+v \neq 2$ *and* $2u-v \neq 1$.

Proof. Let $e_i(a,b,c)$ $(i = 1,\ldots, 10)$ are all the monic cubic monomials and let P_j $(j = 1,\ldots, 10)$ are 10 points in $V_{u,v,w}$. Put $a_{i,j} := e_i(P_j)$ and construct a 10×10 matrix $A := (a_{i,j})$. Then

$$\det A = \pm(u-v)^3(v-1)^3(1-u)^3(u+v+1)^2(w-1)^4\delta_1(u,v,1,w)\delta_2(u,v,1,w)^2.$$

Thus, (2) of Definition 3.2 holds if and only if (1) and (2) of this proposition hold.

It is easy to see that no three points are colinear, if and only if (2) and (3) hold. □

3.2. Sextic polynomial $\mathfrak{f}_{\mathbf{u},w}$.

Definition of the sextic polynomial $\mathfrak{f}_{\mathbf{u},w}$ is somewhat long and complicated. But this polynomial plays main role in this section. Please see Proposition 3.5 about the reason why such polynomial appears.

For $\mathbf{a} = (a,b,c) \in \mathbb{R}^3$ and l, m, $n \in \mathbb{N} \cup \{0\}$ with $l > m > n$, we denote

$$T_{l,m,n} := a^l b^m c^n + a^l b^n c^m + a^m b^l c^n + a^m b^n c^l + a^n b^l c^m + a^n b^m c^l,$$
$$S_{l,m,m} := a^l b^m c^m + a^m b^l c^m + a^m b^m c^l,$$
$$S_{l,l,m} := S_{m,l,l} = a^l b^l c^m + a^l b^m c^l + a^m b^l c^l,$$
$$U_l := a^l b^l c^l = U^l.$$

Note that $T_{l,m,0} = T_{l,m}$, $S_{l,l,0} = S_{l,l}$, $S_{l,0,0} = S_l$ and $U_1 = U$.

We choose $s_0 := S_6 - 3U_2$, $s_1 := T_{5,1} - 6U_2$, $s_2 := T_{4,2} - 6U_2$, $s_3 := S_{3,3} - 3U_2$, $s_4 := S_{4,1,1} - 3U_2$, $s_5 := T_{3,2,1} - 6U_2$ as a base of $\mathcal{H}^{s0}_{3,6}$.

Let $\mathbf{u} = (u_1,u_2,u_3) \in \mathbb{R}^3$ and $w \in \mathbb{R}$. Now we shall construct a polynomial $\mathfrak{f}_{\mathbf{u},w} \in \mathcal{H}^{s0}_{3,6}$ which satisfies

$$\mathfrak{f}_{\mathbf{u},w}(\mathbf{u}) = 0 \quad \text{and} \quad \mathfrak{f}_{\mathbf{u},w}(w,1,1) = 0.$$

Afterward we discuss about the condition for $(\mathbf{u}, w)$ for $\mathfrak{f}_{\mathbf{u},w} \in \mathcal{P}^{s0}_{3,6}$. The polynomial $\mathfrak{f}_{\mathbf{u},w} \in \mathcal{H}_{3,6}$ is defined by

$$\mathfrak{f}_{\mathbf{u},w}(a,b,c) := \sum_{i=0}^{5} p_i^F(\mathbf{u},w)s_i(a,b,c),$$

where we define the coefficients $p_i^F(a,\ b,\ c,\ w)$ $(i = 0,\ldots, 5)$ as the following:

$$\begin{aligned} p_0^F(\mathbf{a},w) := {} & T_{10,2} + (4w^2-w-4)T_{9,3} + (-4w^3-w^2+2w+4)T_{8,4} \\ & + (4+w-4w^2+w^4)T_{7,5} \\ & + (2w^4+8w^3+2w^2-4w-10)S_{6,6} \end{aligned}$$

$$
\begin{aligned}
&+ 2S_{10,1,1} + (-4w^2 - 7w - 8)T_{9,2,1} + (w^2 + 16w + 12)T_{8,3,1} \\
&+ (w^4 + 8w^3 - 9w^2 - 34w - 8)T_{7,4,1} \\
&+ (-13w^4 - 16w^3 + 12w^2 + 25w + 2)T_{6,5,1} \\
&+ (8w^3 + 24w^2 + 60w + 28)S_{8,2,2} \\
&+ (-2w^4 - 16w^3 - 59w^2 - 87w - 36)T_{7,3,2} \\
&+ (26w^4 + 84w^3 + 135w^2 + 88w + 23)T_{6,4,2} \\
&+ (12w^4 - 56w^3 - 168w^2 - 108w - 16)S_{5,5,2} \\
&+ (-26w^4 - 56w^3 + 62w^2 + 102w + 30)S_{6,3,3} \\
&+ (-12w^4 - 48w^3 - 80w^2 - 38w - 2)T_{5,4,3} \\
&+ (30w^4 + 240w^3 + 270w^2 + 60w - 30)U_4, \\
p_1^F(\mathbf{a}, w) := &-2T_{11,1} + (-4w^2 + 2w + 6)T_{10,2} + (-2w^2 - 2w - 2)T_{9,3} \\
&+ (3w^4 + 4w^3 + 5w^2 - 4w - 8)T_{8,4} + (-w^5 - w^4 + 2w^2 + 2w + 4)T_{7,5} \\
&+ (-2w^5 - 8w^4 - 8w^3 - 2w^2 + 4w + 4)S_{6,6} \\
&+ (8w^2 + 8w + 8)S_{10,1,1} + (-2w^2 - 26w - 16)T_{9,2,1} \\
&+ (-2w^3 + 13w^2 + 49w + 18)T_{8,3,1} \\
&+ (-w^5 - 7w^4 - 10w^3 - 17w^2 - 11w - 8)T_{7,4,1} \\
&+ (13w^5 + 25w^4 + 20w^3 - 2w^2 - 20w)T_{6,5,1} \\
&+ (-6w^4 - 12w^3 - 12w^2 - 6w)S_{8,2,2} \\
&+ (2w^5 + 14w^4 + 42w^3 + 27w^2 + 9w + 14)T_{7,3,2} \\
&+ (-26w^5 - 89w^4 - 136w^3 - 87w^2 - 24w - 10)T_{6,4,2} \\
&+ (-12w^5 + 30w^4 + 108w^3 + 132w^2 + 90w + 12)S_{5,5,2} \\
&+ (26w^5 + 68w^4 + 32w^3 - 18w^2 - 60w - 24)S_{6,3,3} \\
&+ (12w^5 + 48w^4 + 72w^3 + 48w^2 + 12w - 6)T_{5,4,3} \\
&+ (-30w^5 - 210w^4 - 300w^3 - 210w^2 - 30w + 60)U_4, \\
p_2^F(\mathbf{a}, w) := &\, S_{12} + (-4w^2 - w)T_{11,1} + (12w^3 + 7w^2 - 2w - 7)T_{10,2} \\
&+ (-9w^4 - 8w^3 - 2w^2 + 4w + 4)T_{9,3} \\
&+ (2w^5 - 4w^4 - 12w^3 - 3w^2 + 6w + 11)T_{8,4} \\
&+ (4w^5 + 4w^4 + 8w^3 + 6w^2 - 3w - 4)T_{7,5} \\
&+ (4w^5 - 2w^4 - 8w^2 - 8w - 10)S_{6,6} + (8w^3 + 5w^2 + 8w)S_{10,1,1} \\
&+ (-15w^4 - 30w^3 - 12w^2 + 18w + 10)T_{9,2,1} \\
&+ (8w^5 + 47w^4 + 54w^3 - 4w^2 - 52w - 20)T_{8,3,1} \\
&+ (-w^6 - 13w^5 - 4w^4 + 10w^3 + 4w^2 - 5w)T_{7,4,1} \\
&+ (-3w^6 - 15w^5 - 24w^4 - 2w^3 + 11w^2 + 32w + 10)T_{6,5,1} \\
&+ (12w^5 + 33w^4 - 12w^3 - 24w^2 - 36w - 9)S_{8,2,2} \\
&+ (-3w^6 - 32w^5 - 44w^4 - 22w^3 + 100w^2 + 79w + 30)T_{7,3,2} \\
&+ (12w^6 + 36w^5 + 42w^4 - 68w^3 - 158w^2 - 110w - 36)T_{6,4,2} \\
&+ (3w^6 + 12w^5 + 36w^4 + 48w^3 + 54w^2 + 102w + 24)T_{5,5,2} \\
&+ (3w^6 + 2w^5 - 10w^4 - 20w^3 + 33w^2 - 24w - 20)S_{6,3,3}
\end{aligned}
$$

$$
\begin{aligned}
&+ (-6w^6 - 3w^5 - 6w^4 + 60w^3 + 57w^2 + 33w + 6)T_{5,4,3} \\
&+ (-12w^6 - 12w^5 - 93w^4 - 84w^3 - 192w^2 - 120w + 18)U_4, \\
p_3^F(\mathbf{a}, w) := {}& (4w^2 - 2)S_{12} + (-8w^3 - 4w^2 + 2w + 4)T_{11,1} \\
&+ (5w^4 + 4w^3 + 5w^2)T_{10,2} + (-w^5 + 3w^4 - 8w^3 - 8w^2 - 2w + 4)T_{9,3} \\
&+ (-2w^5 + 12w^4 + 8w^3 + 6w^2 - 8w - 14)T_{8,4} \\
&+ (-5w^5 + 7w^4 + 16w^3 + 12w^2 - 8)T_{7,5} \\
&+ (-8w^5 - 14w^4 - 24w^3 - 30w^2 + 16w + 32)S_{6,6} \\
&+ (14w^4 + 24w^3 + 12w^2 - 32w - 20)S_{10,1,1} \\
&+ (-7w^5 - 27w^4 - 36w^3 + 4w^2 + 30w + 28)T_{9,2,1} \\
&+ (w^6 + 7w^4 + 16w^3 - 10w^2 - 26w - 20)T_{8,3,1} \\
&+ (3w^6 - 6w^5 - 49w^4 - 80w^3 + 6w^2 + 100w + 32)T_{7,4,1} \\
&+ (4w^6 + 29w^5 + 35w^4 + 4w^3 - 8w^2 - 74w - 24)T_{6,5,1} \\
&+ (2w^6 + 36w^5 + 84w^4 + 144w^3 + 48w^2 - 36w - 38)S_{8,2,2} \\
&+ (-7w^6 - 27w^5 - 122w^4 - 96w^3 - 130w^2 - 2w - 16)T_{7,3,2} \\
&+ (2w^6 + 68w^5 + 175w^4 + 292w^3 + 217w^2 + 92w + 46)T_{6,4,2} \\
&+ (-32w^6 - 108w^5 - 222w^4 - 216w^3 - 48w^2 - 168w - 40)S_{5,5,2} \\
&+ (6w^6 + 46w^5 + 68w^4 + 112w^3 - 120w^2 - 36w + 28)S_{6,3,3} \\
&+ (-56w^5 - 92w^4 - 200w^3 - 92w^2 - 14w + 4)T_{5,4,3} \\
&+ (54w^6 + 144w^5 + 486w^4 + 408w^3 + 414w^2 + 180w - 96)U_4, \\
p_4^F(\mathbf{a}, w) := {}& 2S_{12} + (8w^2 + 2w - 4)T_{11,1} + (-8w^3 - 4w^2 - 4w - 2)T_{10,2} \\
&+ (6w^4 + 10w^3 + 9w^2 + 3w + 4)T_{9,3} \\
&+ (-4w^5 - w^4 - 5w^2 - 2w + 6)T_{8,4} \\
&+ (w^6 - 4w^5 + 5w^4 - 10w^3 - 17w^2 - 5w)T_{7,5} \\
&+ (2w^6 + 24w^4 + 16w^3 + 18w^2 + 12w - 12)S_{6,6} \\
&+ (-48w^3 - 66w^2 - 48w)S_{10,1,1} \\
&+ (42w^4 + 98w^3 + 143w^2 + 89w + 36)T_{9,2,1} \\
&+ (-16w^5 - 88w^4 - 188w^3 - 242w^2 - 140w - 58)T_{8,3,1} \\
&+ (3w^6 + 30w^5 + 69w^4 + 168w^3 + 222w^2 + 156w + 18)T_{7,4,1} \\
&+ (-7w^6 - 22w^5 - 67w^4 - 118w^3 - 65w^2 - 59w + 8)T_{6,5,1} \\
&+ (-24w^5 - 96w^4 - 120w^3 - 138w^2 - 84w - 42)S_{8,2,2} \\
&+ (4w^6 + 56w^5 + 98w^4 + 226w^3 + 103w^2 + 67w - 2)T_{7,3,2} \\
&+ (2w^6 + 32w^5 + 23w^4 + 60w^3 + 67w^2 + 52w + 40)T_{6,4,2} \\
&+ (6w^6 + 24w^5 + 42w^4 - 78w^2 - 240w - 60)S_{5,5,2} \\
&+ (-32w^6 - 108w^5 - 252w^4 - 300w^3 - 306w^2 - 18w + 80)S_{6,3,3} \\
&+ (-42w^5 - 12w^4 - 60w^3 + 12w^2 - 24)T_{5,4,3} \\
&+ (54w^6 + 144w^5 + 396w^4 + 288w^3 + 324w^2 + 180w - 36)U_4, \\
p_5^F(\mathbf{a}, w) := {}& (-4w^2 - 4w - 2)S_{12} + (8w^3 + 10w^2 + 10w + 6)T_{11,1} \\
&+ (-9w^4 - 22w^3 - 24w^2 - 9w - 4)T_{10,2}
\end{aligned}
$$

$$
\begin{aligned}
&+ (5w^5 + 11w^4 + 22w^3 + 19w^2 + 7w - 2)T_{9,3}\\
&+ (-w^6 + w^5 - 8w^4 + 4w^3 + 4w^2 - 2w + 2)T_{8,4}\\
&+ (-w^6 - 21w^4 - 30w^3 - 29w^2 - 17w - 4)T_{7,5}\\
&+ (8w^5 + 14w^4 + 36w^3 + 48w^2 + 30w + 8)S_{6,6}\\
&+ (-6w^4 + 4w^3 - 10w^2 + 2w - 8)S_{10,1,1}\\
&+ (3w^5 + 9w^4 + 14w^3 - 37w^2 - 31w - 20)T_{9,2,1}\\
&+ (-w^6 - 2w^5 + w^4 + 46w^3 + 129w^2 + 81w + 40)T_{8,3,1}\\
&+ (-2w^6 - 7w^5 - 19w^4 - 78w^3 - 119w^2 - 107w - 10)T_{7,4,1}\\
&+ (5w^6 + 2w^5 + 41w^4 + 86w^3 + 27w^2 + 45w - 8)T_{6,5,1}\\
&+ (-18w^5 - 24w^4 - 36w^3 + 42w^2 + 66w + 42)S_{8,2,2}\\
&+ (5w^6 + 9w^5 + 42w^4 - 108w^3 - 70w^2 - 82w - 12)T_{7,3,2}\\
&+ (-7w^6 - 37w^5 - 19w^4 - 22w^3 - 16w^2 - 25w - 30)T_{6,4,2}\\
&+ (6w^6 + 6w^5 - 42w^4 - 60w^3 - 30w^2 + 162w + 48)S_{5,5,2}\\
&+ (6w^6 + 6w^5 + 96w^4 + 184w^3 + 272w^2 + 44w - 56)S_{6,3,3}\\
&+ (24w^5 - 36w^4 + 24w^3 + 6w^2 + 30w + 30)T_{5,4,3}\\
&+ (-24w^6 + 6w^5 - 66w^4 - 48w^3 - 354w^2 - 300w - 24)U_4.
\end{aligned}
$$

Since $\mathfrak{f}_{\lambda\mathbf{u},w}(\mathbf{a}) = \lambda^{12}\mathfrak{f}_{\mathbf{u},w}(\mathbf{a})$ and $\mathfrak{f}_{\mathbf{u},w}(\lambda\mathbf{a}) = \lambda^6\mathfrak{f}_{\mathbf{u},w}(\mathbf{a})$, we may regard $\mathbf{a} \in \mathbb{P}^2_{\mathbb{R}}$ and $\mathbf{u} \in \mathbb{P}^2_{\mathbb{R}}$, when we discuss $\operatorname{sign}\big(\mathfrak{f}_{\mathbf{u},w}(\mathbf{a})\big)$.

Proposition 3.5. *Assume that $u \neq 1$, $v \neq 1$, $w \neq 1$, $u \neq v$ and $p_0^F(u, v, 1, w) \neq 0$. Let $\mathbf{u} = (u\colon v\colon 1)$. If $f \in \mathcal{H}^s_{3,6}$ satisfies the system of equations*

$$f(u, v, 1) = f_a(u, v, 1) = f_b(u, v, 1) = f(w, 1, 1) = f_a(w, 1, 1) = f_b(w, 1, 1) = 0 \quad (*)$$

then there exists $\alpha \in \mathbb{R}$ such that $f = \alpha\mathfrak{f}_{\mathbf{u},w}$. Where $f_a := \partial f(a, b, c)/\partial a$ and $f_b := \partial f(a, b, c)/\partial b$.

In other word if $f \in \mathcal{P}^s_{3,6}$ satisfies

$$f(u, v, 1) = f(w, 1, 1) = 0,$$

then $f = \alpha\mathfrak{f}_{\mathbf{u},w}$.

Proof. It is easy to check that $\mathfrak{f}_{\mathbf{u},w}$ satisfies $(*)$ using PC. Let $\mathbf{s} := (s_0,\dots, s_5)$, $\mathbf{s}_a := \dfrac{\partial}{\partial a}\mathbf{s}$ and so on. Construct 5×6 matrix A aligning $\mathbf{s}(u, v, 1)$, $\mathbf{s}_a(u, v, 1)$, $\mathbf{s}_b(u, v, 1)$, $\mathbf{s}(w, 1, 1)$ and $\mathbf{s}_a(w, 1, 1)$. If f satisfies $(*)$, then $f \in \operatorname{Ker} A$. Put $\mathbf{e}_1 = (1, 0,\dots, 0)$ at the top of A, and construct a 6×6 matrix B. Then

$$\det B = \pm 2(u - v)(v - 1)(1 - u)(w - 1)^4 p_0^F(u, v, 1, w).$$

By our assumption, $\det B \neq 0$. Thus, $\dim(\operatorname{Ker} A) = 1$, and we have the conclusion. □

Note that $V_{\mathbb{R}}(\mathfrak{f}_{\mathbf{u},w}) \supset V_{u,v,w}$ if $\mathbf{u} = (u,\ v,\ 1)$, where $V_{u,v,w}$ was defined in Proposition 3.4.

3.3. When $\mathfrak{f}_{\mathbf{u},w}$ is PSD?

Next our work is to find an open set $\mathcal{U} \subset \mathbb{R}^3$ such that $\mathfrak{f}_{\mathbf{u},w} \in \mathcal{P}_{3,6}$ for every $(u,\ v,\ w) \in \mathcal{U}$ with $\mathbf{u} = (u,\ v,\ 1)$. We have already proved that if $\mathfrak{f}_{\mathbf{u},w}$ is PSD then $\mathfrak{f}_{u,v,w} \in \mathcal{E}(\mathcal{P}_{3,6}) - \Sigma_{3,6}$. We use the next lemma instead of [15].

Lemme 3.6. *Take* $f(a,\ b,\ c) \in \mathcal{H}^s_{3,6}$. *Let* $\sigma_1 := a+b+c$, $\sigma_2 := ab+bc+ca$, $\sigma_3 := abc$, *and denote*

$$f(a,b,c) = g_0\sigma_3^2 + g_1(\sigma_1,\sigma_2)\,\sigma_3 + g_2(\sigma_1,\sigma_2) \quad (g_0 \in \mathbb{R},\ g_1(p,q),\ g_2(p,q) \in \mathbb{R}[p,q]).$$

We also fix the following symbols.

$$D(p,q) := g_1(p,q)^2 - 4g_0g_2(p,q),$$
$$h_1(t) := 2sg_0 + g_1(t+2, 2t+1).$$

If $g_0 \le 0$, *then we put* $I_1 := \emptyset$. *If* $g_0 > 0$, *we put*

$$I_1 := \left\{\ t \in \mathbb{R} \;\middle|\; -2 \le t \le 1 \text{ and } D(t+2,\ 2t+1) > 0\right\}.$$

Then $f \in \mathcal{P}_{3,6}$ *if and only if the following condition* (1) *holds, and for every* $t \in I_1$ *(depending on* t*), one of* (2) *or* (3) *holds.*

(1) $f(0,0,1) \ge 0$ *and* $f(x,1,1) \ge 0$ *for all* $x \in \mathbb{R}$.
(2) $h_1(t) \ge 0$.
(3) $(1+2t)(4-t)h_1\left(\dfrac{4-t}{1+2t}\right) \le 0$.

Proof. We use [1, Theorem 6.1]. In Theorem 6.1, (3) is stated as $h_2((1+2t)/(4-t)) \le 0$, where $h_2(\tau) := 2\tau^2 g_0 + g_1(2\tau+1, \tau^2+2\tau)$. Put $\tau := (1+2t)/(4-t)$. In our case, $h_2(\tau) = \tau^3 h_1(1/\tau)$. Then, $h_2(\tau) \le 0$ if and only if $(1+2t)(4-t)h_1((4-t)/(1+2t)) \le 0$. □

Using the above Lemma, we can theoretically describe the semialgebraic set

$$\mathcal{X} := \left\{(u,v,w) \in \mathbb{R}^3 \;\middle|\; \mathfrak{f}_{u,v,1,w} \in \mathcal{P}_{3,6}\right\}.$$

But, it is not easy to describe this semialgebraic set. On the other hand, $D(t+2,\ 2t+1)$ is not so complicated.

Definition 3.7. As the above lemma, we represent

$$\mathfrak{f}_{\mathbf{u},w}(a,b,c) = g_0(\mathbf{u},w)\sigma_3^2 + g_1(\sigma_1,\sigma_2,\mathbf{u},w)\,\sigma_3 + g_2(\sigma_1,\sigma_2,\mathbf{u},w).$$

where $\mathbf{u} \in \mathbb{R}^3$ and $w \in \mathbb{R}$. Note that

$$g_0 = -9(p_1^F + p_2^F + p_5^F),$$
$$g_1 = (6p_0^F - p_1^F - 2p_2^F + p_4^F)\sigma_1^3 + (-12p_0^F + 7p_1^F + 4p_2^F - 3p_3^F - 3p_4^F + p_5^F)\sigma_1\sigma_2,$$
$$g_2 = p_0^F\sigma_1^6 + (-6p_0^F + p_1^F)\sigma_1^4\sigma_2 + (9p_0^F - 4p_1^F + p_2^F)\sigma_1^2\sigma_2^2 + (-2p_0^F + 2p_1^F - 2p_2^F + p_3^F)\sigma_2^3.$$

Symmetric polynomials δ_1 and δ_2 are defined in Proposition 3.4. We also put

$$\delta_3(a,b,c,w) := S_4 - (w+1)T_{3,1} + (w^2+2w)S_{2,2} - (w^2-1)US_1,$$
$$\delta_4(a,b,c,w) := 2S_5 - (2w+3)T_{4,1} + (-w^2+2w+1)T_{3,2} + 4(w+1)^2US_2 - (2w^2+8w+2)US_{1,1},$$
$$\delta_5(a,b,c,w) := (w+1)S_3 - (w^2+w+1)T_{2,1} + (w^3+3w^2+6w+2)U = ((w+1)a-b-c)((w+1)b-c-a)((w+1)c-a-b),$$
$$h_1(t,\mathbf{u},w) := 2tg_0(\mathbf{u},w) + g_1(t+2, 2t+1, \mathbf{u}, w),$$
$$D_f(t,\mathbf{u},w) := g_1(t+2,2t+1,\mathbf{u},w)^2 - 4g_0(\mathbf{u},w)g_2(t+2,2t+1,\mathbf{u},w).$$

This D_f corresponds to D in Lemma 3.6. We present an important divisor $D_L(t,\ \mathbf{u},\ w)$ of $D_f(t,\ \mathbf{u},\ w)$ as the following:

$$D_L^0(a,b,c,w) := (28w-4)S_{10} + (-56w^2-52w+24)T_{9,1} + (24w^3+128w^2+8w-52)T_{8,2}$$

$$
\begin{aligned}
&+ (4w^4 - 64w^3 - 32w^2 + 24w + 32)T_{7,3}\\
&+ (4w^5 - 16w^4 - 24w^3 - 128w^2 - 36w + 56)T_{6,4}\\
&+ (8w^5 - 40w^4 + 128w^3 + 176w^2 + 56w - 112)S_{5,5}\\
&+ (120w^3 + 200w^2 + 8w - 112)S_{8,1,1}\\
&+ (-60w^4 - 288w^3 - 256w^2 + 296w + 200)T_{7,2,1}\\
&+ (-20w^5 + 164w^4 + 432w^3 + 80w^2 - 416w - 168)T_{6,3,1}\\
&+ (-16w^5 + 8w^4 - 136w^3 + 32w^2 + 164w + 56)T_{5,4,1}\\
&+ (60w^5 + 180w^4 + 288w^3 - 304w^2 - 796w - 292)S_{6,2,2}\\
&+ (-52w^5 - 376w^4 - 232w^3 + 872w^2 + 776w + 200)T_{5,3,2}\\
&+ (140w^5 + 284w^4 - 640w^3 - 1264w^2 - 568w - 112)S_{4,4,2}\\
&+ (-40w^5 + 128w^4 + 680w^3 - 88w^2 - 256w - 64)S_{4,3,3},
\end{aligned}
$$

$$
\begin{aligned}
D_L^1(a,b,c,w) := &(-8w - 40)S_{10} + (16w^2 + 56w + 96)T_{9,1}\\
&+ (-48w^3 - 16w^2 - 100w - 52)T_{8,2}\\
&+ (40w^4 + 8w^3 + 40w^2 + 24w - 40)T_{7,3}\\
&+ (4w^5 + 20w^4 + 48w^3 + 16w^2 + 108w + 92)T_{6,4}\\
&+ (8w^5 - 40w^4 - 16w^3 - 112w^2 - 160w - 112)S_{5,5}\\
&+ (48w^3 - 16w^2 - 280w - 184)S_{8,1,1}\\
&+ (-24w^4 - 72w^3 - 40w^2 + 296w + 56)T_{7,2,1}\\
&+ (-56w^5 - 88w^4 + 72w^3 + 8w^2 - 56w - 24)T_{6,3,1}\\
&+ (-16w^5 - 64w^4 - 208w^3 + 32w^2 - 16w + 56)T_{5,4,1}\\
&+ (96w^5 + 288w^4 + 720w^3 + 272w^2 + 176w + 176)S_{6,2,2}\\
&+ (-16w^5 - 376w^4 - 736w^3 - 640w^2 - 520w - 88)T_{5,3,2}\\
&+ (248w^5 + 1112w^4 + 1520w^3 + 1328w^2 + 296w - 184)S_{4,4,2}\\
&+ (-184w^5 - 376w^4 - 400w^3 - 304w^2 + 392w + 152)S_{4,3,3},
\end{aligned}
$$

$$
\begin{aligned}
D_L^2(a,b,c,w) := &(16w + 8)S_{10} + (-32w^2 - 40w - 12)T_{9,1}\\
&+ (24w^3 + 68w^2 + 29w - 13)T_{8,2}\\
&+ (-8w^4 - 34w^3 - 26w^2 + 6w + 26)T_{7,3}\\
&+ (w^5 - 13w^4 - 24w^3 - 68w^2 - 45w + 5)T_{6,4}\\
&+ (2w^5 - 10w^4 + 68w^3 + 116w^2 + 68w - 28)S_{5,5}\\
&+ (48w^3 + 104w^2 + 74w - 10)S_{8,1,1}\\
&+ (-24w^4 - 126w^3 - 118w^2 + 74w + 86)T_{7,2,1}\\
&+ (4w^5 + 104w^4 + 198w^3 + 38w^2 - 194w - 78)T_{6,3,1}\\
&+ (-4w^5 + 20w^4 - 16w^3 + 8w^2 + 86w + 14)T_{5,4,1}\\
&+ (6w^5 + 18w^4 - 36w^3 - 220w^2 - 442w - 190)S_{6,2,2}\\
&+ (-22w^5 - 94w^4 + 68w^3 + 596w^2 + 518w + 122)T_{5,3,2}\\
&+ (8w^5 - 136w^4 - 700w^3 - 964w^2 - 358w - 10)S_{4,4,2}\\
&+ (26w^5 + 158w^4 + 440w^3 + 32w^2 - 226w - 70)S_{4,3,3},
\end{aligned}
$$

$$
D_L(t,a,b,c,w) := t^2\, D_L^2(a,b,c,w) + t\, D_L^1(a,b,c,w) + D_L^0(a,b,c,w).
$$

Proposiiton 3.8. *Let a, b, c, $w \in \mathbb{R}$ and $\mathbf{u} = (u_1 \colon u_2 \colon u_3) \in \mathbb{P}^2_{\mathbb{R}}$. Then the followings hold.*

(1) $\delta_3(\mathbf{u}, w) \geq 0$.

(2) $g_0(a, b, c, w) = 9(a+b+c)^2(S_2 - S_{1,1})\, \delta_2(a, b, c, w)^2\, \delta_3(a, b, c, w)$.
Especially $g_0(\mathbf{u}, w) \geq 0$.

(3) *For all* $t \in \mathbb{R}$,

$$D_f(t, a, b, c, w) = (w-1)\big((2t+1)S_2 - (t^2+2)S_{1,1}\big)^2 \times \delta_1(a, b, c, w)^2 \delta_2(a, b, c, w)^2 D_L(t, a, b, c, w).$$

Especially $\operatorname{sign}\big(D_f(t, \mathbf{u}, w)\big) = \operatorname{sign}\big((w-1)D_L(t, \mathbf{u}, w)\big)$.

(4) $D_f(1, a, b, c, w) = \Big(9(w-1)(S_2 - S_{1,1})\, \delta_1(a, b, c, w)\, \delta_2(a, b, c, w)\, \delta_4(a, b, c, w)\Big)^2$.
Especially $D_f(1, a, b, c, w) \geq 0$.

(5) $D_f(-2, a, b, c, w) = 972(w-1)(a+b+c)^5(S_2 - S_{1,1})\delta_1(a, b, c, w)^2 \times \delta_2(a, b, c, w)^2\, \delta_3(a, b, c, w)\, \delta_5(a, b, c, w)$.
Especially $\operatorname{sign}\big(D_f(-2, a, b, c, w)\big) = \operatorname{sign}\big((w-1)(a+b+c)\delta_5(a, b, c, w)\big)$.

(6) $h_1(-2, \mathbf{u}, w) = -4g_0(\mathbf{u}, w) \leq 0$.

(7) $h_1(1, a, b, c, w) = 9(1-w)(S_2 - S_{1,1})\, \delta_1(a, b, c, w)\, \delta_2(a, b, c, w)\, \delta_4(a, b, c, w)$.

(8) $\mathfrak{f}_{\mathbf{u},w}(0, -1, 1) = (1-w)(u_1+u_2+u_3)^3 \delta_1(\mathbf{u}, w)^2 \delta_5(\mathbf{u}, w)$.
Especially, if $\mathfrak{f}_{\mathbf{u},w} \in \mathcal{P}_{3,6}$, *then* $D_f(-2, \mathbf{u}, w) \leq 0$.

Proof. (1) $\delta_3(1, 0, 0, w) = 1 > 0$ and $\delta_3(x, 1, 1, w) = (x-1)^2(x-w)^2 \geq 0$. By [1, Proposition 5.1], we have $\delta_3(a, b, c, w) \geq 0$ for all a, b, c, $w \in \mathbb{R}$.

(2)—(8) can be obtained by direct calculations using Mathematica. □

There are some more relations like the above proposition. But we don't use them in this article.

Proposition 3.9. *For a, b, c, $w \in \mathbb{R}$, let*

$$\xi(a, b, c, w) := (a+b+c)(1-w)\delta_1(a, b, c, w)\delta_2(a, b, c, w).$$

Then $p^F_0(a, b, c, w)\mathfrak{f}_{\mathbf{u},w}(x, 1, 1) \geq 0$ for all $x \in \mathbb{R}$, if and only if $\xi(u, v, 1, w) \geq 0$.

Proof. Using PC, we know that $\mathfrak{f}_{\mathbf{u},w}(x, 1, 1)$ can be factored as the form

$$\begin{aligned} &p^F_0(\mathbf{u}, w)\mathfrak{f}_{\mathbf{u},w}(x, 1, 1) \\ &\qquad = (x-1)^2(x-w)^2\big(p^F_0(\mathbf{u}, w)^2x^2 + 2p^F_0(\mathbf{u}, w)f_1(\mathbf{u}, w)x + f_2(\mathbf{u}, w)\big), \end{aligned}$$

where $f_1(\mathbf{u}, w)$ and $f_2(\mathbf{u}, w)$ are certain polynomials. This can be reformed as the form

$$\begin{aligned} p^F_0(\mathbf{u}, w)\, \mathfrak{f}_{\mathbf{u},w}(x, 1, 1) &= (x-1)^2(x-w)^2\Big(\big(p^F_0(\mathbf{u}, w)x + f_1(\mathbf{u}, w)\big)^2 \\ &+ (u_1-u_2)^4(u_2-u_3)^4(u_3-u_1)^4(u_1+u_2+u_3)^3(1-w)\delta_1(\mathbf{u}, w)\delta_2(\mathbf{u}, w)^3\Big). \end{aligned}$$

Thus, we have the conclusion. □

Corollary 3.10. *Take $\mathbf{u} \in \mathbb{R}^3$, and $w \in \mathbb{R}$. Let*

$$I_{\mathbf{u},w} := \big\{t \in [-2, 1] \bigm| (w-1)D_L(t, \mathbf{u}, w) > 0\big\},$$
$$h_3(t, \mathbf{u}, w) := (1+2t)(4-t)h_1\left(\frac{4-t}{1+2t}, \mathbf{u}, w\right).$$

(I) *Assume that $p^F_0(\mathbf{u}, w) > 0$. Then $\mathfrak{f}_{\mathbf{u},w} \in \mathcal{P}_{3,6}$ if and only if $\xi(\mathbf{u}, w) \geq 0$ and for every $t \in I_{\mathbf{u},w}$ one of the following (1) or (2) holds.*

(1) $h_1(t, \mathbf{u}, w) \geq 0$.

(2) $h_3(t, \mathbf{u}, w) \le 0$.

(II) *Assume that* $p_0^F(\mathbf{u}, w) < 0$. *Then* $-\mathfrak{f}_{\mathbf{u},w} \in \mathcal{P}_{3,6}$ *if and only if* $\xi(\mathbf{u}, w) \ge 0$.

It seems that if $p_0^F(\mathbf{u}, w) < 0$ then $\xi(\mathbf{u}, w) < 0$, and $-\mathfrak{f}_{\mathbf{u},w} \notin \mathcal{P}_{3,6}$. But the author does not have a complete proof.

Remark 3.11. If $w = 1$, then

$$\mathfrak{f}_{\mathbf{u},1}(\mathbf{a}) = \big(S_2(\mathbf{u}) - S_{1,1}(\mathbf{u})\big)^3 \Big(\big(T_{2,1}(\mathbf{u}) - 6U(\mathbf{u})\big)S_3(\mathbf{a}) - \big(S_3(\mathbf{u}) - 3U(\mathbf{u})\big)T_{2,1}(\mathbf{a}) + \big(6S_3(\mathbf{u}) - 3T_{2,1}(\mathbf{u})\big)U(\mathbf{a})\Big)^2.$$

Thus, $\mathfrak{f}_{\mathbf{u},1}(\mathbf{a}) \in \mathcal{E}(\mathcal{P}_{3,6}) \cap \Sigma_{3,6}$.

Proof of Theorem 1.5. Put $\mathfrak{f}_{u,v,1,w} = \mathfrak{f}_{\mathbf{u},w}$ ($\mathbf{u} = (u, v, 1)$). This is the polynomial $\mathfrak{f}_{u,v,w}$ in Theorem 1.5. It is easy to see that $\mathfrak{f}_{\mathbf{u},w}(0, 0, 1) = p_0^F(\mathbf{u}, w)$.

Consider the case $\mathbf{u} = (-1/2, -1/3, 1)$ and $w = 9/10$. Then

$$p_0^F\left(-\frac{1}{2}, -\frac{1}{3}, 1, \frac{9}{10}\right) = \frac{2838188587}{147622500} > 0,$$
$$\delta_1\left(-\frac{1}{2}, -\frac{1}{3}, 1, \frac{9}{10}\right) = \frac{722}{135} > 0,$$
$$\delta_2\left(-\frac{1}{2}, -\frac{1}{2}, 1, \frac{9}{10}\right) = \frac{1279}{225} > 0,$$
$$\delta_4\left(-\frac{1}{2}, -\frac{1}{2}, 1, \frac{9}{10}\right) = \frac{255823}{24300} > 0,$$
$$\xi\left(-\frac{1}{2}, -\frac{1}{3}, 1, \frac{9}{10}\right) = \frac{461719}{911250} > 0,$$
$$h_1\left(t, -\frac{1}{2}, -\frac{1}{3}, 1, \frac{9}{10}\right) = \frac{1279}{132860250000}\Big(-4763259726t^3 + 10555137817t^2 + 53854835215t + 1028618365\Big),$$
$$D_L\left(t, -\frac{1}{2}, -\frac{1}{3}, 1, \frac{9}{10}\right) = \frac{1}{5904900000}\Big(398926806344t^2 - 1190526056662t + 202590584357\Big).$$

The least root of $D_L = 0$ is

$$\omega_1 := \frac{595263028331 - 767469\sqrt{464372213673}}{398926806344} = 0.181\cdots.$$

Thus, $D_f(t) \le 0$ if $-2 \le t \le \omega_1$. So, $I_1 \subset (\omega_1, 1]$. It is easy to check that $h_1 > 0$ on $[0, 1]$. Thus, $\mathfrak{f}_{-1/2,-1/3,9/10} \in \mathcal{P}_{3,6}$, by Proposition 3.9 and 3.7. By Theorem 3.1 and Lemma 3.3, $\mathfrak{f}_{-1/2,-1/3,9/10}$ satisfies (1)—(4) of Theorem 1.5.

Since $p_0^F(u, v, 1, w)$, $\xi(u, v, 1, w)$ and $h_1(t, u, v, 1, w)$ are continuous with respect to (u, v, v), there exists an open neighborhood $\mathcal{U}$ of $(-1/2, -1/3, 9/10)$ such that $p_0^F(u, v, 1, w) > 0$, $\xi(u, v, 1, w) > 0$ and $h_1(t, u, v, 1, w) > 0$ for all $-2 \le t \le 1$ and $(u, v, w) \in \mathcal{U}$. Thus, we have the conclusion. □

By numerical analysis, it seems that the above $\mathcal{U}$ is a small set.

4. Extremal elements of $\mathcal{P}_{3,5}^{s+}$

In §4.1 and 4.2, we prove Theorem 1.4. We sketch our idea of proof here because it is long.

In §4.1, we study properties of five families of polynomials $\mathfrak{e}^A_{t,u}$, $\mathfrak{e}^B_{t,u}$, $\mathfrak{e}^C_t$, $\mathfrak{e}^D_t$ and $\mathfrak{e}^E_t$. We study when these are extremal. We also study the conditions which characterize these polynomials.

In §4.2, we prove that $\mathcal{E}(\mathcal{P}^{s+}_{3,5})$ contains only the above polynomials and $s_3 = U(S_2 - S_{1,1})$. We prove this by a geometric observation of the boundary $\partial\mathcal{P}^{s+}_{3,5}$. In Proposition 4.13, we prove that $\partial\mathcal{P}^{s+}_{3,5}$ has five irreducible components. This sentence means that the Zariski closure of $\partial\mathcal{P}^{s+}_{3,5}$ in $\mathcal{H}^s_{3,5}$ is a union of five irreducible real algebraic varieties. In fact,

$$\mathcal{E}(\mathcal{P}^{s+}_{3,5}) \subset \mathcal{E}(\mathcal{F}(C^b)) \cup \mathcal{E}(\mathcal{F}(C^0)) \cup \mathcal{E}(\mathcal{F}(P_1)) \cup \mathcal{E}(\mathcal{F}(P_2)) \cup \mathcal{E}(\mathcal{F}(P_3)),$$

where symbols are explained in §4.2.1. So, we study $\mathcal{E}(\mathcal{F}(C^b))$, $\mathcal{E}(\mathcal{F}(C^0))$, $\mathcal{E}(\mathcal{F}(P_1))$, $\mathcal{E}(\mathcal{F}(P_2))$ and $\mathcal{E}(\mathcal{F}(P_3))$. Figures 4.2—4.10 show the places where the above extremal polynomials exist. These figures also show geometric structure of $\mathcal{E}(\mathcal{P}^{s+}_{3,5})$.

In §4.3, we present some applications. In §4.3.1 and §4.3.2, we prove that $\mathfrak{e}^B_{t,u}(a^2, b^2, c^2) \in \mathcal{E}(\mathcal{P}_{3,10}) - \Sigma_{3,10}$ and $\mathfrak{e}^A_{t,u}(a^2,\ b^2,\ c^2) \notin \Sigma_{3,10}$ under certain conditions. In §4.3.3, we study $\mathcal{P}^{s0+}_{3,5}$.

In this section, we use the following symbols as §3. We denote the coordinate system of $\mathbb{P}^2_{\mathbb{R}}$ by $(a\colon b\colon c)$, and put

$$\begin{aligned}
S_{m,n} &= S_{m,n}(a,b,c) := a^m b^n + b^m c^n + c^m a^n,\\
S_n &:= S_n(a,b,c) = S_{n,0}(a,b,c) = a^n + b^n + c^n,\\
T_{m,n} &:= S_{m,n}(a,b,c) + S_{n,m}(a,b,c),\\
U &= U(a,b,c) := abc.
\end{aligned}$$

To state the structure of $\mathcal{E}(\mathcal{P}^{s+}_{3,5})$, it will be convenient to use the following symbols to describe a base of a cone. For a closed convex cone $\mathcal{P} \subset \mathcal{H}$, we denote

$$\mathbb{P}\mathcal{H} := (\mathcal{H} - \{0\})/\mathbb{R}_{++} \supset \mathbb{P}\mathcal{P} := (\mathcal{P} - \{0\})/\mathbb{R}_{++} \supset \mathbb{P}\mathcal{E}(\mathcal{P}) := \mathcal{E}(\mathcal{P})/\mathbb{R}_{++} = \mathcal{E}(\mathbb{P}\mathcal{P}).$$

Note that $\mathbb{P}\mathcal{H}$ is not a projective space $\mathbb{P}(\mathcal{H}) := (\mathcal{H} - \{0\})/\mathbb{R}^\times$. There exists a natural $2:1$ map $\rho\colon \mathbb{P}\mathcal{H} \to \mathbb{P}(\mathcal{H})$, but $\mathbb{P}\mathcal{H}$ is not a semialgebraic variety. Neverthless, when we discuss about $\mathbb{P}\mathcal{P}^{s+}_{3,5}$, we may regard $\mathbb{P}\mathcal{P}^{s+}_{3,5}$ as a semialgebraic subvariety of $\mathbb{P}(\mathcal{H})$, because $\rho\colon \mathbb{P}\mathcal{P}^{s+}_{3,5} \to \mathbb{P}(\mathcal{H})$ is injective.

For $f \in \mathcal{H} - \{0\}$, we denote its equivalence class by $[f] := \mathbb{R}_{++} \cdot f \in \mathbb{P}\mathcal{H}$. Note that $[\alpha f] = [f]$ if $\alpha > 0$, but $[\alpha f] \neq [f]$ if $\alpha < 0$.

We choose $s_0 := S_5 - US_{1,1}$, $s_1 := T_{4,1} - 2US_{1,1}$, $s_2 := T_{3,2} - 2US_{1,1}$, $s_3 := US_2 - US_{1,1}$, $s_4 := US_{1,1}$ as a base of $\mathcal{H}^s_{3,5}$. Note that $\{s_0,\ s_1,\ s_2,\ s_3\}$ is a base of $\mathcal{H}^{s0}_{3,5}$.

The definitions of extremal polynomians $\mathfrak{e}^A_{t,u}$, $\mathfrak{e}^B_{t,u}$, $\mathfrak{e}^C_t$, $\mathfrak{e}^D_t$ and $\mathfrak{e}^E_t$ are as the following. We study these in §4.1 respectively. Let

$$\begin{aligned}
\mu_L(t) &:= 9(t-1)^2,\\
\mu_H(t) &:= (t+2)(7-t),\\
\mu_A(t) &:= \min\{\mu_L(t),\ \mu_H(t)\} = \begin{cases} \mu_L(t) & (\text{if } 0 \le t \le 5/2),\\ \mu_H(t) & (\text{if } 5/2 < t \le 7),\end{cases}\\
\mu_R(t) &:= 2 - t^2 + t\sqrt{(t-1)(t+2)},\\
\mu_B(t) &:= (1/2)\bigl(\mu_R(t) - \sqrt{\mu_R(t)^2 - 4}\bigr),\\
\mu_Z(t,u) &:= \frac{(t+2)(7-t) - u}{(t+2)(5t+1)}.
\end{aligned}$$

The polynomial $\mathfrak{e}^A_{t,u}$ *is defined by*

$$\begin{aligned}
\mathfrak{e}^A_{t,u}(a,b,c) &:= s_0 + \sum_{i=1}^{4} p^A_i(t,u) s_i \quad (\text{if } u > 0),\\
\mathfrak{e}^A_{t,0}(a,b,c) &:= (5t+1)^2 s_1 + (t-1)^2(t^2-12t-1)s_2\\
&\qquad - 2(t^4+36t^3+34t^2+60t+13)s_3 + 24(t-1)^4 s_4,
\end{aligned}$$

where

$$\begin{aligned}
p^A_1(t,u) &:= \frac{u^2-(t+2)(5t^2+t+9)u+9(t-1)^2(t+2)^2}{(5t+1)(t+2)u},\\
p^A_2(t,u) &:= \frac{1}{(5t+1)^3(t+2)u}\Big(- t^2u^3 + (t-1)(7t^3-t^2+11t+1)u^2\\
&\quad + (t+2)(17t^5-25t^4+199t^3-59t^2+76t+8)u\\
&\quad + 9(t-1)^4(t+2)^2(t^2-12t-1)\Big),\\
p^A_3(t,u) &:= \frac{1}{(t+2)^2(5t+1)^3u}\Big(2t^3+4t^2+5t+1)u^3\\
&\quad - 2(t+2)(7t^4+42t^3+37t^2+48t+10)u^2\\
&\quad + (t+2)^2(91t^5+125t^4+682t^3+182t^2+523t+125)u\\
&\quad - 18(t-1)^2(t+2)^3(t^4+36t^3+34t^2+60t+13)\Big),\\
p^A_4(t,u) &:= \frac{(t-1)^3(6t^2+6t-12+u)^3}{(t+2)^2(5t+1)^3u}.
\end{aligned}$$

Similarly, $\mathfrak{e}^B_{t,u}$, $\mathfrak{e}^C_t$, $\mathfrak{e}^D_t$ *and* $\mathfrak{e}^E_t$ *are defined as the following:*

$$\begin{aligned}
p^B_1(t,w) &:= -2w-3,\\
p^B_2(t,w) &:= w^2+2w+2,\\
p^B_3(t,w) &:= -\frac{2t^3+4t^2+5t+1}{t^2(t+2)}w^2 + \frac{2(4t^2+5t+3)}{t+2}w - \frac{3t^3-7t^2-12t-8}{t+2},\\
p^B_4(s,w) &:= \frac{(t-1)^3\big(-w^2-2t^2w+t^2(t-2)\big)}{t^2(t+2)},\\
\omega(u) &:= u + \frac{1}{u} - 2 = \frac{(u-1)^2}{u},\\
\mathfrak{e}^B_{t,u}(a,b,c) &:= s_0 + \sum_{i=1}^{4} p^B_i\big(t,\ \omega(u)\big)s_i,\\
\mathfrak{e}^C_t(a,b,c) &:= s_0 - (t+1)s_1 + ts_2 + (t+1)^2 s_3,\\
\mathfrak{e}^D_t(a,b,c) &:= s_1 + (t^2-1)s_2 - 2(t+1)^2 s_3,\\
\mathfrak{e}^D_\infty(a,b,c) &:= s_2 - 2s_3,\\
\mathfrak{e}^E_t(a,b,c) &:= s_1 - s_2 - \frac{4t^2+5t+3}{t+2}s_3 + \frac{(t-1)^3}{t+2}s_4,\\
\mathfrak{e}^E_\infty(a,b,c) &:= s_4.
\end{aligned}$$

4.1. Extremality.

In this subsection, we prove that $\mathcal{E}(\mathcal{P}^{s+}_{3,5})$ contains $\mathfrak{e}^A_{t,u}$ $(0 \le t \le 7, 0 \le u \le \mu_A(t))$, $\mathfrak{e}^B_{t,u}$ $(t \ge 2,\ \mu_B(t) \le u \le 1)$, $\mathfrak{e}^C_t$ $(0 \le t \le 2)$, $\mathfrak{e}^D_t$ $(t \ge 0)$, $\mathfrak{e}^D_\infty = s_2 - 2s_3$, $\mathfrak{e}^E_t$ $(t \ge 7)$, $\mathfrak{e}^E_\infty = s_4$, and s_3. We also study some more properties of these polynomials.

4.1.1. Some PSD conditions.

We start from the following Lemma:

Lemme 4.1. *Take $f \in \mathcal{H}^s_{3,5}$. If $f(x,1,1) \ge 0$ and $f(0,x,1) \ge 0$ for all $x \ge 0$, then $f \in \mathcal{P}^{s+}_{3,5}$.*

This lemma is a very special case of theory of *test set* for symmetric polynomials. See [15, Corollary 1.3] or [16, Corollary 2.1]. See also [2, Proposition 5.1] or [15, Theorem 1.1].

For $f(a,b,c) \in \mathcal{H}^s_{3,5}$, we denote

$$f_a(a,b,c) := \frac{\partial}{\partial a} f(a,b,c), \quad f_{ab}(a,b,c) := \frac{\partial^2}{\partial a \partial b} f(a,b,c),$$

and so on.

Proposition 4.2. (1) *Let $x > 0$ and $y > 0$ be constants. If $f \in \mathcal{P}^{s+}_{3,5}$ satisfies $f(x, y, 1) = 0$, then*

$$f_a(x,y,1) = f_b(x,y,1) = f_c(x,y,1) = 0.$$

(2) *Let $x > 0$. If $f \in \mathcal{P}^{s+}_{3,5}$ satisfies $f(0, x, 1) = 0$, then*

$$f_b(0,x,1) = f_c(0,x,1) = 0.$$

(3) *If $f \in \mathcal{H}_{3,5}$ satisfies $f(x,y,z) = f_a(x,y,z) = f_b(x,y,z) = 0$ and $z \ne 0$, then $f_c(x,y,z) = 0$.*

Proof. Easy exercise. □

4.1.2. Properties of polynomial $\mathfrak{e}^C_t$.

Since $\mathfrak{e}^A_{t,u}$ and $\mathfrak{e}^B_{t,u}$ are complicated polynomials, we treat other polynomials before them. To begin with, we study $\mathfrak{e}^C_t$, and next we will study $\mathfrak{e}^D_t$ and $\mathfrak{e}^E_t$.

Theorem 4.3. (1) *If $0 \le t \le 2$, then*

$$\mathfrak{e}^C_t = s_0 - (t+1)s_1 + ts_2 + (t+1)^2 s_3 \in \mathcal{E}(\mathcal{P}^{s+}_{3,5}).$$

Moreover $\mathfrak{e}^C_t$ is characterized by the following conditions:

(2) *If $0 < t \le 2$, $t \ne 1$ and if $f \in \mathcal{P}^{s+}_{3,5}$ satisfies*

$$f(t,1,1) = f(1,1,1) = f(0,1,1) = 0,$$

then $[f] = [\mathfrak{e}^C_t]$.

(3) *If $f \in \mathcal{P}^{s+}_{3,5}$ satisfies*

$$f(1,1,1) = f(0,1,1) = f_a(0,1,1) = f_{aa}(0,1,1) = 0,$$

then $[f] = [\mathfrak{e}^C_0]$.

(4) *If $f \in \mathcal{P}^{s+}_{3,5}$ satisfies*

$$f(0,1,1) = f(1,1,1) = f_{aa}(1,1,1) = 0,$$

then $[f] = [\mathfrak{e}^C_1]$.

Proof. (0) We shall show $\mathfrak{e}_t^C \in \mathcal{P}_{3,5}^{+}$ if $0 \le t \le 2$. Note that

$$\mathfrak{e}_t^C(x,1,1) = x(x-1)^2(x-t)^2 \ge 0,$$
$$\mathfrak{e}_t^C(0,x,1) = (x-1)^2(x+1)\big((x-1)^2 + (2-t)x\big) \ge 0,$$

if $x \ge 0$. Thus, $\mathfrak{e}_t^C \in \mathcal{P}_{3,5}^{s+}$ by Lemma 4.1. We prove $\mathfrak{e}_t^C \in \mathcal{E}(\mathcal{P}_{3,5}^{s+})$ later. Note that $\mathfrak{e}_t^C \notin \mathcal{P}_{3,5}^{+}$ if $t > 2$, because $\mathfrak{e}_t^C(0,1+x,1) < 0$ for $0 < x \ll 1$.

(2) Consider the case $0 < t \le 2$, $t \ne 1$. If $f \in \mathcal{P}_{3,5}^{s+}$ satisfies $f(t,1,1) = 0$, then $f_a(t,1,1) = 0$ by Proposition 4.2(2). Solve the following system of function equations for $f \in \mathcal{H}_{3,5}^{s}$:

$$f(t,1,1) = f_a(t,1,1) = f(1,1,1) = f(0,1,1) = 0. \qquad (*)$$

Denote $f = \sum_{i=0}^{4} p_i s_i$. Let A be the following matrix:

$$\begin{pmatrix} (t-1)^2(t+1)(t^2+t+2) & 2(t^2-1)^2 & 2(t-1)^2(t+1) & t(t-1)^2 & t(2t+1) \\ (t-1)(5t^3+5t^2+5t+1) & 8t(t^2-1) & 2(t-1)(3t+1) & 3t^2-4t+1 & 4t+1 \\ 0 & 0 & 0 & 0 & 3 \\ 2 & 2 & 2 & 0 & 0 \end{pmatrix}$$

and $\mathbf{p} := {}^t(p_0,\, p_1,\, p_2,\, p_3,\, p_4)$. The equation $(*)$ can be represented by $A\mathbf{p} = \mathbf{0}$. Thus, the solution space of $(*)$ is $\operatorname{Ker} A$. Since $\mathfrak{e}_t^C(t,1,1) = 0$, $\mathfrak{e}_t^C(1,1,1) = 0$ and $\mathfrak{e}_t^C(0,1,1) = 0$, we have $\mathfrak{e}_t^C \in \operatorname{Ker} A$.

Let $\mathbf{e}_1 := (1,0,0,0,0)$, and let A_1 be the square matrix obtained by putting $\mathbf{e}_1$ at the top line above A. Since $\det A_1 = -12t^2(t-1)^4$, we have $\dim \operatorname{Ker} A = 1$. Thus, $\operatorname{Ker} A = \mathbb{R} \cdot \mathfrak{e}_t^C$. This implies $\mathfrak{e}_t^C \in \mathcal{E}(\mathcal{P}_{3,5}^{s+})$.

(3) Consider the case $t = 0$. In this case, we consider the following, instead of $(*)$.

$$f(1,1,1) = f(0,1,1) = f_a(0,1,1) = f_{aa}(0,1,1) = 0.$$

The left part is same with (2).

(4) Consider the case $t = 1$. If $f \in \mathcal{P}_{3,5}^{s+}$ satisfies $f(1,1,1) = f_{aa}(1,1,1) = 0$, then $f_a(1,1,1) = f_{aaa}(1,1,1) = 0$. In this case, we consider the following, instead of $(*)$.

$$f(0,1,1) = f(1,1,1) = f_{aa}(1,1,1) = f_{aaa}(1,1,1) = 0.$$

The left part is same with (2).

(1) We prove $\mathfrak{e}_t^C \in \mathcal{E}(\mathcal{P}_{3,5}^{s+})$. Assume that $0 < t \le 2$, $t \ne 1$ and $\mathfrak{e}_t^C = f + g$ by a certain $f,\, g \in \mathcal{P}_{3,5}^{s+}$. Then f and g satisfy the equalities $(*)$ in the proof of (2), by Proposition 4.2. Thus, $f,\, g \in \mathbb{R} \cdot \mathfrak{e}_t^C$. This implies $\mathfrak{e}_t^C \in \mathcal{E}(\mathcal{P}_{3,5}^{s+})$.

We can prove $\mathfrak{e}_0^C,\, \mathfrak{e}_1^C \in \mathcal{E}(\mathcal{P}_{3,5}^{s+})$ using (3) and (4) similarly. □

4.1.3. Properties of polynomial $\mathfrak{e}_t^D$.

Note that $\lim_{t\to+\infty} [\mathfrak{e}_t^D] = [\mathfrak{e}_\infty^D] = [s_2 - 2s_3]$ in $\mathbb{P}\mathcal{H}_{3,5}$.

Theorem 4.4. (1) *If $t \ge 0$, then*

$$\mathfrak{e}_t^D = s_1 + (t^2-1)s_2 - 2(t+1)^2 s_3 \in \mathcal{E}(\mathcal{P}_{3,5}^{s+}).$$

But $\mathfrak{e}_t^D \notin \mathcal{E}(\mathcal{P}_{3,5}^{+})$.

(2) *If $t > 0$, $t \ne 1$ and $f \in \mathcal{P}_{3,5}^{s}$ satisfies*

$$f(t,1,1) = f(1,1,1) = f(0,0,1) = 0,$$

then $[f] = [\mathfrak{e}_t^D]$.

(3) *If* $f \in \mathcal{P}^{s+}_{3,5}$ *satisfies*

$$f(1,1,1) = f_{aa}(1,1,1) = f(0,0,1) = 0,$$

then $[f] = [\mathfrak{e}^D_1]$. *Especially,* $\mathfrak{e}^D_1 = s_1 - 8s_3 \in \mathcal{E}(\mathcal{P}^{s+}_{3,5})$.

(4) *If* $f \in \mathcal{P}^{s+}_{3,5}$ *satisfies*

$$f(1,1,1) = f(0,0,1) = f_a(0,0,1) = f_{aa}(0,0,1) + f_{ab}(0,0,1) = 0,$$

then $[f] = [\mathfrak{e}^D_\infty]$. *Especially,* $\mathfrak{e}^D_\infty = s_2 - 2s_3 \in \mathcal{E}(\mathcal{P}^{s+}_{3,5})$.

Proof. (0) We shall show $\mathfrak{e}^D_t \in \mathcal{P}^{s+}_{3,5}$ if $t \geq 0$. Note that

$$\begin{aligned}\mathfrak{e}^D_t(a,b,c) &= a(b-c)^2((t+1)a-b-c)^2 \\ &\quad + b(c-a)^2((t+1)b-c-a)^2 + c(a-b)^2((t+1)c-a-b)^2.\end{aligned}$$

Thus $\mathfrak{e}^D_t \in \mathcal{P}^{s+}_{3,5}$, but $\mathfrak{e}^D_t \notin \mathcal{E}(\mathcal{P}^{+}_{3,5})$. We also note that

$$\mathfrak{e}^D_t(x,1,1) = 2(x-1)^2(x-t)^2, \quad \mathfrak{e}^D_t(0,x,1) = x(x+1)\big((x-1)^2 + t^2x\big).$$

(2) If $f \in \mathcal{P}^{s+}_{3,5}$ satisfies $f(t,1,1) = 0$, then $f_a(t,1,1) = 0$. Take $f = \sum_{i=0}^{4} p_i s_i \in \mathcal{H}^s_{3,5}$, and put $\mathbf{p} := {}^t(p_0, p_1, p_2, p_3, p_4)$. Let A be the following matrix:

$$\begin{pmatrix} (t-1)^2(t+1)(t^2+t+2) & 2(t^2-1)^2 & 2(t-1)^2(t+1) & t(t-1)^2 & t(2t+1) \\ (t-1)(5t^3+5t^2+5t+1) & 8t(t^2-1) & 2(t-1)(3t+1) & (t-1)(3t-1) & 4t+1 \\ 0 & 0 & 0 & 0 & 3 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

The system of equations $f(t,1,1) = f_a(t,1,1) = f(1,1,1) = f(0,0,1) = 0$ is equivalent to $A\mathbf{p} = \mathbf{0}$. Using Mathematica, we can check $\operatorname{Ker} A = \mathbb{R} \cdot \mathfrak{e}^D_t$.

(3) Consider $f(1,1,1) = f_{aa}(1,1,1) = f_{aaa}(1,1,1) = f(0,0,1) = 0$.

(1) $\mathfrak{e}^D_t \in \mathcal{E}(\mathcal{P}^{s+}_{3,5})$ $(t \geq 0)$ follows from (0), (2) and (3).

(4) $\mathfrak{e}^D_\infty \in \mathcal{P}^{s+}_{3,5}$, because

$$\mathfrak{e}^D_\infty(x,1,1) = (x+1)x^2, \quad \mathfrak{e}^D_\infty(0,x,1) = 2(x-1)^2.$$

The system of equations $f(1,1,1) = f(0,0,1) = f_a(0,0,1) = f_{aa}(0,0,1) + f_{ab}(0,0,1) = 0$ is represented by the matrix

$$A = \begin{pmatrix} 0 & 0 & 0 & 0 & 3 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 2 & 1 & 0 \end{pmatrix}.$$

It is easy to see that $\operatorname{Ker} A = \mathbb{R} \cdot \mathfrak{e}^D_\infty$. □

4.1.4. Properties of polynomials $\mathfrak{e}^E_t$ and s_3.

Note that $\lim_{t \to +\infty} [\mathfrak{e}^E_t] = [\mathfrak{e}^E_\infty] = [s_4]$ in $\mathbb{P}\mathcal{H}_{3,5}$.

Theorem 4.5. (1) *If* $t \geq 7$, *then*

$$\mathfrak{e}_t^E := s_1 - s_2 - \frac{4t^2+5t+3}{t+2}s_3 + \frac{(t-1)^3}{t+2}s_4 \in \mathcal{E}(\mathcal{P}_{3,5}^{s+}).$$

(2) *If* $t \geq 7$ *and* $f \in \mathcal{P}_{3,5}^{s+}$ *satisfies*

$$f(t,1,1) = f(0,1,1) = f(0,0,1) = 0,$$

then $[f] = [\mathfrak{e}_t^E]$.

(3) $\mathfrak{e}_\infty^E = s_4 \in \mathcal{E}(\mathcal{P}_{3,5}^{s+})$.

(4) *If* $f \in \mathcal{P}_{3,5}^{s+}$ *satisfies*

$$f(0,1,1) = f(0,0,1) = f_a(0,0,1) = f_{ab}(0,0,1) = 0,$$

then $[f] = [\mathfrak{e}_\infty^E]$.

Proof. (0) We shall show $\mathfrak{e}_t^E \in \mathcal{P}_{3,5}^{s+}$ if $t \geq 7$. Note that

$$\mathfrak{e}_t^E(0,x,1) = x(x+1)(x-1)^2 \geq 0,$$
$$\mathfrak{e}_t^E(x,1,1) = x(x-t)^2\left(2x + \frac{t-7}{t+2}\right) \geq 0,$$

if $x \geq 0$. Thus, $\mathfrak{e}_t^E \in \mathcal{P}_{3,5}^{s+}$. Note that $\mathfrak{e}_t^E \notin \mathcal{P}_{3,5}^{+}$ if $t < 7$.

(2) Consider the system of equations

$$f(t,1,1) = f_a(t,1,1) = f(0,1,1) = f(0,0,1) = 0$$

for $f \in \mathcal{H}_{3,5}^s$. The solution space of this system of equations is the kernel of the following matrix A:

$$\begin{pmatrix} (t-1)^2(t+1)(t^2+t+2) & 2(t^2-1)^2 & 2(t-1)^2(t+1) & t(t-1)^2 & t(2t+1) \\ (t-1)(5t^3+5t^2+5t+1) & 8t(t^2-1) & 2(t-1)(3t+1) & (t-1)(3t-1) & 4t+1 \\ 2 & 2 & 2 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

It is easy to see that $\operatorname{Ker} A = \mathbb{R}\cdot\mathfrak{e}_t^E$.

(1) $\mathfrak{e}_t^E \in \mathcal{E}(\mathcal{P}_{3,5}^{s+})$ follows from (0) and (2).

(3) $\mathfrak{e}_\infty^E \in \mathcal{P}_{3,5}^{s+}$, since $\mathfrak{e}_\infty^E(0,x,1) = 0$ for all $x \in \mathbb{R}_+$, and $\mathfrak{e}_\infty^E(x,1,1) = x(2x+1) \geq 0$.

(4) Consider $f(0,1,1) = f(0,0,1) = f_a(0,0,1) = f_{ab}(0,0,1) = 0$ for $f \in \mathcal{H}_{3,5}^s$. This system of equations is equivalent to

$$\begin{pmatrix} 2 & 2 & 2 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}\mathbf{p} = \mathbf{0}.$$

The solution space is $\mathbb{R}\cdot s_4$. □

Theorem 4.6. (1) $s_3 \in \mathcal{E}(\mathcal{P}_{3,5}^{s+})$.

(2) *If* $f \in \mathcal{P}_{3,5}^{s+}$ *satisfies* $f(0,x,1) = 0$ *for all* $x \geq 0$, $f_a(0,0,1) = 0$ *and* $f(1,1,1) = 0$, *then* $[f] = [s_3]$.

Proof. (0) $s_3 \in \mathcal{P}_{3,5}^{s+}$ follows from $s_3(0,x,1) = 0$ and $s_3(x,1,1) = x(x-1)^2 \geq 0$ for all $x \in \mathbb{R}_+$.

(2) The solution space $f(1,1,1) = f(0,1,1) = f(0,0,1) = f_a(0,0,1) = 0$ for $f \in \mathcal{H}_{3,5}^s$ is $\mathbb{R}\cdot s_3$.

(1) $s_3 \in \mathcal{E}(\mathcal{P}_{3,5}^{s+})$ follows from (0) and (2). □

4.1.5. Properties of polynomial $\mathfrak{e}^A_{t,u}$.

Now, we observe $\mathfrak{e}^A_{t,u}$. Since $\mathfrak{e}^A_{t,u}(t,1,1) = \mathfrak{e}^A_{t,u}(\mu_Z(t,u),1,1) = 0$, $\mathfrak{e}^A_{t,u}$ has at least six zeros interior of $\mathbb{P}^3_+$. Note that $\lim_{u\to+0}[\mathfrak{e}^A_{t,u}] = [\mathfrak{e}^A_{t,0}]$ in $\mathbb{P}\mathcal{H}_{3,5}$, and $\mathfrak{e}^A_{1,0} = 36(s_1 - 8s_3) = 36\mathfrak{e}^D_1 \in \mathcal{E}(\mathcal{P}^{s+}_{3,5})$.

Theorem 4.7. (1) *If $0 \le t \le 7$, $t \ne 1$ and $0 \le u \le \mu_A(t)$, then $\mathfrak{e}^A_{t,u} \in \mathcal{E}(\mathcal{P}^{s+}_{3,5})$.*
(2) *Assume that $0 \le t \le 7$, $t \ne 1$ and $0 \le u \le \mu_A(t)$. Then $t \ne \mu_Z(t,u)$. If $f \in \mathcal{P}^{s+}_{3,5}$ satisfies*

$$f(t,1,1) = f(\mu_Z(t,u),1,1) = 0,$$

then $[f] = [\mathfrak{e}^A_{t,u}]$.
(3) $\mathfrak{e}^A_{7,0} = 1296\mathfrak{e}^A_{0,0} = 1296\mathfrak{e}^E_7$.

Proof. (0) We shall show $\mathfrak{e}^A_{t,u} \in \mathcal{E}(\mathcal{P}^{s+}_{3,5})$ if $0 \le t \le 7$, $t \ne 1$ and $0 \le u \le \mu_A(t)$.

(0-i) We shall prove $\mathfrak{e}^A_{t,u}(0,x,1) \ge 0$ for $x \ge 0$. Let

$$\begin{aligned} h^A(t,u) &:= u^2 - (t+2)(5t^2 - 14t + 6)u + (t+2)^2\mu_L(t), \\ g^A(t,u,w) &:= (t+2)(5t+1)^3uw^2 \\ &\qquad + (5t+1)^2h^A(t,u)w + t^2(\mu_H(t) - u)^2(\mu_L(t) - u). \end{aligned}$$

Then $\mathfrak{e}^A_{t,u}(0,x,1) = \dfrac{x^2(x+1)g^A(t,u,x+1/x-2)}{u(t+2)(5t+1)^3}$. To prove $\mathfrak{e}^A_{t,u}(0,x,1) \ge 0$ for all $x \ge 0$, it is enough to show $h^A(t,u) \ge 0$. If $0 \le t \le 8/5$, then

$$h^A(t,u) = (u + 3(t-1)(t+2))^2 + t(t+2)(8-5t)u \ge 0.$$

If $8/5 < t \le (10 + 2\sqrt{10})/5$, then $5t^2 - 20t + 12 \le 0$. Thus,

$$h^A(t,u) = \big(u - 3(t-1)(t+2)\big)^2 - (t+2)(5t^2 - 20t + 12)u \ge 0.$$

If $t > (10 + 2\sqrt{10})/5$, then

$$\frac{(t+2)(5t^2 - 14t + 6)}{2} - \mu_H(t) = \frac{(t+2)(5t^2 - 12t - 8)}{2} > 0.$$

Thus, $h^A(t,u)$ is decreasing on $0 \le u \le \mu_H(t)$, and

$$h^A(t,u) \ge h^A(t,\mu_H(t)) = (t-4)^2(t+2)^2(5t+1) \ge 0.$$

Thus, we have $\mathfrak{e}^A_{t,u}(0,x,1) \ge 0$ for $x \ge 0$.

(0-ii) Assume t and u are the same as in (0). Then,

$$\mathfrak{e}^A_{t,u}(x,1,1) = (x-t)^2\big(x - \mu_Z(t,u)\big)^2\left(x + \frac{2(t+2)(\mu_L(t) - u)}{(5t+1)u}\right) \ge 0$$

for all $x \ge 0$. Thus, $\mathfrak{e}^A_{t,u} \in \mathcal{P}^{s+}_{3,5}$.

(2) Let $0 \le t \le 7$, $t \ne 1$ and $0 \le u \le \mu_A(t)$. It is easy to see that $t = \mu_Z(t,u)$ if and only if $u = -(t-1)(t+2)(5t+7)$. If $t > 1$, then $-(t-1)(t+2)(5t+7) < 0 \le u$. If $0 \le t < 1$, then $u \le \mu_A(t) = \mu_L(t) < -(t-1)(t+2)(5t+7)$. Thus we have $t \ne \mu_Z(t,u)$.

Assume that $f \in \mathcal{H}^s_{3,5}$ satisfies

$$f(t,1,1) = f_a(t,1,1) = f(\mu_Z(t,u),1,1) = f_a(\mu_Z(t,u),1,1) = 0.$$

Construct the 4×5 matrix A from these equalities as before. Put the vector $\mathbf{e}_4 = (0,$ $0,$ $0,$ $1,$ $0)$ above the top line of A, and construct the 5×5 matrix B. Then,

$$\det B = -\frac{(4(t-1)^4(u + 6(t-1)(t+2))^4(u + (t-1)(t+2)(5t+7))^4)}{(t+2)^8(5t+1)^8}.$$

Note that $u + (t-1)(t+2)(5t+7) \neq 0$ if $t \neq \mu_Z(t,u)$. It is easy to see that $u + 6(t-1)(t+2) \neq 0$ if $0 \leq u \leq \mu_A(t)$. Thus, $\det B \neq 0$ and $\operatorname{Ker} A = \mathbb{R} \cdot \mathfrak{e}^A_{t,u}$.

(3) follows from a direct calculation.

(1) $\mathfrak{e}^A_{t,u} \in \mathcal{E}(\mathcal{P}^{s+}_{3,5})$ follows from (0) and (2). □

Remark 4.8. (1) Let $u_0 = 3(t-1)^2(t+2)/(2t+1)$. Then

$$\mathfrak{e}^A_{t,u_0}(a,b,c) = (a+b+c)\left(S_2(a,b,c) - \frac{S_2(t,1,1)}{S_{1,1}(t,1,1)} S_{1,1}(a,b,c)\right)^2.$$

(2) The following typical polynomials often appear:

$$\mathfrak{e}^A_{t,\mu_L(t)} = s_0 - \frac{t^2+5}{t+2} s_1 + \frac{t^2-t+3}{t+2} s_2 + \frac{t^4-6t^3+10t^2+18t+13}{(t+2)^2} s_3 + 3\frac{(t-1)^4}{(t+2)^2} s_4,$$

when $0 \leq t \leq 5/2$.

$$\mathfrak{e}^A_{t,\mu_H(t)} = s_0 + \frac{t^2-5t-5}{7-t} s_1 - \frac{t^2-6t+2}{7-t} s_2 - \frac{(t+2)(t^2-3t-2)}{7-t} s_3 + \frac{(t-1)^3}{7-t} s_4,$$

when $5/2 \leq t < 7$.

4.1.6. Properties of polynomial $\mathfrak{e}^B_{t,u}$.

The polynomial $\mathfrak{e}^B_{t,u}$ is hard to treat. But $\mathfrak{e}^B_{t,u}$ will be the most important element in $\mathcal{E}(\mathcal{P}^{s+}_{3,5})$. The fact $\mathfrak{e}^B_{t,u} \in \mathcal{E}(\mathcal{P}^{+}_{3,5})$ will be proved in Theorem 4.28. To treat $\mathfrak{e}^B_{t,u}$, we need the following lemma. We denote the discriminant of $c_n x^n + c_{n-1}x^{n-1} + \cdots + c_1 x + c_0 = 0$ by $\mathrm{Disc}_n(c_n, c_{n-1}, \ldots, c_0)$.

Lemma 4.9. *Let $f(x) = x^3 + ax^2 + bx + c$. Then $f(x) \geq 0$ for all $x \geq 0$ if and only if one of (1), (2) or (3) holds:*
(1) $a \geq 0$, $b \geq 0$ *and* $c \geq 0$.
(2) $c = 0$ *and* $a^2 - 4b \leq 0$.
(3) $c > 0$ *and* $\mathrm{Disc}_3(1,a,b,c) = a^2b^2 - 4b^3 - 4a^3c + 18abc - 27c^2 \leq 0$.

Proof. If $f(x) \geq 0$ for all $x \geq 0$, then $c = f(0) \geq 0$.

(i) If $c = 0$, considering a condition for that $x^2 + ax + b \geq 0$ for all $x \geq 0$, we have (1) "$a \geq 0$ and $b \geq 0$", or (2) $a^2 - 4b \leq 0$.

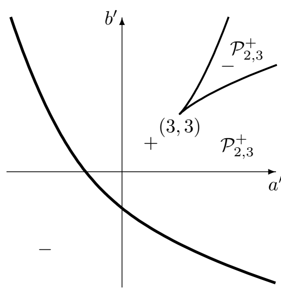

Fig. 4.1. Graph of $D_3(1,a',b',1) = 0$

(ii) Assume $c > 0$. If we consider $a' := a/\sqrt[3]{c}$, $b' := b/\sqrt[3]{c^2}$, $c' := c/\sqrt[3]{c^3} = 1$, and $x' := x/\sqrt[3]{c}$, we can reduce to the case $c = 1$. Then $\mathrm{Disc}_3(1,a,b,1) = \mathrm{disc}^{c+}_3(a,b)$, where disc^{c+}_3 was defined in [1, Theorem 3.1] (see Fig.4.1). Thus, by the same argument with the proof of [1, Theorem 3.1], we have the conclusion. □

Theorem 4.10. (1) *If $t \geq 2$ and $\mu_B(t) \leq u \leq 1$, then $\mathfrak{e}^B_{t,u} \in \mathcal{E}(\mathcal{P}^{s+}_{3,5})$.*
(2) *Let t, u be constants such that $t \geq 2$ and $\mu_B(t) \leq u < 1$. If $f \in \mathcal{P}^s_{3,5}$ satisfies*
$$f(t,1,1) = f(0,u,1) = f_b(0,u,1) = 0,$$
then $[f] = [\mathfrak{e}^B_{t,u}]$.
(3) *Assume that $t \geq 2$. If $f \in \mathcal{P}^s_{3,5}$ satisfies*
$$f(t,1,1) = f(0,1,1) = f_{bb}(0,1,1) = 0,$$
then $[f] = [\mathfrak{e}^B_{t,1}]$.
(4) $\mathfrak{e}^B_{2,1} = \mathfrak{e}^C_2$.

Proof. (0) We shall show that $\mathfrak{e}^B_{t,u} \in \mathcal{P}^{s+}_{3,5}$ if $t \geq 2$ and $\mu_B(t) \leq u \leq 1$. Using computer, we have
$$\mathfrak{e}^B_{t,u}(0,x,1) = (x+1)(x-u)^2(x-1/u)^2 \geq 0,$$
if $x \geq 0$. But, our proof of $\mathfrak{e}^B_{t,u}(x,1,1) \geq 0$ is not so easy. We shall prove this as the following steps (0-i)—(0-iv).

(0-i) Put $\omega(u) = u - 2 + 1/u$. Note that $\mu_B(t) \leq u \leq 1$ is equivalent to $0 \leq \omega(u) \leq \mu_R(t) - 2$. Let
$$\begin{aligned}
f^B_{t,w}(a,b,c) &:= s_0 + \sum_{i=1}^{4} p^B_i(t,w)s_i,\\
c^B_0(t,w) &:= t^2(t+2),\\
c^B_1(t,w) &:= 2t^2(t+2)(-2w+t-3),\\
c^B_2(t,w) &:= -(5t+1)w^2 - 2t^2(t-7)w + t^2(t-4)^2,\\
c^B_3(t,w) &:= 2(t+2)w^2,\\
g^B(t,w,x) &:= c^B_0(t,w)x^3 + c^B_1(t,w)x^2 + c^B_2(t,w)x + c^B_3(t,w).
\end{aligned}$$
Then $\mathfrak{e}^B_{t,u}(x,1,1) = f^B_{t,\omega(u)}(x,1,1) = \dfrac{(x-t)^2}{t^2(t+2)} g^B\,(t,\ \omega(u),\ x)$. Thus, $\mathfrak{e}^B_{t,u}(t,1,1) \geq 0$ is equivalent to $g^B(t,w,x) \geq 0$ for $w = \omega(u)$.

Note that $c^B_0(t,w) > 0$ and $c^B_3(t,w) \geq 0$. We also note that $\mathfrak{e}^B_{t,u}(t,1,1) = 0$.

(0-ii) We shall show that $c^B_2(t,w) \geq 0$ if $t \geq 2$ and $0 \leq w \leq \mu_R(t) - 2$.

$c^B_2(t,w)$ is a concave quadric function on w, and $c^B_2(t,0) = t^2(t-4)^2 \geq 0$. Since
$$c^B_2\big(t, \mu_R(t) - 2\big) = t^2(t+2)\big(8t\sqrt{(t-1)(t+2)} - (8t^2+4t-9)\big),$$
$$\big(8t\sqrt{(t-1)(t+2)}\big)^2 - (8t^2+4t-9)^2 = 9(8t-9) > 0,$$
we have $c^B_2(t,w) \geq 0$.

(0-iii) Consider the case $c^B_1(t,w) \geq 0$, $t \geq 2$ and $0 \leq w \leq \mu_R(t) - 2$. Then $g^B(t,w,x) \geq g^B(t,w,0) = c^B_3(t,w) \geq 0$. By Lemma 4.9(1), we have $g^B(t,w,x) \geq 0$.

(0-iv) We assume $c^B_1(t,w) < 0$. Then $w > (t-3)/2$.

(0-iv-a) Consider the case $w = 0$. Since $0 = w > (t-3)/2$, we have $2 \leq t < 3$. Then $t^2 - 3t - 1 < 0$. Thus
$$c^B_1(t,0)^2 - 4c^B_0(t,0)c^B_2(t,0) = 4t^4(t^2-4)(t^2-3t-1) < 0.$$
By Lemma 4.9(2), we have $g^B(t,w,x) \geq 0$.

(0-iv-b) Consider the case $0 < w \leq \mu_R(t) - 2$ under assumtions $t \geq 2$ and $c^B_1(t,w) < 0$. It is enough to show $\mathrm{Disc}_3(c^B_0, c^B_1, c^B_2, c^B_3) \geq 0$ by Lemma 4.9(3). Using PC, we have
$$\mathrm{Disc}_3\big(c^B_0(t,w),\ c^B_1(t,w),\ c^B_2(t,w),\ c^B_3(t,w)\big) = 4t^2(t+2)b^B_1(t,w)^2 b^B_2(t,w) b^B_3(t,w),$$

$$b_1^B(t,w) := (2t+1)w - t(t-4),$$
$$b_2^B(t,w) := -w^2 - 2t^2w + t^2(t-2),$$
$$b_3^B(t,w) := (t+1)(5t+1)^2w^2 + 2t(t^4-13t^3+25t^2+27t-4)w - t^2(t-4)^2(t^2-3t-1).$$

Note that $b_2^B\big(t,\,\mu_R(t)-2\big)=0$. Since $b_2^B(t,w)$ is a concave quadric function on w, it is easy to see that $b_2^B(t,w)\ge 0$, if $t\ge 2$ and $0\le w\le \mu_R(t)-2$. Thus, to show $\mathrm{Disc}_3(c_0^B,c_1^B,c_2^B,c_3^B)\ge 0$, it is enough to show $b_3^B(t,w)\ge 0$.

(0-iv-b-1) We shall show $b_3^B(t,w)>0$ if $2\le t<3$ and $0\le w$. Let

$$w_3(t) := \frac{-t(t^4-13t^3+25t^2+27t-4)}{(t+1)(5t+1)^2},\quad v_3(t) := \frac{t^3(3-t)^3(t+2)^4}{(t+1)(5t+1)^2}.$$

Note that $b_3^B(t,w)=(t+1)(5t+1)^2\big(w-w_3(t)\big)^2+v_3(t)$. Thus $b_3^B(t,w)\ge v_3(t)>0$ if $2\le t<3$,

(0-iv-b-2) We shall show $b_3^B(t,w)>0$ if $t\ge 3$ and $(t-3)/2\le w$. Let

$$u_3(t) := 2t^5-t^4+10t^3-40t^2-40t-3,\quad d_3(t) := 5t^5-15t^4-6t^3-26t^2+141t+9.$$

Then

$$\frac{t-3}{2}-w_3(t)=\frac{u_3(t)}{2(t+1)(5t+1)^2}.$$

Since $u_3(s+3)=2s^5+29s^4+178s^3+536s^2+692s+192>0$ when $s\ge 0$, we have $(t-3)/2>w_3(t)$ if $t\ge 3$. Thus $b_3^B(t,w)$ is strictly increasing on $w\ge (t-3)/2$. Therefore

$$b_3^B(t,w) > b_3\left(t,\frac{t-3}{2}\right)=\frac{d_3(t)}{4}.$$

Since $d_3(s+3)=5s^5+60s^4+264s^3+460s^2+228s+36>0$ if $s\ge 0$, we have $b_3^B(t,w)>0$. Thus, $\mathrm{Disc}_3\ge 0$.

(1) Thus, we have $\mathfrak{e}_{t,u}^B\in\mathcal{P}_{3,5}^{s+}$. We obtain $\mathfrak{e}_{t,u}^B\in\mathcal{E}(\mathcal{P}_{3,5}^{s+})$, if we prove (2) and (3).

(2) Consider $f(t,1,1)=f_a(t,1,1)=f(0,u,1)=f_b(0,u,1)=0$ for $f\in\mathcal{H}_{3,5}^s$. The solution space is $\mathrm{Ker}\,A$, where

$$A := \begin{pmatrix} t^5-2t^2-t+2 & 2(t^2-1)^2 & 2(t-1)^2(t+1) & t(t-1)^2 & t(2t+1)\\ 5t^4-4t-1 & 8t(t^2-1) & 6t^2-4t-2 & 3t^2-4t+1 & 4t+1\\ u^5+1 & u^4+u & u^3+u^2 & 0 & 0\\ 5u^4 & 4u^3+1 & 3u^2+2u & 0 & 0\end{pmatrix}.$$

Put $(1,0,0,0,0)$ above A, and make a square matrix B. Then

$$\det B = 2u^2(u-1)(u+1)^3t^2(t-1)(t+2).$$

Since $t\ge 2$ and $0<\mu_B(t)\le u<1$, we have $\det B\ne 0$. Thus, $\mathrm{Ker}\,A=\mathbb{R}\cdot\mathfrak{e}_{t,u}^B$.

(3) Consider $f(t,1,1)=f_a(t,1,1)=f(0,1,1)=f_{bb}(0,1,1)=0$.

(4) follows from a direct calculation. □

Remark 4.11. (1) If $t\ge 2$, then

$$\mathfrak{e}_{t,\mu_B(t)}^B = s_0+(1-2\mu_R(t))s_1+(t^3+2t^2-2-2(t^2-1)\mu_R(t))s_2 - ((t+1)^2(2t+3)-4(t+1)^2\mu_R(t))s_3.$$

(2) Since $\lim_{t\to+\infty}[\mathfrak{e}_{t,u}^B]=[s_4]$ for any $\mu_B(t)\le u\le 1$, we regard $\mathfrak{e}_{\infty,u}^B := s_4=\mathfrak{e}_\infty^E$.

(3) If $b_1^B(t,\,\omega(u))=(2t+1)\omega(u)-t(t-4)=0$, then

$$\mathfrak{e}_{t,u}^B(a,b,c)=S_1(S_2-kS_{1,1})^2,$$

where $k=\dfrac{S_2(t,1,1)}{S_{1,1}(t,1,1)}=\dfrac{S_2(0,u,1)}{S_{1,1}(0,u,1)}$.

4.2. Structure of $\mathcal{E}(\mathbb{P}\mathcal{P}^{s+}_{3,5})$.

We define $\Phi : \mathbb{P}^2_+ \longrightarrow \mathbb{P}\big((\mathcal{H}^s_{3,5})^\vee\big)$ by $\Phi(\mathbf{a}) = \big(s_0(\mathbf{a})\colon s_1(\mathbf{a})\colon s_2(\mathbf{a})\colon s_3(\mathbf{a})\colon s_4(\mathbf{a})\big)$. The semialgebraic set $X = X^{s+}_{3,5} := \Phi(\mathbb{P}^2_+)$ is called the characteristic variety of $\mathcal{P}^{s+}_{3,5}$ (see [1, §1.2]). We regard X as a semialgebraic variety. About the definition of semialgebraic varieties, please see [2, §5] or [3, §2].

The symmetric group $\mathfrak{S}_3$ acts on $\mathbb{P}^2_+$ naturally. Let $\sigma_1(a, b, c) = a+b+c$, $\sigma_2(a, b, c) = ab+bc+ca$, $\sigma_3(a, b, c) = abc$, and define $\pi\colon \mathbb{P}^2_+ \longrightarrow \mathbb{P}^2_+/\mathfrak{S}_3 \subset \mathbb{P}_{\mathbb{R}}(1:2:3)$ by $\pi(\mathbf{a}) = \big(\sigma_1(\mathbf{a})\colon \sigma_2(\mathbf{a})\colon \sigma_3(\mathbf{a})\big)$, where $\mathbb{P}_{\mathbb{R}}(1:2:3)$ is the weighted projective space. Note that $\mathbb{P}^2_{\mathbb{C}}/\mathfrak{S}_3 \cong \mathbb{P}_{\mathbb{C}}(1:2:3)$, but $\mathbb{P}^2_{\mathbb{R}}/\mathfrak{S}_3 \subsetneq \mathbb{P}_{\mathbb{R}}(1:2:3)$. There exists a natural rational map $\Psi\colon \mathbb{P}^2_+/\mathfrak{S}_3 \cdots\to X$ such that $\Phi = \Psi\circ\pi$. By [2, Proposition 2.14] and [1, Proposition 2.12—2.14], $\Psi\colon \mathbb{P}^2_+/\mathfrak{S}_3 \longrightarrow X$ is a regular map and is an isomorphism.

4.2.1. Structure of $\partial\mathbb{P}\mathcal{P}^{s+}_{3,5}$.

For a semialgebraic variety Y, we denote its boundary by ∂Y. At [1, Definition 1.5], we defined a critical decomposition $\Delta(Y) = \bigsqcup_{i=0}^{\dim Y} \Delta^i(Y)$ of Y. If $\Delta(Y) = \big\{D_1, \ldots, D_r\big\}$, then all D_j are non-singular irreducible semialgebraic varieties with $\partial D_j = \emptyset$ and $Y = \bigsqcup_{j=1}^{r} D_j$ (disjoint union). If $D_j \in \Delta^i(Y)$, then $\dim D_j = i$.

Since $\Psi\colon \mathbb{P}^2_+/\mathfrak{S}_3 \longrightarrow X$ is an isomorphism, we have $\Delta^i(\mathbb{P}^2_+/\mathfrak{S}_3) \cong \Delta^i(X)$. The critical decomposition of $\mathbb{P}^2_+/\mathfrak{S}_3$ is given in [1, Proposition 2.13]. Using this, we shall describe the critical decomposition of X. Let

$$\begin{aligned}
C^b &:= \big\{\Phi(t\colon 1\colon 1) \in X \;\big|\; 0<t<1 \text{ or } 1<t\big\},\\
C^0 &:= \big\{\Phi(0\colon t\colon 1) \in X \;\big|\; 0<t<1\big\},\\
P_1 &:= \Phi(0\colon 0\colon 1) = (1\colon 0\colon 0\colon 0\colon 0),\\
P_2 &:= \Phi(0\colon 1\colon 1) = (1\colon 1\colon 2\colon 0\colon 0),\\
P_3 &:= \Phi(1\colon 1\colon 1) = (0\colon 0\colon 0\colon 0\colon 1).
\end{aligned}$$

Proposition 4.12. *The critical decomposition of X is given by*

$$\Delta^2(X) = \big\{\operatorname{Int}(X)\big\},\quad \Delta^1(X) = \big\{C^b, C^0\big\},\quad \Delta^0(X) = \big\{P_1, P_2, P_3\big\}.$$

Proof. This follows from [1, Proposition 2.13, 2.14] or [2, Proposition 2.14]. □

For $D \in \Delta(X)$, a semialgebraic variety $\mathcal{F}(D) \subset \partial\mathcal{P}^{s+}_{3,5}$ is defined as [1, Definition 1.19] (see also [1, Theorem 1.18] or [2, Theorem 2.6]). Roughly speaking, $\mathcal{F}(D)$ is obtained from the dual semialgebraic variety of D ([1, Theorem 1.18]). Note that $\mathcal{F}(P_3) = \mathcal{P}^{s0+}_{3,5}$.

Proposition 4.13.

$$\partial\mathcal{P}^{s+}_{3,5} = \mathcal{F}(C^b) \cup \mathcal{F}(C^0) \cup \mathcal{F}(P_1) \cup \mathcal{F}(P_2) \cup \mathcal{F}(P_3).$$

Proof. By [1, Theorem 1.18] or [2, Theorem 2.6], we have
$$\partial\mathcal{P}^{s+}_{3,5} = \bigcup_{D\in\Delta(X)} \mathcal{F}(D).$$
But $\mathcal{F}(\mathrm{Int}(X))$ is not a face component of $\mathcal{P}^{s+}_{3,5}$ by [2, Theorem 2.21]. □

For $D \in \Delta(X)$, we denote
$$\mathcal{F}_D := \big(\mathcal{F}(D) - \{0\}\big)/\mathbb{R}_{++} \subset \partial\mathbb{P}\mathcal{P}^{s+}_{3,5} \subset \mathbb{P}\mathcal{H}^s_{3,5}.$$
Since $\mathcal{P}^{s+}_{3,5}$ is a convex set, $\mathcal{F}_D$ is also a convex set.

For a subset A of $\mathbb{R}^m$ or $\mathbb{P}^m_{\mathbb{R}}$, we denote its Zariski closure by $\mathrm{Zar}(A)$. For $i = 1, 2$ and 3, $\mathrm{Zar}(\mathcal{F}_{P_i})$ is a hypersurface of $\mathbb{P}\mathcal{H}^s_{3,5}$ (see [1, Remark 1.21(3)]). This sentence means that $\mathrm{Zar}\left(\mathcal{F}(P_i)\right)$ is a hypersurface of $\mathcal{H}^s_{3,5}$. So, we may regard $\mathcal{F}_{P_i}$ as a compact convex domain in $\mathbb{R}^3$. Since
$$\mathbb{P}\mathcal{E}(\mathcal{P}^{s+}_{3,5}) = \mathcal{E}(\mathbb{P}\mathcal{P}^{s+}_{3,5}) \subset \mathcal{E}(\mathcal{F}_{C^b}) \cup \mathcal{E}(\mathcal{F}_{C^0}) \cup \mathcal{E}(\mathcal{F}_{P_1}) \cup \mathcal{E}(\mathcal{F}_{P_2}) \cup \mathcal{E}(\mathcal{F}_{P_3}),$$
we need to study $\mathcal{F}_{C^b}$, $\mathcal{F}_{C^0}$, $\mathcal{F}_{P_1}$, $\mathcal{F}_{P_2}$ and $\mathcal{F}_{P_3}$ to prove Theorem 1.4.

For $f \neq g \in \mathcal{H}^s_{4,5}$, the line segment connecting $[f]$ and $[g] \in \mathbb{P}\mathcal{H}^s_{4,5}$ is denoted by
$$\mathrm{L}[f, g] := \big\{[(1-t)f + tg] \in \mathbb{P}\mathcal{H}^s_{4,5} \bigm| 0 \le t \le 1\big\} \subset \mathbb{P}\mathcal{H}^s_{4,5}.$$
Since $\dim \mathcal{F}_D \le 3$, a line segment $\mathrm{L}[f, g]$ often appears in the irreducible components of $\mathcal{F}_{D_1} \cap \mathcal{F}_{D_2} \cap \mathcal{F}_{D_3}$.

4.2.2. Structure of $\mathcal{F}_{C^b}$.

For $t > 0$, we put
$$\mathcal{L}^b_t := \big\{[f] \in \mathcal{F}_{C^b} \bigm| f(t, 1, 1) = 0\big\}, \quad \mathcal{L}^b_\infty := \lim_{t\to+\infty} \mathcal{L}^b_t, \quad \mathcal{L}^b_0 := \lim_{t\to+0} \mathcal{L}^b_t.$$
Note that $\dim \mathcal{F}_{C^b} = 3$ and $\dim \mathcal{L}^b_t \le 2$. If $[f]$, $[g] \in \mathcal{L}^b_t$, then $\mathrm{L}[f, g] \subset \mathcal{L}^b_t$. Thus $\mathrm{Zar}(\mathcal{L}^b_t)$ is included in a two dimensional plane in $\mathbb{P}\mathcal{H}^s_{3,5}$.

If $P \in C^b$, then there exists $t > 0$ such that $P = \Phi(t\colon 1\colon 1)$. Thus, if $f \in \mathcal{F}(C^b)$ and $f(1, 0, 0) \neq 0$, then there exists $t \ge 0$ such that $f(t, 1, 1) = 0$. Therefore,
$$\mathcal{F}_{C^b} = \bigcup_{t\in[0,\infty]} \mathcal{L}^b_t.$$
This implies
$$\mathcal{E}(\mathcal{F}_{C^b}) \subset \bigcup_{t\in[0,\infty]} \mathcal{E}(\mathcal{L}^b_t).$$
Thus, we shall study $\mathcal{E}(\mathcal{L}^b_t)$.

For $\mathbf{a} \in \mathbb{R}^3$, $\big\{f \in \mathcal{F}(D) \bigm| f(\mathbf{a}) = 0\big\}$ is a linear subset of $\mathcal{H}^s_{3,5}$. So, we may regard $\mathcal{L}^b_t$ as a compact convex domain in $\mathbb{R}^2$. We study the shape of $\mathcal{L}^b_t$.

Theorem 4.14. (1)*If* $0 \le t \le 2$, *then*
$$\mathcal{E}(\mathcal{L}^b_t) = \big\{[\mathfrak{e}^A_{t,u}] \bigm| 0 \le u \le \mu_L(t)\big\} \cup \big\{[\mathfrak{e}^C_t],\ [\mathfrak{e}^D_t]\big\}.$$
(2) *If* $2 < t \le 5/2$, *then*
$$\mathcal{E}(\mathcal{L}^b_t) = \big\{[\mathfrak{e}^A_{t,u}] \bigm| 0 \le u \le \mu_L(t)\big\} \cup \big\{[\mathfrak{e}^B_{t,u}] \bigm| \mu_B(t) \le u \le 1\big\} \cup \big\{[\mathfrak{e}^D_t]\big\}.$$
(3) *If* $5/2 < t < 7$, *then*
$$\mathcal{E}(\mathcal{L}^b_t) = \big\{[\mathfrak{e}^A_{t,u}] \bigm| 0 \le u \le \mu_H(t)\big\} \cup \big\{[\mathfrak{e}^B_{t,u}] \bigm| \mu_B(t) \le u \le 1\big\} \cup \big\{[\mathfrak{e}^D_t]\big\}.$$
(4) *If* $t \ge 7$, *then*
$$\mathcal{E}(\mathcal{L}^b_t) = \big\{[\mathfrak{e}^B_{t,u}] \bigm| \mu_B(t) \le u \le 1\big\} \cup \big\{[\mathfrak{e}^D_t,\ \mathfrak{e}^E_t]\big\}.$$

(5) $\mathcal{E}(\mathcal{L}^b_\infty) = \{[\mathfrak{e}^D_\infty],\, [\mathfrak{e}^E_\infty]\}$.

(6)

$$\mathcal{E}(\mathcal{F}_{C^b}) = \{[\mathfrak{e}^A_{t,u}] \mid 0 \le t \le 7,\, 0 \le u \le \mu_A(t)\} \cup \{[\mathfrak{e}^B_{t,u}] \mid t \in [2,\infty],\, \mu_B(t) \le u \le 1\}$$
$$\cup \{[\mathfrak{e}^C_t] \mid 0 \le t \le 2\} \cup \{[\mathfrak{e}^D_t] \mid t \in [0,\infty]\} \cup \{[\mathfrak{e}^C_t] \mid t \in [7,\infty]\}.$$

Proof. Put

$$S := \{[\mathfrak{e}^A_{t,u}] \in \mathcal{F}_{C^b} \mid 0 \le t \le 7,\, t \ne 1 \text{ and } 0 \le u \le \mu_A(t)\}.$$

Since $[\mathfrak{e}^A_{t,u}] \in \mathcal{L}^b_t \cap \mathcal{L}^b_{\mu_Z(t,u)}$ by Theorem 4.7, and since $\mathcal{L}^b_t \ne \mathcal{L}^b_{\mu_Z(t,u)}$, we have $S \subset \operatorname{Sing}(\mathcal{F}_{C^b})$. Note that

$$\mathcal{E}(\mathcal{L}^b_t) \subset \partial\mathcal{L}^b_t \subset S \cup \mathcal{F}_{C^0} \cup \mathcal{F}_{P_1} \cup \mathcal{F}_{P_2} \cup \mathcal{F}_{P_3}.$$

So, we study $S \cap \mathcal{L}^b_t$, $\mathcal{F}_{C^0} \cap \mathcal{L}^b_t$, $\mathcal{F}_{P_1} \cap \mathcal{L}^b_t$, $\mathcal{F}_{P_2} \cap \mathcal{L}^b_t$ and $\mathcal{F}_{P_3} \cap \mathcal{L}^b_t$.

(1) We consider the case $0 \le t \le 2$ and $t \ne 1$. (Fig.4.2)

By Theorem 4.7, we have $S \cap \mathcal{L}^b_t = \{[\mathfrak{e}^A_{t,u}] \mid 0 \le u \le \mu_L(t)\}$.

By Theorem 4.10, we have $\mathcal{F}_{C^0} \cap \mathcal{L}^b_t = \emptyset$.

Since $\operatorname{Zar}(\mathcal{F}_{P_i})$ and $\operatorname{Zar}(\mathcal{L}^b_t)$ are linear subspaces of $\mathbb{P}\mathcal{H}^s_{3,5}$ of dimensions 3 and 2, we have $\dim(\mathcal{F}_{P_i} \cap \mathcal{L}^b_t) \le 1$. Note that $\mathcal{F}_{P_1} \cap \mathcal{F}_{P_2} \cap \mathcal{L}^b_t = \emptyset$, $\mathcal{F}_{P_2} \cap \mathcal{F}_{P_3} \cap \mathcal{L}^b_t = \{[\mathfrak{e}^C_t]\}$, and $\mathcal{F}_{P_1} \cap \mathcal{F}_{P_3} \cap \mathcal{L}^b_t = \{[\mathfrak{e}^D_t]\}$, by Theorem 4.3, 4.4 and 4.7. Since $\operatorname{Zar}(\mathcal{F}_{P_i})$ is a hyperplane of $\mathbb{P}\mathcal{H}^s_{3,5}$, and $\operatorname{Zar}(\mathcal{L}^b_t)$ is a two dimensional plane in $\mathbb{P}\mathcal{H}^s_{3,5}$, we have

$$\mathcal{F}_{P_1} \cap \mathcal{L}^b_t = \mathrm{L}[\mathfrak{e}^A_{t,0},\, \mathfrak{e}^D_t], \quad \mathcal{F}_{P_2} \cap \mathcal{L}^b_t = \mathrm{L}[\mathfrak{e}^A_{t,\mu_L(t)},\, \mathfrak{e}^C_t], \quad \mathcal{F}_{P_3} \cap \mathcal{L}^b_t = \mathrm{L}[\mathfrak{e}^C_t,\, \mathfrak{e}^D_t].$$

When we draw these boundary components of $\mathcal{L}^b_t$, we obtain Fig.4.2. Since $\mathcal{L}^b_t$ is a convex set, (1) is proved.

When $t = 1$, (1) can be obtained if we take a limit $t \to 1$.

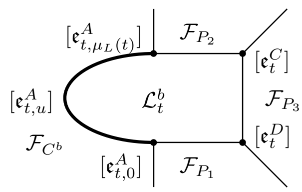

Fig.4.2. $\mathcal{L}^b_t$ $(0 \le t \le 2)$

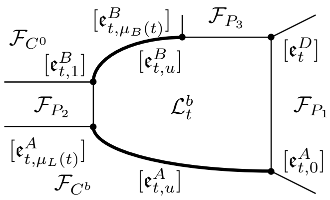

Fig.4.3. $\mathcal{L}^b_t$ $(2 < t \le 5/2)$

(2) We consider the case $2 < t \le 5/2$. (Fig.4.3)

By similar arguments as in (1), we conclude that

(i) $S \cap \mathcal{L}^b_t = \{[\mathfrak{e}^A_{t,u}] \mid 0 \le u \le \mu_L(t)\}$. (Theorem 4.7)

(ii) $\mathcal{F}_{C^0} \cap \mathcal{L}^b_t = \{[\mathfrak{e}^B_{t,u}] \mid \mu_B(t) \le u \le 1\}$. (Theorem 4.10)

(iii) $\mathcal{F}_{P_1} \cap \mathcal{L}^b_t = \mathrm{L}[\mathfrak{e}^A_{t,0}, \mathfrak{e}^D_t]$. (Theorem 4.7, 4.4)

(iv) $\mathcal{F}_{P_2} \cap \mathcal{L}^b_t = \mathrm{L}[\mathfrak{e}^A_{t,\mu_L(t)},\, \mathfrak{e}^B_{t,1}]$. (Theorem 4.7, 4.10)

(v) $\mathcal{F}_{P_3} \cap \mathcal{L}^b_t = \mathrm{L}[\mathfrak{e}^B_{t,\mu_B(t)},\, \mathfrak{e}^D_t]$. (Theorem 4.10, 4.4)

Thus, we have Fig.4.3, and (2) is proved.

(3) We consider the case $5/2 < t < 7$. (Fig.4.4)

By similar arguments as in (1), we conclude that

(i) $S \cap \mathcal{L}^b_t = \{[\mathfrak{e}^A_{t,u}] \mid 0 \le u \le \mu_H(t)\}$. (Theorem 4.7)

(ii) $\mathcal{F}_{C^0} \cap \mathcal{L}^b_t = \{[\mathfrak{e}^B_{t,u}] \mid \mu_B(t) \le u \le 1\}$. (Theorem 4.10)

(iii) $\mathcal{F}_{P_1} \cap \mathcal{L}^b_t = \mathrm{L}[\mathfrak{e}^A_{t,0}, \mathfrak{e}^D_t]$. (Theorem 4.4, 4.7)

(iv) $\mathcal{F}_{P_2} \cap \mathcal{L}^b_t = \mathrm{L}[\mathfrak{e}^A_{t,\mu_H(t)},\, \mathfrak{e}^B_{t,1}]$. (Theorem 4.7, 4.10)

(v) $\mathcal{F}_{P_3} \cap \mathcal{L}^b_t = \mathrm{L}[\mathfrak{e}^B_{t,\mu_B(t)},\, \mathfrak{e}^D_t]$. (Theorem 4.4, 4.10)

Thus, we have Fig.4.4, and (3) is proved.

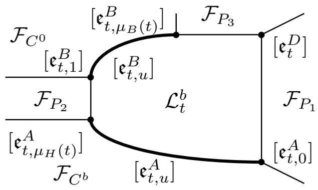

Fig.4.4. $\mathcal{L}^b_t$ $(5/2 < t < 7)$

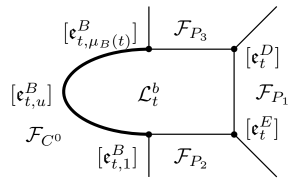

Fig.4.5. $\mathcal{L}^b_t$ $(t \geq 7)$

(4) We consider the case $t \geq 7$. (Fig.4.5)
By similar arguments as in (1), we conclude that
(i) $S \cap \mathcal{L}^b_t = \emptyset$. (Theorem 4.7)
(ii) $\mathcal{F}_{C^0} \cap \mathcal{L}^b_t = \{[\mathfrak{e}^B_{t,u}] \mid \mu_B(t) \leq u \leq 1\}$. (Theorem 4.10)
(iii) $\mathcal{F}_{P_1} \cap \mathcal{L}^b_t = \mathrm{L}[\mathfrak{e}^D_t, \mathfrak{e}^E_t]$. (Theorem 4.4, 4.5)
(iv) $\mathcal{F}_{P_2} \cap \mathcal{L}^b_t = \mathrm{L}[\mathfrak{e}^B_{t,1}, \mathfrak{e}^E_t]$. (Theorem 4.5, 4.10)
(v) $\mathcal{F}_{P_3} \cap \mathcal{L}^b_t = \mathrm{L}[\mathfrak{e}^B_{t,\mu_B(t)}, \mathfrak{e}^D_t]$. (Theorem 4.4, 4.10)
Thus, we have Fig.4.5, and (4) is proved.
(5) follows from $\lim\limits_{t\to\infty}[\mathfrak{e}^B_{t,u}] = [\mathfrak{e}^E_\infty]$, if $\mu_B(t) \leq u \leq 1$.
(6) By (1)—(4), we have $\mathcal{E}(\mathcal{F}_{C^b}) \supset \bigcup\limits_{t\in[0,\infty]} \mathcal{E}(\mathcal{L}^b_t)$. The inclusion $\subset$ is clear. □

4.2.3. Structures of $\mathcal{F}_{P_1}$ and $\mathcal{F}_{P_2}$.

We start from $\mathcal{F}_{P_1}$.

Theorem 4.15.

$$\mathcal{E}(\mathcal{F}_{P_1}) = \{[\mathfrak{e}^A_{t,0}] \mid 0 \leq t \leq 7\} \cup \{[\mathfrak{e}^D_t] \mid t \in [0,\infty]\} \cup \{[\mathfrak{e}^E_t] \mid t \in [7,\infty]\} \cup \{[s_3]\}.$$

Proof. Since $\mathrm{Zar}(\mathcal{F}_{P_1}) \cong \mathbb{P}^3_{\mathbb{R}}$, $\mathcal{F}_{P_1}$ is non-singular. Thus,

$$\mathcal{E}(\mathcal{F}_{P_1}) \subset \mathcal{F}_{C^b} \cup \mathcal{F}_{C^0} \cup \mathcal{F}_{P_2} \cup \mathcal{F}_{P_3}.$$

By Theorem 4.14, we have

$$\mathcal{E}(\mathcal{F}_{C^b} \cap \mathcal{F}_{P_1}) = \{[\mathfrak{e}^A_{t,0}] \mid 0 \leq t \leq 7\} \cup \{[\mathfrak{e}^D_t] \mid t \in [0,\infty]\} \cup \{[\mathfrak{e}^E_t] \mid t \in [7,\infty]\}.$$

We need to observe $\mathcal{F}_{C^0} \cap \mathcal{F}_{P_1}$, $\mathcal{F}_{P_2} \cap \mathcal{F}_{P_1}$ and $\mathcal{F}_{P_3} \cap \mathcal{F}_{P_1}$.

(1) It is easy to see that $\mathcal{F}_{C^0} \cap \mathcal{F}_{P_1}$ is a triangle whose vertices are $[\mathfrak{e}^D_\infty] = [s_2 - s_3]$, $[s_3]$ and $[s_4] = [\mathfrak{e}^E_\infty] = [\mathfrak{e}^B_{\infty,1}]$ (Fig.4.6).

(2) We observe $\mathcal{F}_{P_2} \cap \mathcal{F}_{P_1}$. (Fig.4.7)
As the proof of (1) in Theorem 4.14, we obtain:
(i) $\mathcal{F}_{C^b} \cap \mathcal{F}_{P_2} \cap \mathcal{F}_{P_1} = \mathrm{L}[\mathfrak{e}^D_0, \mathfrak{e}^E_7] \cup \{[\mathfrak{e}^E_t] \mid t \in [7,\infty]\}$. (Theorem 4.4, 4.5)
(ii) $\mathcal{F}_{C^0} \cap \mathcal{F}_{P_2} \cap \mathcal{F}_{P_1} = \mathrm{L}[s_3, \mathfrak{e}^E_\infty]$. (Theorem 4.5, 4.6)
(iii) $\mathcal{F}_{P_3} \cap \mathcal{F}_{P_2} \cap \mathcal{F}_{P_1} = \mathrm{L}[\mathfrak{e}^D_0, s_3]$. (Theorem 4.4, 4.6)
Thus, we can draw Fig.4.7. This implies

$$\mathcal{E}(\mathcal{F}_{P_2} \cap \mathcal{F}_{P_1}) = \{[\mathfrak{e}^E_t] \mid t \in [7,\infty]\} \cup \{[\mathfrak{e}^D_0], [s_3]\}.$$

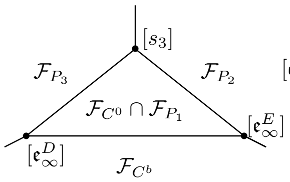

Fig.4.6. $\mathcal{F}_{C^0} \cap \mathcal{F}_{P_1}$

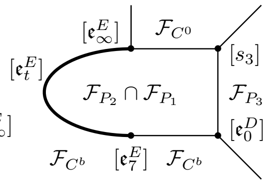

Fig.4.7. $\mathcal{F}_{P_2} \cap \mathcal{F}_{P_1}$

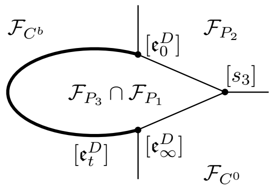

Fig.4.8. $\mathcal{F}_{P_3} \cap \mathcal{F}_{P_1}$

(3) We observe $\mathcal{F}_{P_3} \cap \mathcal{F}_{P_1}$. (Fig.4.8)
As the proof of (1) in Theorem 4.14, we obtain:
(i) $\mathcal{F}_{C^b} \cap \mathcal{F}_{P_3} \cap \mathcal{F}_{P_1} = \{[\mathfrak{e}^D_t] \mid t \in [0,\infty]\}$. (Theorem 4.4)
(ii) $\mathcal{F}_{C^0} \cap \mathcal{F}_{P_3} \cap \mathcal{F}_{P_1} = \mathrm{L}[\mathfrak{e}^D_\infty, s_3]$. (Theorem 4.4, 4.6)
(iii) $\mathcal{F}_{P_2} \cap \mathcal{F}_{P_3} \cap \mathcal{F}_{P_1} = \mathrm{L}[\mathfrak{e}^D_0, s_3]$. (Theorem 4.4, 4.6)
Thus, we can draw Fig.4.8, and we have

$$\mathcal{E}(\mathcal{F}_{P_3} \cap \mathcal{F}_{P_1}) = \{[\mathfrak{e}^D_t] \mid t \in [0,\infty]\} \cup \{[s_3]\}.$$

Thus, we complete the proof of the theorem. □

Next, we observe $\mathcal{F}_{P_2}$.

Theorem 4.16.

$$\mathcal{E}(\mathcal{F}_{P_2}) = \{[\mathfrak{e}^A_{t,0}] \mid 0 \le t \le 7\} \cup \{[\mathfrak{e}^B_{t,1}] \mid t \in [2,\infty]\} \cup \{[\mathfrak{e}^C_t] \mid t \in [0,2]\} \cup \{[\mathfrak{e}^D_0]\} \cup \{[\mathfrak{e}^E_t] \mid t \in [7,\infty]\} \cup \{[s_3]\}.$$

Proof. Since $\mathrm{Zar}(\mathcal{F}_{P_2})$ is 3 dimensional affine space, we have

$$\mathcal{E}(\mathcal{F}_{P_2}) \subset \mathcal{F}_{C^b} \cup \mathcal{F}_{C^0} \cup \mathcal{F}_{P_1} \cup \mathcal{F}_{P_3}.$$

By Theorem 4.14, we have

$$\mathcal{E}(\mathcal{F}_{C^b} \cap \mathcal{F}_{P_2}) = \{[\mathfrak{e}^A_{t,\mu_A(t)}] \mid 0 \le t \le 7\} \cup \{[\mathfrak{e}^B_{t,1}] \mid t > 2\} \cup \{[\mathfrak{e}^C_t] \mid 0 \le t \le 2\} \cup \{[\mathfrak{e}^E_t] \mid t \in [7,\infty]\}.$$

By Theorem 4.15, $\mathcal{E}(\mathcal{F}_{P_1} \cap \mathcal{F}_{P_2})$ is as Fig.4.7. Thus, we need to observe $\mathcal{E}(\mathcal{F}_{C^0} \cap \mathcal{F}_{P_2})$ and $\mathcal{E}(\mathcal{F}_{P_3} \cap \mathcal{F}_{P_2})$.

(1) We observe $\mathcal{F}_{C^0} \cap \mathcal{F}_{P_2}$. (Fig.4.9)
As the proof of (1) in Theorem 4.14, we obtain:
(i) $\mathcal{F}_{C^b} \cap \mathcal{F}_{C^0} \cap \mathcal{F}_{P_2} = \{[\mathfrak{e}^B_{t,1}] \mid t \in [2,\infty]\}$. (Theorem 4.10)
(ii) $\mathcal{F}_{P_1} \cap \mathcal{F}_{C^0} \cap \mathcal{F}_{P_2} = \mathrm{L}[s_3, \mathfrak{e}^E_\infty]$, where $[\mathfrak{e}^E_\infty] = [\mathfrak{e}^B_{\infty,1}]$. (Theorem 4.5, 4.6)
(iii) $\mathcal{F}_{P_3} \cap \mathcal{F}_{C^0} \cap \mathcal{F}_{P_2} = \mathrm{L}[\mathfrak{e}^B_{2,1}, s_3]$. (Theorem 4.6, 4.10)
Thus, we obtain Fig.4.9.

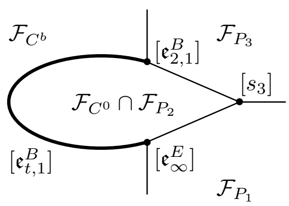

Fig.4.9. $\mathcal{F}_{C^0} \cap \mathcal{F}_{P_2}$

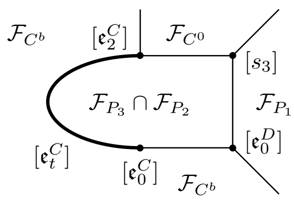

Fig.4.10. $\mathcal{F}_{P_3} \cap \mathcal{F}_{P_2}$

(2) We observe $\mathcal{F}_{P_3} \cap \mathcal{F}_{P_2}$. (Fig.4.10)
As the proof of (1) in Theorem 4.14, we obtain:
(i) $\mathcal{F}_{C^b} \cap \mathcal{F}_{P_3} \cap \mathcal{F}_{P_2} = \mathrm{L}[\mathfrak{e}_0^C,\, \mathfrak{e}_0^D] \cup \big\{[\mathfrak{e}_t^C] \;\big|\; 0 \le t \le 2\big\}$. (Theorem 4.3, 4.4)
(ii) $\mathcal{F}_{C^0} \cap \mathcal{F}_{P_3} \cap \mathcal{F}_{P_2} = \mathrm{L}[\mathfrak{e}_2^C,\, s_3]$. (Theorem 4.3, 4.6)
(iii) $\mathcal{F}_{P_1} \cap \mathcal{F}_{P_3} \cap \mathcal{F}_{P_2} = \mathrm{L}[\mathfrak{e}_0^D,\, s_3]$. (Theorem 4.4, 4.6)
Thus, we obtain Fig.4.10.
By these observations, we obtain the theorem. □

4.2.4. Discriminants of $\mathcal{P}_{3,5}^{s+}$.

To determine $\mathcal{E}(\mathcal{F}_{C^0})$, we need to prove that $\mathcal{F}_{C^0}$ is non-singular.

An element $f \in \mathcal{H}_{3,5}^s$ is represented by $f = \sum_{i=0}^{4} p_i s_i$. We use $(p_0,\dots, p_4)$ as a coordinate system of $\mathcal{H}_{3,5}^s$, and write $f = (p_0,\dots, p_4)$. We represent discriminants using this coordinate system.

If $D \in \Delta(X)$ satisfies $\dim \mathcal{F}(D) = \dim \mathcal{P}_{3,5}^{s+} - 1$, the defining equation of $\mathrm{Zar}(\mathcal{F}(D))$ in $\mathcal{H}_{3,5}^s$ is called a discriminant of $\mathcal{P}_{3,5}^{s+}$, and is written by $\mathrm{disc}(D)$, disc_D or $\mathrm{disc}_D(\mathbf{p})$. To describe $\mathrm{disc}_{C^b}(\mathbf{p})$, we put

$$\begin{aligned} &c_5(\mathbf{p}) := p_0, \quad c_4(\mathbf{p}) := 2p_1, \quad c_3(\mathbf{p}) := 2p_2 + p_3, \\ &c_2(\mathbf{p}) := -2(p_0 + 2p_1 + p_2 + p_3 - p_4), \\ &c_1(\mathbf{p}) := -p_0 - 2p_2 + p_3 + p_4, \quad c_0(\mathbf{p}) := 2(p_0 + p_1 + p_2). \end{aligned}$$

Note that if $f = (p_0,\dots, p_4)$, then $f(x, 1, 1) = \sum_{i=0}^{5} c_i(\mathbf{p}) x^i$.

Theorem 4.17. *All the discriminants of* $\mathcal{P}_{3,5}^{s+}$ *are*

$$\begin{aligned} &\mathrm{disc}_{P_1}(\mathbf{p}) = p_0, \\ &\mathrm{disc}_{P_2}(\mathbf{p}) = p_0 + p_1 + p_2, \\ &\mathrm{disc}_{P_3}(\mathbf{p}) = p_4, \\ &\mathrm{disc}_{C^0}(\mathbf{p}) = 5p_0^2 + 2p_0p_1 + p_1^2 - 4p_0p_2, \\ &\mathrm{disc}_{C^b}(\mathbf{p}) = \frac{\mathrm{Disc}_5\left(c_5(\mathbf{p}), c_4(\mathbf{p}), c_3(\mathbf{p}), c_2(\mathbf{p}), c_1(\mathbf{p}), c_0(\mathbf{p})\right)}{16p_4}. \end{aligned}$$

Proof. We obtain disc_{P_i} $(i = 1, 2, 3)$ by [1, Remark 1.21(3)]. Discriminants disc_{C^0} and disc_{C^b} can be obtained by the calculation explained in [1, Remark 1.21(1)]. We can obtain disc_{C^0} without a computer, but the calculation of disc_{C^b} took very long time even if we used a computer. So, we present an alternative method to justify the above disc_{C^b} is really discriminant.

Let $F(p_0,\dots, p_4)$ be the right hand side of $\mathrm{disc}_{C^b}(\mathbf{p})$ presented in the theorem. $V_{\mathbb{C}}(\mathrm{disc}_{C^b})$ must contains $\mathfrak{e}_{t,u}^A$ $(0 \le t \le 7,\ 0 \le u \le \mu_A(t))$, $\mathfrak{e}_{t,u}^B$ $(t \ge 2,\ \mu_B(t) \le u \le 1)$, $\mathfrak{e}_t^C$ $(0 \le t \le 2)$, $\mathfrak{e}_t^D$ $(t \ge 0)$, and $\mathfrak{e}_t^E$ $(t \ge 7)$, by Theorem 4.14(6).

Using computer, it is easy to see all of these are on $V_{\mathbb{C}}(F)$. An irreducible polynomial $G(p_0,\dots, p_4)$ such that $V_{\mathbb{C}}(G)$ contains all the above $\mathfrak{e}$ is unique up to constant multiplication. It is easy to check that F is irreducible. Thus, F is a discriminant. □

Corollary 4.18. *The real algebraic variety* $\mathrm{Zar}(\mathcal{F}_{C^0}) = V_{\mathbb{R}}(\mathrm{disc}_{C^0})$ *is non-singular.*

4.2.5. Proof of Theorem 1.4.

We shall determine $\mathcal{E}(\mathcal{F}_{C^0})$ and $\mathcal{E}(\mathcal{F}_{P_3})$.

Corollary 4.19.

$$\mathcal{E}(\mathcal{F}_{C^0}) = \left\{[\mathfrak{e}^B_{t,u}] \;\middle|\; t \in [2,\infty],\, \mu_B(t) \le u \le 1\right\} \cup \left\{[\mathfrak{e}^D_\infty],\, [s_3]\right\}.$$

Proof. Since $\mathrm{Zar}(\mathcal{F}_{C^0})$ is non-singular, we have

$$\mathcal{E}(\mathcal{F}_{C^0}) \subset \mathcal{F}_{C^b} \cup \mathcal{F}_{P_1} \cup \mathcal{F}_{P_2} \cup \mathcal{F}_{P_3}.$$

By Theorem 4.14, we have

$$\mathcal{E}(\mathcal{F}_{C^b} \cap \mathcal{F}_{C^0}) = \left\{[\mathfrak{e}^B_{t,u}] \;\middle|\; t \in [2,\infty],\, \mu_B(t) \le u \le 1\right\}.$$

$\mathcal{F}_{P_1} \cap \mathcal{F}_{C_0}$ is given in Fig.4.6, and $\mathcal{F}_{P_2} \cap \mathcal{F}_{C^0}$ is given in Fig.4.9.

Thus, it is enough to observe $\mathcal{F}_{P_3} \cap \mathcal{F}_{C^0}$. By the proofs of Theorem 4.14, 4.15 and 4.16, we have

$$\mathcal{E}(\mathcal{F}_{P_3}) \cap \mathcal{F}_{C^0} = \left\{[\mathfrak{e}^B_{t,\mu_B(t)}] \;\middle|\; t \in [2,\infty]\right\} \cup \left\{[\mathfrak{e}^D_\infty],\, [s_3]\right\}.$$

Here, note that $\mu_B(2) = 1$, $\mathfrak{e}^B_{\infty,1} = \mathfrak{e}^E_\infty$ and $\mathfrak{e}^B_{2,1} = \mathfrak{e}^C_2$. Thus, we have the conclusion. □

Corollary 4.20.

$$\mathcal{E}(\mathcal{F}_{P_3}) = \left\{[\mathfrak{e}^B_{t,\mu_B(t)}] \;\middle|\; t \in [2,\infty]\right\} \cup \left\{[\mathfrak{e}^C_t] \;\middle|\; 0 \le t \le 2\right\}$$
$$\cup \left\{[\mathfrak{e}^D_t] \;\middle|\; t \in [0,\infty]\right\} \cup \{[s_3]\}.$$

Proof. This is already proved in the proofs till here. □

Corollary 4.21. *All the elements of* $\mathcal{E}(\mathbb{P}\mathcal{P}^{s+}_{3,5})$ *are* $[\mathfrak{e}^A_{t,u}]$ *(*$0 \le t \le 7$, $0 \le u \le \mu_A(t)$*),* $[\mathfrak{e}^B_{t,u}]$ *(*$t \in [2,\infty]$, $\mu_B(t) \le u \le 1$*),* $[\mathfrak{e}^C_t]$ *(*$0 \le t \le 2$*),* $[\mathfrak{e}^D_t]$ *(*$t \in [0,\infty]$*),* $[\mathfrak{e}^E_t]$ *(*$t \in [7,\infty]$*), and* $[s_3]$.

Thus, we complete the proof of Theorem 1.4.

4.3. Application.

4.3.1. Reducible extremal elements.

In this subsection, we study when $f \in \mathcal{E}(\mathcal{P}^{s+}_{3,5})$ is irreducible. We need some lemmata for it.

Lemma 4.22. *Assume that* $f \in \mathcal{E}(\mathcal{P}^{s+}_{3,5})$ *is reducible in* $\mathbb{C}[a,b,c]$*. Then, there exists an integer* $d \in \{1,\, 2\}$*,* $g \in \mathcal{E}(\mathcal{P}^{s+}_{3,d})$ *and* $h \in \mathcal{E}(\mathcal{P}^{s+}_{3,5-d})$ *such that* $f = gh$*.*

Proof. (1) We shall show that if $f \in \mathcal{E}(\mathcal{P}^{s+}_{3,5})$ is reducible in $\mathbb{C}[a,b,c]$, then there exists $g,\ h \in \mathbb{R}[a,b,c] - \mathbb{R}$ such that $f = gh$.

Assume that $f/g \in \mathbb{C}[a,b,c]$ by non-constant $g \in \mathbb{C}[a,b,c]$. We may assume g is irreducible in $\mathbb{C}[a,b,c]$ and $\deg g$ is odd. If $\alpha g \notin \mathbb{R}[a,b,c]$ for any $\alpha \in \mathbb{C}^{\times}$, then the complex conjugate $\overline{g}$ divides f. Then $g\overline{g} \in \mathbb{R}[a,b,c]$ and $f/(g\overline{g}) \in \mathbb{R}[a,b,c]$.

(2) We shall show that if f is reducible, we can find a symmetric divisor of f. Assume that $f = gh$ by $g,\ h \in \mathbb{R}[a,b,c]$. We may assume that g is irreducible in $\mathbb{R}[a,b,c]$. If g is not symmetric, $\sigma(g)$ is also a divisor of f by any $\sigma \in \mathfrak{S}_3$. So, there exists a symmetric divisor of f.

Assume that $f = gh$ by $g \in \mathcal{P}^{s+}_{3,d}$ and $h \in \mathcal{P}^{s+}_{3,5-d}$. If $g \notin \mathcal{E}(\mathcal{P}^{s+}_{3,d})$, then $f \notin \mathcal{E}(\mathcal{P}^{s+}_{3,5})$. □

Lemma 4.23. (1) $\mathbb{P}\mathcal{E}(\mathcal{P}^{s+}_{3,1}) = \big\{[S_1]\big\}$.
(2) $\mathbb{P}\mathcal{E}(\mathcal{P}^{s+}_{3,2}) = \big\{[S_2 - S_{1,1}]\big\}$.
(3) $\mathbb{P}\mathcal{E}(\mathcal{P}^{s+}_{3,3}) = \big\{[S_3 + 3U - T_{2,1}],\ [T_{2,1} - 6U],\ [U]\big\}$.

Proof. (1) and (2) are trivial. (3) is proved in [1, Corollary 3.4]. □

Lemma 4.24. *Let*

$$\mathfrak{g}_t(a,b,c) := S_4 - (t+1)T_{3,1} + (t^2+2t)S_{2,2} - (t^2-1)T_{2,1,1}.$$

Then

$$\mathbb{P}\mathcal{E}(\mathcal{P}^{s+}_{3,4}) = \big\{[\mathfrak{g}_t] \bigm| t \geq 0\big\} \cup \big\{[(S_2 - tS_{1,1})^2] \bigm| t \geq 1\big\} \cup \big\{[T_{3,1} - 2S_{2,2}]\big\}.$$

Proof. This is a corollary of [1, Theorem 4.10]. This $\mathfrak{g}_t$ is equal to $\mathfrak{g}^A_{-t-1,-t-1}$ in [1, Theorem 4.10]. Note that $\mathfrak{g}_t(x,1,1) = (x-1)^2(x-t)^2$. □

Theorem 4.25. *Let $b^B_1(t,w) := (2t+1)w - t(t-4)$ as in the proof of Theorem 4.10. If $f \in \mathcal{E}(\mathcal{P}^{s+}_{3,5})$ is reducible in $\mathbb{C}[a,b,c]$, then f is a positive multiple of one of the following polynomials.*
(1) $S_1\left(S_2 - \dfrac{t^2+2}{2t+1}S_{1,1}\right)^2 = \mathfrak{e}^A_{t,\mu_0(t)} = \mathfrak{e}^B_{t,\alpha}$, *where* $\mu_0(t) := \dfrac{3(t-1)^2(t+2)}{2t+1}$ *and* α *is a root of* $b^B_1\big(t,\ \omega(\alpha)\big) = 0$.
(2) $(S_2 - S_{1,1})(S_3 + 3U - T_{2,1}) = \mathfrak{e}^C_1$.
(3) $(S_2 - S_{1,1})(T_{2,1} - 6U) = \mathfrak{e}^D_1$.
(4) $(S_2 - S_{1,1})U = s_3$.

Proof. If $f \in \mathcal{E}(\mathcal{P}^{s+}_{3,5})$ is reducible, then there exists an integer $d \in \{1,\ 2\}$, $g \in \mathcal{E}(\mathcal{P}^{s+}_{3,d})$ and $h \in \mathcal{E}(\mathcal{P}^{s+}_{3,5-d})$ such that $f = gh$.

(I) Consider the case $\deg g = 1$. Then, we may assume $g = S_1 = a + b + c$.

By the previous lemma, $h = (S_2 - tS_{1,1})^2$ $(t \geq 1)$, or $h = \mathfrak{g}_t$ $(t \geq 0)$, or $h = T_{3,1} - 2S_{2,2}$. If $h = (S_2 - tS_{1,1})^2$, this is the case (1).

If $h = \mathfrak{g}_t$, then $f = S_1\mathfrak{g}_t = \mathfrak{e}^C_t + \mathfrak{e}^D_t \notin \mathcal{E}(\mathcal{P}^{s+}_{3,5})$.

If $h = (S_2 - tS_{1,1})^2$, then $f = S_1(T_{3,1} - 2S_{2,2}) = \mathfrak{e}^D_0 + 4s_3 \notin \mathcal{E}(\mathcal{P}^{s+}_{3,5})$.

(II) Consider the case $\deg g = 2$. Then, we may assume $g = S_2 - S_{1,1}$.

Since $h \in \mathcal{E}(\mathcal{P}^{s+}_{3,3})$, we have $h = S_3 + 3U - T_{2,1}$ or $f = T_{2,1} - 6U$ or $f = U$. Thus, we have (2), (3) or (4). □

4.3.2. $\mathfrak{e}^B_{t,u}(a^2,b^2,c^2) \in \mathcal{E}(\mathcal{P}_{3,10}) - \Sigma_{3,10}$ and $\mathfrak{e}^A_{t,u}(a^2,b^2,c^2) \notin \Sigma_{3,10}$.

Theorem 4.26. (1) *If* $0 < t < 7$, $t \neq 1$, $0 < u < \mu_A(t)$, *and* $u \neq \dfrac{3(t-1)^2(t+2)}{2t+1}$, *then* $\mathfrak{e}^A_{t,u}(a^2,\, b^2,\, c^2) \notin \Sigma_{3,10}$.
(2) *If* $t > 2$ *and* $\mu_B(t) < u < 1$ *and* $b^B_1(t,\, \omega(u)) \neq 0$, *then* $\mathfrak{e}^B_{t,u}(a^2,\, b^2,\, c^2) \notin \Sigma_{3,10}$.

Proof. (1) Let $0 < t < 7$, $t \neq 1$, $0 < u < \mu_A(t)$, $p := \sqrt{\mu_Z(t,u)}$, $q := \sqrt{t}$, and $F(a,b,c) := \mathfrak{e}^A_{t,u}(a^2,\, b^2,\, c^2)$. Note that $p > 0$, $p^2 \neq 1$, $q > 0$, $q^2 \neq 1$ and $p^2 \neq q^2$. Consider the zero point set $Z := V_{\mathbb{R}}(F) \subset \mathbb{P}^2_{\mathbb{R}}$. Remember that $\mathfrak{e}^A_{t,u}(p^2,1,1) = \mathfrak{e}^A_{t,u}(q^2,1,1) = 0$. Thus, $F(\pm p,\, \pm 1,\, 1) = F(\pm q,\, \pm 1,\, 1) = F(\pm 1,\, \pm p,\, 1) = F(\pm 1,\, \pm q,\, 1) = F(1,\, \pm 1,\, \pm p) = F(1,\, \pm 1,\, \pm q) = 0$. Therefore $\#Z \geq 24$.

Assume that $F \in \Sigma_{3,10}$. Then, there exists $r \in \mathbb{N}$ and $g_1, \ldots, g_r \in \mathcal{H}_{3,5}$ such that $F = g_1^2 + \cdots + g_r^2$. If $\mathbf{a} \in Z$, then $g_1(\mathbf{a}) = \cdots = g_r(\mathbf{a}) = 0$. Note that $\dim \mathcal{H}_{3,5} = 21$. So, let's find 21 points $\mathbf{a}_i \in Z$ $(1 \leq i \leq 21)$ such that there exists no $g \in \mathcal{H}_{3,5} - \{0\}$ which satisfy $g(\mathbf{a}_i) = 0$ for all $1 \leq i \leq 21$.

Let $\mathbf{a}_1 := (-p\colon 1\colon 1)$, $\mathbf{a}_2 := (p\colon -1\colon 1)$, $\mathbf{a}_3 := (p\colon 1\colon -1)$, $\mathbf{a}_4 := (1\colon p\colon 1)$, $\mathbf{a}_5 := (-1\colon p\colon 1)$, $\mathbf{a}_6 := (1\colon -p\colon 1)$, $\mathbf{a}_7 := (1\colon p\colon -1)$, $\mathbf{a}_8 := (1\colon 1\colon p)$, $\mathbf{a}_9 := (-1\colon 1\colon p)$, $\mathbf{a}_{10} := (1\colon -1\colon p)$, $\mathbf{a}_{11} := (q\colon 1\colon 1)$, $\mathbf{a}_{12} := (-q\colon 1\colon 1)$, $\mathbf{a}_{13} := (q\colon -1\colon 1)$, $\mathbf{a}_{14} := (q\colon 1\colon -1)$, $\mathbf{a}_{15} := (1\colon q\colon 1)$, $\mathbf{a}_{16} := (-1\colon q\colon 1)$, $\mathbf{a}_{17} := (1\colon -q\colon 1)$, $\mathbf{a}_{18} := (1\colon q\colon -1)$, $\mathbf{a}_{19} := (1\colon 1\colon q)$, $\mathbf{a}_{20} := (-1\colon 1\colon q)$, $\mathbf{a}_{21} := (1\colon -1\colon q)$. Take 21 monic monomials $e_1, \ldots, e_{21}$ as a base of $\mathcal{H}_{3,5}$, and denote $g = c_1e_1 + \cdots + c_{21}e_{21} \in \mathcal{H}_{3,5}$. Let $A = (a_{i,j})$ be the 21×21-matrix such that $a_{i,j} = e_j(\mathbf{a}_i)$. Then

$$\det A = \pm 262144 p^4 (p^2-1)^6 q^5 (q^2-1)^7 (p^2-q^2)^{10} (2p^2q^2 + p^2 + q^2 - 4)^3.$$

Note that $2p^2q^2 + p^2 + q^2 - 4 = 0$, if and only if $u = \dfrac{3(t-1)^2(t+2)}{2t+1}$. Thus, $\det A \neq 0$, and we obtain (1).

(2) Let $t > 2$, $\mu_B(t) < u < 1$, $p := \sqrt{t}$, $q := \sqrt{u}$, and $F(a,b,c) := \mathfrak{e}^B_{t,u}(a^2,\, b^2,\, c^2)$. Note that $0 < q < 1$ and $p > \sqrt{2}$. By the same argument with (1), it is enough to find 21 points $\mathbf{b}_i \in Z$ $(1 \leq i \leq 21)$ such that there exists no $g \in \mathcal{H}_{3,5} - \{0\}$ which satisfy $g(\mathbf{b}_i) = 0$ for all $1 \leq i \leq 21$.

Let $\mathbf{b}_1 := (p\colon 1\colon 1)$, $\mathbf{b}_2 := (-p\colon 1\colon 1)$, $\mathbf{b}_3 := (p\colon -1\colon 1)$, $\mathbf{b}_4 := (p\colon 1\colon -1)$, $\mathbf{b}_5 := (1\colon p\colon 1)$, $\mathbf{b}_6 := (-1\colon p\colon 1)$, $\mathbf{b}_7 := (1\colon -p\colon 1)$, $\mathbf{b}_8 := (1\colon p\colon -1)$, $\mathbf{b}_9 := (1\colon 1\colon p)$, $\mathbf{b}_{10} := (-1\colon 1\colon p)$, $\mathbf{b}_{11} := (1\colon -1\colon p)$, $\mathbf{b}_{12} := (q\colon 1\colon 0)$, $\mathbf{b}_{13} := (q\colon 0\colon 1)$, $\mathbf{b}_{14} := (-q\colon 1\colon 0)$, $\mathbf{b}_{15} := (-q\colon 0\colon 1)$, $\mathbf{b}_{16} := (1\colon q\colon 0)$, $\mathbf{b}_{17} := (1\colon 0\colon q)$, $\mathbf{b}_{18} := (1\colon 0\colon -q)$, $\mathbf{b}_{19} := (0\colon -q\colon 1)$, $\mathbf{b}_{20} := (0\colon 1\colon q)$, $\mathbf{b}_{21} := (0\colon 1\colon -q)$.

Let $B = (b_{i,j})$ be the 21×21-matrix such that $b_{i,j} = e_j(\mathbf{b}_i)$. Then

$$\begin{aligned}\det B = \pm 16384 p^6 (p^2-1)^7 (p^2+2) q^8 (q-1)^5 (q+1)^4 (q^2+1)^4 ((q+1)^2 + p^2 q) \\ \times (q^2(p^2+1) - 1)(p^4q^2 - 2p^2q^4 - 2p^2 - (q^2-1)^2)^3.\end{aligned}$$

Thus, if $\det B = 0$, then $q^2(p^2+1) - 1 = 0$ or $p^4q^2 - 2p^2q^4 - 2p^2 - (q^2-1)^2 = 0$. If $q^2(p^2+1) - 1 = 0$, then $\mu_B(t) < u = 1/(t+1) < \mu_B(t)$. A contradiction.

On the other hand, $p^4q^2 - 2p^2q^4 - 2p^2 - (q^2-1)^2 = 0$ is equivalent to $b^B_1(t,\, \omega(u)) = 0$. Thus, we have $\det B \neq 0$. □

It seems that $\mathfrak{e}^A_{t,u} \notin \mathcal{E}(\mathcal{P}^+_{3,5})$ and $\mathfrak{e}^A_{t,u}(a^2,\, b^2,\, c^2) \notin \mathcal{E}(\mathcal{P}_{3,10})$. But the author does not have a proof. We can prove the following:

Corollary 4.27. *Assume that* $0 < t < 7$, $t \neq 1$, $0 < u < \mu_A(t)$, *and* $u \neq \dfrac{3(t-1)^2(t+2)}{2t+1}$, *If* $\mathfrak{e}^A_{t,u}(a^2,\, b^2,\, c^2) = f_1 + \cdots + f_r$ *by certain* $f_1, \ldots, f_r \in \mathcal{P}_{3,10}$, *then* $f_1, \ldots, f_r \notin \Sigma_{3,10}$.

Theorem 4.28. *Assume $t \geq 2$, $\mu_B(t) \leq u < 1$ and $b_1^B\bigl(t,\ \omega(u)\bigr) \neq 0$. Then $\mathfrak{e}_{t,u}^B \in \mathcal{E}(\mathcal{P}_{3,5}^+)$ and $\mathfrak{e}_{t,u}^B(a^2, b^2, c^2) \in \mathcal{E}(\mathcal{P}_{3,10})$.*

Proof. Put $p := \sqrt{t}$, $q := \sqrt{u}$. When we discuss $\mathcal{H}_{3,10}$, we denote the coordinate system of $\mathbb{P}^2_{\mathbb{R}}$ by $(x\colon y\colon x)$. When we discuss $\mathcal{H}_{3,5}$, we denote the coordinate system of $\mathbb{P}^2_{\mathbb{R}}$ by $(a\colon b\colon c)$, with $x = a^2$, $y = b^2$, $z = c^2$.

Let $e_1,\dots, e_{66}$ are all the monic monomials of $\mathcal{H}_{3,10}$. We choose these so that $e_{66} = z^{10}$. Then $e_1,\dots, e_{66}$ is a base of the vector space $\mathcal{H}_{3,10}$. We define $\tau \in \mathrm{Aut}(\mathbb{P}^2_{\mathbb{R}})$ by $\tau(x\colon y\colon z) = (-x\colon y\colon z)$. Let $G \subset \mathrm{Aut}(\mathbb{P}^2_{\mathbb{R}})$ be the subgroup generated by τ and the symmetric group $\mathfrak{S}_3$. Put

$$\mathcal{Z} := \bigl\{\sigma(p\colon 1\colon 1),\, \sigma(q\colon 1\colon 0) \bigm| \sigma \in G\bigr\}.$$

Note that $\mathcal{Z}$ consists of 24 points. Align these points as $\mathcal{Z} = \{\mathbf{c}_1,\dots, \mathbf{c}_{24}\}$. Let

$$a_{3i-2,j} := e_j(\mathbf{c}_i), \quad a_{3i-1,j} := \frac{\partial e_j}{\partial x}(\mathbf{c}_i), \quad a_{3i,j} := \frac{\partial e_j}{\partial y}(\mathbf{c}_i),$$

and construct the 72×66 matrix $A = (a_{i,j})$. By Theorem 4.10(2), we have $\mathfrak{e}_{p^2,q^2}^B(a^2, b^2, c^2) \in \mathrm{Ker}\, A$. Thus, if $\mathrm{rank}\, A = 65$, then $\mathrm{Ker}\, A = \mathbb{R} \cdot \mathfrak{e}_{p^2,q^2}^B(a^2, b^2, c^2)$, and $\mathfrak{e}_{t,u}^B(a^2, b^2, c^2) \in \mathcal{E}(\mathcal{P}_{3,10})$. Note that if $\mathfrak{e}_{t,u}^B(a^2, b^2, c^2) \in \mathcal{E}(\mathcal{P}_{3,10})$, then $\mathfrak{e}_{t,u}^B \in \mathcal{E}(\mathcal{P}_{3,5}^+)$.

We choose a 65×65 minor of A as the following way. We delete the column corresponding to $e_{66} = z^{10}$. Next, we delete the seven lines corresponding to $\dfrac{\partial f}{\partial y}(p\colon 1\colon 1)$, $\dfrac{\partial f}{\partial y}(-p\colon 1\colon 1)$, $\dfrac{\partial f}{\partial y}(p\colon 1\colon\ -1)$, $\dfrac{\partial f}{\partial y}(q\colon 1\colon 0)$, $\dfrac{\partial f}{\partial x}(-q\colon 1\colon 0)$, $\dfrac{\partial f}{\partial y}(1\colon q\colon 0)$, $\dfrac{\partial f}{\partial y}(1\colon -q\colon 0)$, where $f = (e_1,\dots, e_{65})$. We denote this 65×65 square matrix by B. Let

$$\begin{aligned}
f_1^C(p, q) &:= p^2 - q^2 + 1,\\
f_2^C(p, q) &:= p^2q^2 + p^2 - 1,\\
f_3^C(p, q) &:= q^2 - (p^2 + 2)q + 1,\\
f_4^C(p, q) &:= (2p^2 + 1)(q^2 - 1)^2 - p^2q^2(p^2 - 4).
\end{aligned}$$

Using Mathematica, we obtain

$$\begin{aligned}
\det B = \pm&2417851639229258349412352\\
&\times p^{45}(p^2-1)^{38}(p^2+2)^5 q^{61}(1-q^2)^{28}\bigl(p^2q + (q+1)^2\bigr)^5\\
&\times f_1^C(p,q)^3 f_2^C(p,q)^3 f_3^C(p,q)^5 f_4^C(p,q)^9.
\end{aligned}$$

Since $p = \sqrt{t} \geq \sqrt{2}$, $0 < \sqrt{u} = q < 1$, we have

$$p^{45}(p^2-1)^{38}(p^2+2)^5 q^{61}(1-q^2)^{28}\bigl(p^2q + (q+1)^2\bigr)^5 > 0.$$

Since $p^2 > 1 > q^2$, we have $f_1^C(p, q) = p^2 - q^2 + 1 > 0$, and $f_2^C(p, q) = p^2q^2 + p^2 - 1 > 0$. Note that

$$f_4^C(p, q) = q^2 b_1^B\bigl(p^2,\ \omega(q^2)\bigr) \neq 0,$$

by the assumption. Thus, it is enough to show $f_3^C(p, q) \neq 0$. Put

$$g_3^C(t) := \sqrt{\frac{t + 2 \pm p\sqrt{t+2}}{2}}.$$

When $p \geq 2$ and $0 < q < 1$, $f_3^C(p, q) = 0$ is equivalent to $u = g_3^c(t)$. It is easy to see $g_3^C(t) < \mu_B(t)$ if $t \geq 2$. Thus, we have $f_3^C(p, q) \neq 0$. □

Note that if $(2t+1)(u-1)^2 - t(t-4)u = 0$, then $\mathfrak{e}^B_{t,u} \notin \mathcal{E}(\mathcal{P}^+_{3,5})$, by Theorem 4.26(1).

4.3.3. Extremal elements of $\mathcal{P}^{s0+}_{3,5}$.

The author should apologize for that [1, Corollary 5.7] is not correct. It must be replaced by the following:

Theorem 4.29. *All the extremal rays of $\mathcal{P}^{s0+}_{3,5}$ are generated by $\mathfrak{e}^B_{t,\mu_B(t)}$ ($t \in [2,\infty]$), $\mathfrak{e}^C_t$ ($0 \le t \le 2$), $\mathfrak{e}^D_t$ ($t \in [0,\infty]$) or s_3.*

Proof. Since $\mathcal{P}^{s0+}_{3,5} = \mathcal{F}(P_3)$, this follows from Theorem 4.20. □

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