Errata and Comments
of Discriminants of Cyclic Homogeneous
Inequalities of Three Variables

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• p.385. Line 8 from the bottom.
  Error: and disc$c^0_0 + 3 \geq 0$ determine the PSD cone.
  Correction: and disc$c^0_0 + 4 \geq 0$ determine the PSD cone.

• p.385. Theorem 0.2 (2).
  Comment: Let $D_3(c_1, c_2, c_3)$ be the discriminant of the cubic equation $x^3 + c_1 x^2 + c_2 x + c_3 = 0$. Then $4p^3 + 4q^3 + 27 - p^2q^2 - 18pq = -D_3(p, q, 1)$.

• p.386. Theorem 0.3.
  Comment: Let $D_4(c_1, c_2, c_3, c_4)$ be the discriminant of the quartic equation $g(x) := x^4 + c_1 x^3 + c_2 x^2 + c_3 x + c_4 = 0$. Then $\varphi(p, q, r) = D_4(p, r, q, 1)$.

• p.387. Theorem 0.4.
  Comment: $d_5(p, q, r)$ is the longest irreducible factor of the discriminant of the cubic equation $f(x, 1, 1)/(x - 1)^2 = 0$. In other word,

$$d_5(p, q, r) = \frac{27}{4}D_3(2 + 2p, 3 + 4p + 2q + r, 2 + 2p + 2q).$$

• p.390. Definition 1.7.
I want to change the definition of signed linear system as the following.

**Definition 1.7.** (Signed linear system) Let $A$ be a semialgebraic quasi-variety, $\mathcal{R}_A^{an}$ be the sheaf the germs of real analytic functions on $A$. Assume that there exists an invertible $\mathcal{R}_A$-sheaf $\mathcal{J}$ and an invertible $\mathcal{R}_A^{an}$-sheaf $\mathcal{J}$ such that $\mathcal{J} \otimes_{\mathcal{R}_A} \mathcal{R}_A^{an} = \mathcal{J} \otimes_{\mathcal{R}_A^{an}} \mathcal{J}$. For any point $a \in A$, we assume that we can take an affine open subset $a \in U \subset A$ such that $\mathcal{J}|_U = \mathcal{R}_A|_U \cdot e_U^2$ by a certain $e_U \in H^0(U, \mathcal{J})$. Then, for $f \in H^0(A, \mathcal{H})$, there exists $g_U \in H^0(U, \mathcal{R}_A)$ such that $f|_U = g_Ue_U^2$. We define sign$(f(a)) \in \{ 0, \pm 1 \}$ by sign$(f(a)) = \text{sign}(g_U(a))$. A finite dimensional subspace $\mathcal{H} \subset H^0(A, \mathcal{J})$ is called a signed linear system on $A$.

For example, when $A = \mathbb{P}_+^n$, $\mathcal{H} = \mathcal{H}_{n+1,d}$ and $U = \{ (x_0, \cdots, x_n) \in \mathbb{P}_+^n \mid x_0 \neq 0 \}$, we can take $\mathcal{J}$ so that $\mathcal{J}|_U = \mathcal{R}_A|_U \cdot \sqrt{x_0^2}$. So, $\mathcal{H}$ is a signed linear system.

• p.393. Proposition 1.16.
The statement and the proof of this proposition are too rough. Please replace by:

**Proposition 1.16.** (Boundary Theorem) Let $A$ be a compact semialgebraic quasi-variety, and $\mathcal{H}$ be a signed linear system on $A$. Assume that $\mathcal{P} := \mathcal{P}(A, \mathcal{H}) \subset \mathcal{H}$ is non-degenerate, and $\dim \mathcal{P} \geq 2$. Let $f \in \mathcal{P}$.
(1) If \( f(a) = 0 \) for a certain \( a \in A - B_\varepsilon \mathcal{H} \), then \( f \in \partial \mathcal{P} \).

(2) If \( f \in \partial \mathcal{P} \), then there exists \( a \in A \) such that \( f(a) = 0 \).

**Proof.** (1) We can reduce to the case \( A \) is irreducible, and \( \mathcal{H} \subset \text{Rat}(A) \), since \( \mathcal{P}(A_1 \cup A_2, \mathcal{H}) = \mathcal{P}(A_1, \mathcal{H}) \cap \mathcal{P}(A_2, \mathcal{H}) \). Since \( a \notin B_\varepsilon \mathcal{H} \), there exists \( g \in \mathcal{P} \) such that \( g(a) > 0 \). Then for all \( \varepsilon > 0 \), \( f(a) - \varepsilon g(a) < 0 \). This means \( f - \varepsilon g \notin \mathcal{P} \). Thus \( f \in \partial \mathcal{P} \).

(2) Let \( \{s_0, ..., s_N\} \) be a base of \( \mathcal{H} \) such that \( s_0, ..., s_N \in \mathcal{P} \), and define \( \Phi_\mathcal{H} : A \cdots \rightarrow X \subset \mathbb{P}^N_R \) by \( s_0, ..., s_N \). We may assume that \( A = X \). Put

\[
W_i := \left\{(X_0, \cdots, X_N) \in \mathbb{P}^N_R \mid X_0^2 + \cdots + X_N^2 \leq 3X_0^2\right\}.
\]

Then \( W_0 \cup \cdots \cup W_N = \mathbb{P}^N_R \).

Assume that \( f \in \mathcal{P} \) satisfies \( f(a) > 0 \) for all \( a \in A = X \). Take \( g \in \text{Int}(\mathcal{P}) \). We can regard \( f_i := f/X_i \) and \( g_i := g/X_i \) as holomorphic functions on \( W_i \). Since \( W_i \) is compact, there exists \( \varepsilon_i > 0 \) such that \( f_i(a) \pm \varepsilon_i g_i(a) > 0 \) for all \( a \in X \cap W_i \). Put \( \varepsilon := \min\{\varepsilon_0, ..., \varepsilon_N\} \). Then \( f \pm \varepsilon g \in \mathcal{P} \). Thus \( f \notin \partial \mathcal{P} \). \( \square \)

**p.397. Proposition 1.27.**

Error: (1) If \( \mathcal{P}_x \neq 0 \), then \( \dim \mathcal{P}_x = N - r \).

Correction: (1) \( \dim \mathcal{P}_x \leq N - r \).

In the proof, let \( \mathcal{L} := \left\{ f \in \mathcal{H} \mid T_{D,x} \subset H_f \right\} \). As the original proof, \( \dim \mathcal{L} = \dim \mathcal{H} - (r + 1) = N - r \). Since \( \mathcal{P}_x = \mathcal{P} \cap \mathcal{L} \), we have \( \dim \mathcal{P}_x \leq N - r \).

This proposition is used in some places. But only the fact \( \dim \mathcal{P}_x \leq N - r \) is used in this article.

**p.399. Proposition 1.36.**

Error: Let \( A = \mathbb{P}_R^n \) or \( A = \mathbb{P}_+ \).

Correction: Let \( A = \mathbb{P}_R^n \) or \( A = \mathbb{P}_+ \).

**p.403. Proposition 2.10.**

Comments: \( \text{disc}^+_d \) agrees with the discriminant of the equation \( x^d + p_0 x^{d-1} + \cdots + p_{d-1} x + 1 \) under a suitable base of \( \mathcal{H}_d^0 \). Please see Theorem 6.11 in this paper.

**p.406. line 23.**

Error: Note that if they are not 0, then \( \dim \mathcal{L}^c_{0,s} = 2 \), and \( \dim \mathcal{L}^{0+}_{0,s} = 1 \) by Proposition 2.7(1).

Correction: Note that if they are not 0, then \( \dim \mathcal{L}^c_{0,s} \leq 2 \), and \( \dim \mathcal{L}^{0+}_{0,s} = 1 \) by Proposition 2.7(1).

**p.411. line 8. (Line 2 after the proof of Proposition 4.1.)**

Error: Then \( \dim \mathcal{L}^c_{s,t} = 2 \) and \( \dim \mathcal{L}^{0+}_{s,t} = 1 \) for any \( (s, t) \in \mathbb{P}^2_R \) by Proposition 2.7(1).

Correction: Then \( \dim \mathcal{L}^c_{s,t} \leq 2 \) and \( \dim \mathcal{L}^{0+}_{s,t} = 1 \) for any \( (s, t) \in \mathbb{P}^2_R \) by Proposition 2.7(1).

**p.411. line 14.**

Comment: \( g_{p,q}^X(a, b, c) \) is not irreducible in \( \mathbb{C}[a, b, c] \) since the curve defined by \( g_{p,q}^X = 0 \) in \( \mathbb{P}^2_C \) has 4 nodes. There must be a conic \( h \in \mathbb{C}[a, b, c] \) such that \( g_{p,q}^X = h \bar{h} \).

**p.413. Proof of Theorem 4.4. line 4-5.**
Improvement:
\[ h_t(a, b, c) = ab(a - sb - c + sc)^2 + bc(b - sc - a + sa)^2 + ca(c - sa - b + sb)^2 \geq 0 \]

  Error: \( \dim \mathcal{L}_{s,t} = N - 2 = 2 \) if \((s, t) \neq (1, 1)\).
  Correction: \( \dim \mathcal{L}_{s,t} = N - 2 \leq 2 \) if \((s, t) \neq (1, 1)\).

\*p.421. Proof of Theorem 4.11.
  Comment: Let \( d_4(p, r, v) \) be the discriminant of the quartic equation \( f(x, 1, 1) = 0 \). Then
  \[ \text{disc}_4(1, p, p, r, v) := \frac{d_4(p, r, v)}{16(1 + 2p + r + v)} \]
  \( \text{disc}_4(1, p, p, r, v) \) consists of 44 terms. When we choose \( t_0 := S_1^4, t_1 := S_1^2 S_1, t_2 := S_1^1, t_3 := U S_1 \) as a base of \( \mathcal{H}_{3,4}^4 \), and present \( f = \sum_{i=0}^3 q_i t_i \), \( \text{disc}_4(1, p, p, r, v) \) become shorter. It consists of only 14 terms:
  \[ d_4(1, q_0, q_1, q_2, q_3) = 27q_1^2 q_2 - 216q_0 q_1 q_2^2 + 432q_0^2 q_2^3 + 36q_1 q_2 q_3 - 144q_0 q_1 q_2 q_3 + 16q_1^2 q_2 q_3 - 64q_0 q_2 q_3^2 + q_1^3 q_3^2 - 36q_0 q_1 q_2 q_3^2 + 8q_1^2 q_2 q_3^2 - 48q_0 q_2 q_3^3 + q_1^3 q_3^3 - 12q_0 q_2 q_3^4 - q_0 q_3^4. \]
  Note that \( \text{disc}_4(1, p, p, r, v) = d_4(-4 + p, 2 - 2p + r, 3 - 3p - 3r + v) \).

\*p.422. Proof of Proposition 4.12. (2)
  Comment: It is better to choose \( g^A_{t,1} \) (\( t \geq 0 \)) in stead of \( g^X_{p,p} \) (\( p \in \mathbb{R} \)), since
  \[ g^A_{t,1} = s_0 - (t + 1)s_1 + (t^2 + 2t)s_2. \]

  Comment: The author referd \[13\]. This proposition is also a corollary of Theorem 1.1 of \[23\].

\*p.428. Theorem 5.6. line 2 from the bottom.
  Error: \( d_5(p, q, r) = \text{disc}(C_2), 4q - (p + 1)^2 - 4 = \text{disc}(C_1) \).
  Correction: \( d_5(p, q, r) = \text{disc}(C_1), 4q - (p + 1)^2 - 4 = \text{disc}(C_2) \).

\*p.429. Corollary 5.7.
  Original Corollary 5.7 is incorrect. It must be replaced by the following:

Lemma 5.6b. (1) Let \( I \subset A \). If \( 0 \neq f \in \mathcal{P}_I \) and \( \dim \mathcal{P}_I = 1 \), then \( f \) is an extremal element of \( \mathcal{P} \).
  (2) Let \( a_1, \ldots, a_r \in A \). If \( \dim(\mathcal{P}_{a_1} \cap \cdots \cap \mathcal{P}_{a_r}) = 1 \) and \( f \in \mathcal{P} \) satisfies \( f(a_1) = \cdots = f(a_r) = 0 \), then \( f \) is an extremal element of \( \mathcal{P} \).
  Proof. (1) Assume that \( f = \alpha g + \beta h \) (\( g, h \in \mathcal{P}, \alpha, \beta \in \mathbb{R}_+ \)). Take \( a \in I \). Since \( 0 = f(a) = \alpha g(a) + \beta h(a), g(a) \geq 0 \) and \( h(a) \geq 0 \), we have \( g(a) = h(a) = 0 \). Thus \( g, h \in \mathcal{P}_I = \mathbb{R} \cdot f \). Thus \( f \) is an extremal element of \( \mathcal{P} \).
Let $I = \{a_1, \ldots, a_r\}$, and apply (1).

**Corollary 5.7.** Let
\[
\begin{align*}
f_p^A(a, b, c) &:= s_0 + ps_1 - (p + 1)s_2 + p^2s_3, \\
\ell(t) &:= 2 - t^2 + t\sqrt{(t-1)(t+2)}, \\
s_m(t) &:= (1/2)(\ell(t) - \sqrt{\ell(t)^2 - 4}, \\
f_t^B(a, b, c) &:= s_0 + (1 - 2\ell(t))s_1 + (t^3 + 2t^2 - 2 - 2(t^2 - 1)\ell(t))s_2 \\
&\quad - ((t+1)^2(2t+3) - 4(t+1)^2\ell(t))s_3, \\
g_t(a, b, c) &:= s_1 + (t^2 - 1)s_2 - 2(t+1)^2s_3.
\end{align*}
\]

(1) For all $t \geq 0$, $g_t$ is an extremal element of $\mathcal{P}_5^{s_0+}$, and $g_t \in \mathcal{L}_{t, 1}^{s_0+} \cap \mathcal{L}_{0, 0}^{s_0+}$.

(2) Let $t \geq 2$, and put $s := s_m(t)$. Then $0 < s \leq 1$, and $f_t^B \in \mathcal{L}_{t, 1}^{s_0+} \cap \mathcal{L}_{0, 0}^{s_0+}$, $f_t^B$ is an extremal element of $\mathcal{P}_5^{s_0+}$.

(3) Let $0 \leq t \leq 2$, and put $p := -t - 1$. Then $f_p^A \in \mathcal{L}_{t, 1}^{s_0+} \cap \mathcal{L}_{0, 0}^{s_0+}$, and $f_p^A$ is an extremal element of $\mathcal{P}_5^{s_0+}$.

(4) All the extremal elements of $\mathcal{P}_5^{s_0+}$ are positive multiples of $f_p^A$ ($-3 \leq p \leq -1$), $f_t^B$ ($t \geq 2$), $g_t$ ($t \geq 0$), $s_2 - 2s_3$ and $s_3$.

**Proof.** (1) $f \in \mathcal{L}_{0, 0}^{s_0+}$ implies the coefficient of $s_0$ in $f$ is equal to zero. Since,
\[
\begin{align*}
g_t(s, 1, 1) &= 2(s-1)^2(s-t)^2, \\
g_t(0, s, 1) &= s(s+1)((s-1)^2 + t^2s),
\end{align*}
\]
we have $g_t \in \mathcal{L}_{t, 1}^{s_0+}$ by Proposition 5.1.

(2) It is easy exercise to verify that $s_m(t)$ varies $(0, 1]$ when $t \geq 2$. Since
\[
\begin{align*}
f_t^B(s, 1, 1) &= (s-t)^2(s-1)^2(s+2(t-\sqrt{(t-1)(t+2)})^2), \\
f_t^B(0, s, 1) &= (s+1)(s^2 - (2 - t^2) + t\sqrt{(t-1)(t+2)}s+1)^2,
\end{align*}
\]
we have $f_t^B \in \mathcal{L}_{t, 1}^{s_0+} \cap \mathcal{L}_{0, 0}^{s_0+}$.

(3) follow from
\[
\begin{align*}
f_p^A(t, 1, 1) &= t(t-1)^2(t+p+1)^2, \\
f_p(0, t, 1) &= (t+1)(t-1)^2(t^2 + (p+1)t + 1).
\end{align*}
\]

(4) All the extremal elements of $\mathcal{L}_{0, 0}^{s_0+}$ are positive multiples of $g_t$ ($0 \geq 0$) and $g_\infty := s_2$. $\mathcal{L}_{0, 0}^{s_0+} \cap \mathcal{L}_{0, 0}^{s_0+} = \mathbb{R}_+ \cdot s_3$. Thus we obtain (4).

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**p.430. Lemme 5.8.** line 2.

**Error:** Note that $\dim \mathcal{L}_s^{c_0+} = 6 - 2 = 4$ by Proposition 2.7(1).

**Correction:** Note that $\dim \mathcal{L}_s^{c_0+} = 6 - 2 \leq 4$ by Proposition 2.7(1).

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**p.431. Theorem 5.9.**

**Comment:** Let $D_n(c_1, \ldots, c_n)$ be the discriminant of $f(x) = x^n + \sum_{i=1}^n c_i x^{n-i}$. Then,
\[
\text{disc}^+_5(x, y, z, w) = D_5(x, z, w, y).
\]

**p.434 line 14.**
Error: Assume that $g_0(S + 2, 2S) \leq 0$.
Correction: Assume that $g_0(S + 2, 2S + 1) \leq 0$.

Error: Note that $\dim \mathcal{L}^c_{\alpha} = 9 - 2 = 7$ by Proposition 2.7(1).
Correction: Note that $\dim \mathcal{L}^c_{\alpha} = 9 - 2 \leq 7$ by Proposition 2.7(1).

Comment: Let $D_n(c_1, \ldots, c_n)$ be the discriminant of $f(x) = x^n + \sum_{i=1}^{n} c_i x^{n-i}$. Then,
\[
\text{disc}^c_{b}(x, y, z, w, u) = D_6(x, z, u, w, y).
\]
In general,

Theorem 6.11. Take the base of $\mathcal{H}^c_n$ so that $s_0 = S_n$, $s_1 = S_{n-1,1}$, $s_2 = S_{n-2,2}$, \ldots, $s_{n-1} = S_{1,n-1}$, \ldots. Here, if $i \geq n$, then $s_i$ is a multiple of $U$. We represent $f \in \mathcal{H}^c_n$ as $f = \sum p_i s_i$. Then, the edge discriminant of $\mathcal{P}^c_n$ agrees with $D_n(p_1, \ldots, p_{n-1}, 1)$.

Proof. Take $f \in \mathcal{L}^c_{0,t} \subset \mathcal{E}^c_n$, where $t > 0$. Then $f(0, t, 1) = 0$. Since $f(0, x, 1) \geq 0$ for all $x > 0$, the equation $f(0, x, 1) = 0$ has a multiple root at $x = t$. Thus, the discriminant of $f$ is equal to 0. Since $S_{i,n-1}(0, x, 1) = x^i (1 \leq i \leq n - 1)$, $S_n(0, x, 1) = x^n + 1$ and $U(0, x, 1) = 0$, we have $f(0, x, 1) = x^n + p_1 x^{n-1} + \cdots + p_{n-1} x + 1$.

Since $D_n$ and $\text{disc}^c_{n}$ are irreducible, we have the conclusion. 

\[\square\]
Additional Results.

After the end of §4, the following new result may be added. This will be published somewhere else.

4.6. The PSD cones $\mathcal{P}_4^+$. We choose $s_0 := S_4 - U S_1$, $s_1 := T_{3,1} - 2 U S_1$, $s_2 := S_{2,2} - U S_1$, $s_3 := U S_1$ as a base of $\mathcal{H}_4^4$.

Theorem 4.15. Take $f(a_0, a_1, a_2) := s_0 + \sum_{i=1}^{3} p_i s_i = S_4 + p_1 T_{3,1} + p_2 S_{2,2} + (p_3 - 1 - 2 p_1 - p_2) U S_1 \in \mathcal{H}_4^4$. Let $d_4(p_1, p_2, p_3)$ be the discriminant of the quartic equation $f(x, 1, 1) = 0$, and take it's irreducible factor $\text{disc}_4^3(p_1, p_2, p_3) := d_4(p_1, p_2, p_3)/(16(1+2p_1+p_2+p_3))$. Then, $f(a_0, a_1, a_2) \geq 0$ for all $a_0, a_1, a_2 \in \mathbb{R}^+$. If and only if one of the $(1) - (7)$ holds.

By Proposition 2.13, 2.14 and §4.5, we conclude that $\Psi : X \to \mathcal{H}_4^4$ is decomposed as $\Phi := \Phi(1 : 1) = (0 : 0 : 1)$. By Remark 1.21 (3), $\text{disc}(\Phi) = 2^{16}(1+2p_1+p_2+p_3)$.

This theorem will be proved at the end of this subsection. $\Phi := \Phi(1 : 1) = (0 : 0 : 1)$.

Theorem 4.16. $\Delta^2(X) = \{ X^0 \}$, $\Delta^1(X) = \{ \Phi(L_{F,+}^b), \Phi(L_{F,+}^b) \}$, $\Delta^0(X) = \{ \Phi(0 : 0) \}$. Put $C^b := (\Phi(L_{F,+}^b), C^0 := \Phi(L_{F,+}^b)$, $P_1 := \Phi(0 : 0 : 1) = (1 : 0 : 0 : 0), P_2 := \Phi(0 : 0 : 1) = (2 : 2 : 1 : 0)$, and $P_3 := \Phi(1 : 1) = (0 : 0 : 0 : 1)$. By Remark 1.21 (3), $\text{disc}(P_1) = x_0$, $\text{disc}(P_2) = 2 x_0 + 2 x_1 + x_2$ and $\text{disc}(P_3) = x_3$. Thus $\mathcal{F}(P_1)$ is at infinity, and $\mathcal{F}(P_3) = \mathcal{P}_4^{b}$.

Thus, $\mathcal{F}(C^b)$ and $\mathcal{F}(C^0)$ are essential for $\partial \mathcal{P}_4^{b+}$.

On $\mathcal{H}_4^4$, $\mathcal{G}$ become very simple:

\[ g_t(a, b, c) := g_{t,1}^1(a, b, c) = s_0 - (t + 1) s_1 + (t^2 + 2t) s_2, \]
\[ c^X_t(a, b, c) := (S_2 - 1 t S_3, 1)^{-z} = s_0 - \frac{2}{1} s_1 + \frac{2t^2 + 1}{1} s_2 + 3 \left( \frac{1}{t} - 1 \right)^2 s_3, \]
\[ \Psi(s, t) = \frac{S_{1,1}(s, t, 1)}{S_2(s, t, 1)} \in [0, 1], \text{ } e_{s,t}^d(a, b, c) := e_{s,t}^d(a, b, c), \]
where $s \in [0, \infty]$ and $k \in [0, 1]$. By the next Proposition 4.17 (1), $\mathcal{F}(X^s)$ is not a face component. Since $g_s(s, 1, 1) = 0$ and $e^A_{s,1}(s, 1, 1) = 0$, we have $g_s, \ e^A_{s,1} \in \mathcal{F}(C^0)$. Since $e^A_{0,s}(0, s, 1) = 0$ and $US_1(0, s, 1) = 0$, we have $e^A_{0,s}, \ US_1 \in \mathcal{F}(C^0)$.

**Proposition 4.17.** Let $\mathcal{L}^{s,t+}_s$ be the local cone of $\mathcal{P}^{s+}_4$ at $(s : t : 1) \in A = \mathbb{P}^2_+$. 

1. If $0 < s \neq 1, 0 < t \neq 1$ and $s \neq t$, then $\mathcal{L}^{s,t+}_s = \mathbb{R} \cdot e^A_{s,t}$.

2. If $0 < t \neq 1$ then $\mathcal{L}^{s,t+}_s = \mathbb{R} \cdot g_s + \mathbb{R} \cdot e^A_{t,1}$.

3. If $0 < t \neq 1$ then $\mathcal{L}^{s,t+}_0 = \mathbb{R} \cdot e^A_{0,t} + \mathbb{R} \cdot US_1$.

4. $\mathcal{L}^{s,t+}_0 = \mathbb{R} \cdot (S_1 + US_1 - 2S_{2,2}) + \mathbb{R} \cdot e^A_{0,1} + \mathbb{R} \cdot (T_{3,1} - 2S_{2,2}) + \mathbb{R} \cdot US_1$.

**Proof.** (1) When $0 < s \neq 1, t \neq 1$, and $s \neq t$, dim $\mathcal{L}^{s,t+}_s \leq 3 - 2 = 1$. On the other hand, $e^A_{s,t} \in \mathcal{L}^{s,t+}_s$. Thus, (1) holds.

(2) Assume that $0 < t \neq 1$, dim $\mathcal{L}^{s,t+}_t \leq 3 - 1 = 2$. Since $g_t, e^A_{t,1} \in \mathcal{L}^{s,t+}_t$, any $f \in \mathcal{L}^{s,t+}_t$ can be expressed as $f = \alpha g_t + \beta e^A_{t,1}$ by certain $\alpha, \beta \in \mathbb{R}$. Note that $g_t(1, 1, 1) = 0$. Since $t \neq 1$, $e^A_{t,1}(1, 1, 1) > 0$. Since $0 \leq f(1, 1, 1) = \beta e^A_{t,1}(1, 1, 1)$, we have $\beta \geq 0$. On the other hand, there exists $a' = (s', t', 1) \in \mathbb{P}^2_+$ such that $e^A_{s',t'} = e^A_{s,t}$ and $\pi(a') \in \text{Int}(\mathbb{P}^2_+ / \mathbb{G}_3)$. Then $g_s(s', t', 1) > 0$. Since $0 \leq f(s', t', 1) = \alpha e^A_{t,1}(s', t', 1)$, we have $\alpha > 0$.

(3) Assume that $0 < t \neq 1$, dim $\mathcal{L}^{s,t+}_0 \leq 3 - 1 = 2$. Since $e^A_{0,t}, US_1 \in \mathcal{L}^{s,t+}_0$, any $f \in \mathcal{L}^{s,t+}_0$ can be expressed as $f = \alpha e^A_{0,t} + \beta US_1$ by certain $\alpha, \beta \in \mathbb{R}$. Since $0 \geq f(0, 0, 1) = \alpha e^A_{0,t}$ and $e^A_{0,t} > 0$, we have $\alpha > 0$. There exists $a' = (s', t', 1) \in \mathbb{P}^2_+$ such that $e^A_{s',t'} = e^A_{0,t}$ and $\pi(a') \in \text{Int}(\mathbb{P}^2_+ / \mathbb{G}_3)$. Since $0 \leq f(s', t', 1) = \beta s't'(s' + t' + 1)$, we have $\beta > 0$.

(4) Note that $\mathcal{L}^{s+}_0 = \mathcal{F}(P_2) \subset V(2x_0 + 2x_1 + x_2)$. By Proposition 1.33, $\partial \mathcal{L}^{s+}_0 \subset \mathcal{F}(P_2) \cap (V(x_0) \cup V(x_4) \cup \mathcal{F}(C^0) \cup \mathcal{F}(C^0))$. $\mathcal{F}(P_2) \cap \mathcal{F}(C^0) = \lim_{t \to 0} \mathcal{L}^{s+}_t = \mathbb{R} \cdot g_0 + \mathbb{R} \cdot e^A_{0,1} = \mathbb{R} \cdot (S_1 + US_1 - 2S_{2,2}) + \mathbb{R} \cdot e^A_{0,1}$. $\mathcal{F}(P_2) \cap \mathcal{F}(C^0) = \lim_{t \to 1} \mathcal{L}^{s+}_t = \mathbb{R} \cdot e^A_{0,1} + \mathbb{R} \cdot US_1$. By Theorem 0.3, we have $\mathcal{F}(P_2) \cap V(x_3) = \mathbb{R} \cdot g_0 + \mathbb{R} \cdot (T_{3,1} - 2S_{2,2})$. Now, it is easy to see that $\mathcal{F}(P_2) \cap V(x_0) = \mathbb{R} \cdot US_1 + \mathbb{R} \cdot (T_{3,1} - 2S_{2,2})$. Thus, we have (4).

**Corollary 4.18.** All the extremal elements of $\mathcal{P}^{s+}_4$ are positive multiple of following polynomials: $e^X_s (0 < k \leq 1)$, $g_s (s \geq 0)$, $s_0 - 2s_2 = S_1 + US_1 - 2S_{2,2}$, $s_1 - 2s_2 = T_{3,1} - 2S_{2,2}$ and $s_3 = US_1$.

**Corollary 4.19.** For $f := s_0 + \sum_{i=1}^3 p_is_i$, disc$(C^0) = \text{disc}^s_4(p_1, p_2, p_3)$ and disc$(C^0) = 4p_2 - 8 - p_1^2$.

**Proof.** Since disc$^s_4$ is irreducible, and $g_s, \ e^A_{t,1} \in \text{disc}^s_4$ for all $t \in \mathbb{R}$, we have disc$(C^0) = \text{disc}^s_4$. Since $US_1, \ e^A_{0,t} \in V(4p_2 - 8 - p_1^2)$ for all $t \in \mathbb{R}$, we have disc$(C^0) = 4p_2 - 8 - p_1^2$.

Note that $e^X_s \in \mathcal{F}(C^0) \cap \mathcal{F}(C^0)$ if and only if $0 \leq k \leq 1/2$. When $1/2 < k \leq 1$, $e^X_s \in \mathcal{F}(C^0) \setminus \mathcal{F}(C^0)$.

**Proof of Theorem 4.15.** Put $\mathcal{P} := \mathcal{P}^{s+}_4$.
Theorem 0.3 implies there exists a curve defined by $\text{disc}^2(x, y, v) = 0$ on $H_v$. 

Note that $\text{disc}(C)$, $C$, $L$ represent the zero loci of $\text{disc}(C^0)$, $\text{disc}(P^0)$ respectively. When $v = 0$, (1) follows from Theorem 0.3.

(II) Put $L(x \geq c) := \{(x, y) \in L \mid x \geq c\}$. If $x \geq 0$, The point $(x, -2x - 2) \in L$ corresponds to $(S_1 + US_1 - 2S_{2,2}) + vUS_1 \in \partial\mathcal{P}$. Thus $L(x \geq 0) \subset \partial\mathcal{P}$.

Note that $L$ tangents to $C_v$ at $(x, y) = (-v - 1, 2v)$ with the multiplicity 2, and $L$ tangent to $C$ at $(-4, 6)$. When $0 \leq v \leq 3$, the point $(-v - 1, 2v)$ corresponds to $\frac{v + 1}{4} e_{t/2}^X + \frac{3 - v}{4} (S_1 + US_1 - 2S_{2,2}) + \frac{v + 1}{4} US_1 \in \partial\mathcal{P}$. Thus $L(x \geq -v - 1) \subset \partial\mathcal{P}$.

When $v \geq 3$, the point $(-4, 6)$ corresponds to $e_{t/2}^X + (v - 3)US_1 \in \partial\mathcal{P}$. Thus $L(x \geq -4) \subset \partial\mathcal{P}$.

This implies (2) and (4).

(III) When $v > 0$, the curve $C_v$ has a node at $P_v : (x, y) = \left(-2\sqrt{\frac{v}{3}}, -2, \frac{v + 2\sqrt{3v} + 9}{3}\right)$. $P_v$ corresponds to extremal polynomials $e_k^X$, where $k = \frac{1}{\sqrt{v/3 + 1}}$, $v = 3(k - 1)^2/k^2$ ($0 \leq k \leq 1$). Moreover $P_v, Q_v \in C \cap C_v$.

When $v \geq 3$, we put $C[P_v, -4] := \{(x, y) \in C \mid -2\sqrt{v/3} - 2 \leq x \leq -4\}$. Consider $f := e_{t/2}^X + (v - 3(1/k - 1)^2)US_1 \in C$. $f$ corresponds to $(x, y) = (-2/k, 1/k^2 + 2) \in C$. Since $0 \leq k \leq 1/2$, we have $x \leq -4$. $v - 3(1/k - 1)^2 \geq 0$ is equivalent to $x = -2/k \geq -2(\sqrt{v/3} + 1)$. Thus $C[P_v, -4] \subset \partial\mathcal{P}$, if $v \geq 3$. This implies (6).

(IV) We consider the cases (3), (6) and (7). Put $C_v' := \begin{cases} \{(x, y) \in C_v \mid x \leq -1 - v, y \geq x^2 - (v + 2\sqrt{3v} + 1)\} & (0 < v \leq 3) \\ \{(x, y) \in C_v \mid x \leq -2\sqrt{v/3} - 2, y \geq x^2 - (v + 2\sqrt{3v} + 1)\} & (2 \leq v \leq 27) \\ \{(x, y) \in C_v \mid x \leq -2\sqrt{v/3} - 2, y \geq (8 + x^2)/4\} & (v \geq 27) \end{cases}$

Theorem 0.3 implies there exists a $f \in \mathcal{F}_1$ of the form $f = \alpha e_{t/2}^A + (1 - \alpha)\mathcal{g}^A$ ($\exists \alpha \in [0, 1]$, $\exists t \in \mathbb{R}$). The $x$-coordinate of $f$ is equal to $\alpha \frac{-2(t^2 + 2)}{2t + 1} + (1 - \alpha)(-t - 1)$. Since this is negative on $C_v'$, we have $t \geq 0$. Thus $C_v' \subset \partial\mathcal{P}$. \[\square\]