# Errata and Comments of Discriminants of Cyclic Homogeneous Inequalities of Three Variables

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•p.385. Line 8 from the bottom.

*Error*: and $\mathrm{disc}_3^{c+} \geq 0$ determine the PSD cone.

*Correction*: and $\mathrm{disc}_4^{c0} \geq 0$ determine the PSD cone.

•p.385. Theorm 0.2 (2).

*Comment*: Let $D_3(c_1, c_2, c_3)$ be the discriminant of the cubic equation $x^3 + c_1x^2 + c_2x + c_3 = 0$. Then $4p^3 + 4q^3 + 27 - p^2q^2 - 18pq = -D_3(p, q, 1)$.

•p.386. Theorem 0.3.

*Comment*: Let $D_4(c_1, c_2, c_3, c_4)$ be the discriminant of the quartic equation $g(x) := x^4 + c_1x^3 + c_2x^2 + c_3x + c_4 = 0$. Then $\varphi(p, q, r) = D_4(p, r, q, 1)$.

Using this fact, we can give an equivalent condition for that $g(x) \geq 0$ for all $x \in \mathbb{R}$ (resp. for all $x \geq 0$).

•p.387. **Theorem 0.4.**

*Comment*: $d_5(p, q, r)$ is the longest irreducible factor of the discriminant of the cubic equation $f(x, 1, 1)/(x-1)^2 = 0$. In other word,

$$d_5(p, q, r) = \frac{27}{4} D_3(2 + 2p, 3 + 4p + 2q + r, 2 + 2p + 2q).$$

•p.390. **Definition 1.7.**

I want to cahnge the definition of signed linear system as the following.

**Definition 1.7.**(Signed linear system) Let $A$ be a semialgebraic quasi-variety, $\mathfrak{R}_A^{\mathrm{an}}$ be the sheaf the germs of real analytic functions on $A$. Assume that there exists an invertible $\mathfrak{R}_A$-sheaf $\mathfrak{I}$ and an invertible $\mathfrak{R}_A^{\mathrm{an}}$-sheaf $\mathfrak{J}$ such that $\mathfrak{I} \otimes_{\mathfrak{R}_A} \mathfrak{R}_A^{\mathrm{an}} = \mathfrak{J} \otimes_{\mathfrak{R}_A^{\mathrm{an}}} \mathfrak{J}$. For any point $a \in A$, we assume that we can take an affine open subset $a \in U \subset A$ such that $\mathfrak{I}|_U = \mathfrak{R}_A|_U \cdot e_U^2$ by a certain $e_U \in H^0(U, \mathfrak{J})$. Then, for $f \in H^0(A, \mathfrak{H})$, there exists $g_U \in H^0(U, \mathfrak{R}_A)$ such that $f|_U = g_U e_U^2$. We define $\mathrm{sign}(f(a)) \in \{0, \pm 1\}$ by $\mathrm{sign}(f(a)) = \mathrm{sign}(g_U(a))$. A finite dimensional subspace $\mathfrak{H} \subset H^0(A, \mathfrak{I})$ is called a *signed linear system* on $A$.

For example, when $A = \mathbb{P}_+^n$, $\mathfrak{H} = \mathfrak{H}_{n+1,d}$ and $U = \left\{(x_0 \colon \cdots \colon x_n) \in \mathbb{P}_+^n \;\middle|\; x_0 \neq 0\right\}$, we can take $\mathfrak{I}$ so that $\mathfrak{I}|_U = \mathfrak{R}_A|_U \cdot \sqrt{x_0^d}^2$. So, $\mathfrak{H}$ is a signed linear system.

•p.393. **Proposition 1.16.**

The statement and the proof of this proposition are too rough. Please replace by:

**Proposition 1.16.** (Boundary Theorem) *Let $A$ be a compact semialgebraic quasi-variety, and $\mathfrak{H}$ be a signed linear system on $A$. Assume that $\mathfrak{P} := \mathfrak{P}(A, \mathfrak{H}) \subset \mathfrak{H}$ is non-degenerate, and* $\dim \mathfrak{P} \geq 2$. *Let $f \in \mathfrak{P}$.*

(1) *If* $f(a) = 0$ *for a certain* $a \in A - \mathrm{Bs}\,\mathcal{H}$, *then* $f \in \partial\mathcal{P}$.
(2) *If* $f \in \partial\mathcal{P}$, *then there exists* $a \in A$ *such that* $f(a) = 0$.

*Proof.* (1) We can reduce to the case $A$ is irreducible, and $\mathcal{H} \subset \mathrm{Rat}(A)$, since $\mathcal{P}(A_1 \cup A_2, \mathcal{H}) = \mathcal{P}(A_1, \mathcal{H}) \cap \mathcal{P}(A_2, \mathcal{H})$. Since $a \notin \mathrm{Bs}\,\mathcal{H}$, there exists $g \in \mathcal{P}$ such that $g(a) > 0$. Then for all $\varepsilon > 0$, $f(a) - \varepsilon g(a) < 0$. This means $f - \varepsilon g \notin \mathcal{P}$. Thus $f \in \partial\mathcal{P}$.

(2) Let $\{s_0, \ldots, s_N\}$ be a base of $\mathcal{H}$ such that $s_0, \ldots, s_N \in \mathcal{P}$, and define $\Phi_{\mathcal{H}}\colon A \cdots \to X \subset \mathbb{P}^N_{\mathbb{R}}$ by $s_0, \ldots, s_N$. We may assume that $A = X$. Put

$$W_i := \left\{ (X_0 : \cdots : X_N) \in \mathbb{P}^N_{\mathbb{R}} \;\middle|\; X_0^2 + \cdots + X_N^2 \le 3X_i^2 \right\}.$$

Then $W_0 \cup \cdots \cup W_N = \mathbb{P}^N_{\mathbb{R}}$.

Assume that $f \in \mathcal{P}$ satisfies $f(a) > 0$ for all $a \in A = X$. Take $g \in \mathrm{Int}(\mathcal{P})$. We can regard $f_i := f/X_i$ and $g_i := g/X_i$ as holomorphic functions on $W_i$. Since $W_i$ is compact, there exists $\varepsilon_i > 0$ such that $f_i(a) \pm \varepsilon_i g_i(a) > 0$ for all $a \in X \cap W_i$. Put $\varepsilon := \min\{\varepsilon_0, \ldots, \varepsilon_N\}$. Then $f \pm \varepsilon g \in \mathcal{P}$. Thus $f \notin \partial\mathcal{P}$. □

•p.397. **Proposition 1.27.**

*Error*: (1) *If* $\mathcal{P}_x \neq 0$, *then* $\dim \mathcal{P}_x = N - r$.

*Correction*: (1) $\dim \mathcal{P}_x \le N - r$.

In the proof, let $\mathcal{L} := \left\{ f \in \mathcal{H} \;\middle|\; T_{D,x} \subset H_f \right\}$. As the origibal proof, $\dim \mathcal{L} = \dim \mathcal{H} - (r+1) = N - r$. Since $\mathcal{P}_x = \mathcal{P} \cap \mathcal{L}$, we have $\dim \mathcal{P}_x \le N - r$.

This proposition is used in some places. But only the fact $\dim \mathcal{P}_x \le N - r$ is used in this article.

•p.399. **Proposition 1.36.**

*Error*: *Let* $A = \mathbb{P}^n_{\mathbb{R}}$ *or* $A = \mathbb{P}_+$,

*Correction*: *Let* $A = \mathbb{P}^n_{\mathbb{R}}$ *or* $A = \mathbb{P}^n_+$,

•p.403. **Proposition 2.10.**

*Comments*: $\mathrm{disc}^{c+}_d$ agrees with the discriminant of the equation $x^d + p_0 x^{d-1} + \cdots + p_{d-1}x + 1$ under a suitable base of $\mathcal{H}^{c0}_d$. Please see Theorem 6.11 in this paper.

•p.406. line 23.

*Error*: Note that if they are not 0, then $\dim \mathcal{L}^{c+}_{0,s} = 2$, and $\dim \mathcal{L}^{c0+}_{0,s} = 1$ by Proposition 2.7(1).

*Correction*: Note that if they are not 0, then $\dim \mathcal{L}^{c+}_{0,s} \le 2$, and $\dim \mathcal{L}^{c0+}_{0,s} = 1$ by Proposition 2.7(1).

•p.411. line 8. (Line 2 after the proof of Proposition 4.1.)

*Error*: Then $\dim \mathcal{L}^{c}_{s,t} = 2$ and $\dim \mathcal{L}^{c0}_{s,t} = 1$ for any $(s, t) \in \mathbb{P}^2_{\mathbb{R}}$ by Proposition 2.7(1).

*Correction*: Then $\dim \mathcal{L}^{c}_{s,t} \le 2$ and $\dim \mathcal{L}^{c0}_{s,t} = 1$ for any $(s, t) \in \mathbb{P}^2_{\mathbb{R}}$ by Proposition 2.7(1).

•p.410. The first line of 4.1.

*Error*: Hilbert proved that every element in $\mathcal{P}_4 := \mathcal{P}(\mathbb{P}_{\mathbb{R}}, \mathcal{H}_4)$ can

*Correction*: Hilbert proved that every element in $\mathcal{P}_4 := \mathcal{P}(\mathbb{P}^2_{\mathbb{R}}, \mathcal{H}_4)$ can

•p.411. line 14.

*Comment*: $\mathfrak{g}^X_{p,q}(a,b,c)$ is not irreducible in $\mathbb{C}[a,b,c]$ since the curve defined by $\mathfrak{g}^X_{p,q}=0$ in $\mathbb{P}^2_C$ has 4 nodes. There must be a conic $h\in\mathbb{C}[a,b,c]$ such that $\mathfrak{g}^X_{p,q}=h\overline{h}$. For example,

$$\begin{aligned}\mathfrak{g}^A_{1,2}(x,y,z) &= S_4-3T_{3,1}+8S_{2,2}-3US_1\\ &=(x^2+\omega y^2+\omega^2z^2+3\omega^2xy+3yz+3\omega zx)\\ &\qquad\times(x^2+\omega^2y^2+\omega z^2+3\omega xy+3yz+3\omega^2zx)\end{aligned}$$

where $\omega:=(-1+\sqrt{-3})/2$.

$\mathfrak{g}^X_{p,q}(a,b,c)$ is not extremal in $\mathcal{P}_4$.

•p.413. Proof of **Theorem 4.4.** line 4-5.

*Improvement*:

$$\begin{aligned}\mathfrak{h}_s(a,b,c)=ab(a-sb-c+sc)^2+bc(b-sc-a+sa)^2\\ +ca(c-sa-b+sb)^2\geq 0\end{aligned}$$

•p.417. Proof of **Theorem 4.7.** line 1.

*Error*: $\dim\mathcal{L}^c_{s,t}=N-2=2$ if $(s,\,t)\neq(1,\,1)$.

*Correction*: $\dim\mathcal{L}^c_{s,t}=N-2\leq 2$ if $(s,\,t)\neq(1,\,1)$.

•p.421. Proof of **Theorem 4.11.**

*Comment*: Let $d_4(p,r,v)$ be the discriminant of the quartic equation $f(x,1,1)=0$. Then

$$\mathrm{disc}^c_4(1,p,p,r,v):=\frac{d_4(p,r,v)}{16(1+2p+r+v)}.$$

$\mathrm{disc}^c_4(1,p,p,r,v)$ consists of 44 terms. When we choose $t_0:=S_1^4$, $t_1:=S_1^2S_{1,1}$, $t_2:=S_{1,1}^2$, $t_3:=US_1$ as a base of $\mathcal{H}^s_{3,4}$, and present $f=\sum_{i=0}^{3}q_it_i$, $\mathrm{disc}^c_4(1,p,p,r,v)$ become shorter. It consists of only 14 terms:

$$\begin{aligned}d^s_4(q_0,q_1,q_2,q_3)=&27q_1^4q_2-216q_0q_1^2q_2^2+432q_0^2q_2^3+36q_1^3q_2q_3-144q_0q_1q_2^2q_3+16q_1^2q_2^2q_3\\ &-64q_0q_2^3q_3+q_1^3q_3^2-36q_0q_1q_2q_3^2+8q_1^2q_2q_3^2-48q_0q_2^2q_3^2+q_1^2q_3^3\\ &-12q_0q_2q_3^3-q_0q_3^4.\end{aligned}$$

Note that $\mathrm{disc}^c_4(1,p,p,r,v)=d^s_4(-4+p,2-2p+r,3-3p-3r+v)$.

•p.422. Proof of **Proposition 4.12.** (2)

*Comment*: It is better to choose $\mathfrak{g}^A_{t,1}$ $(t\geq 0)$ in stead of $\mathfrak{g}^X_{p,p}$ $(p\in\mathbb{R})$, since

$$\mathfrak{g}^A_{t,1}(a,b,c)=s_0-(t+1)s_1+(t^2+2t)s_2.$$

•p.425. **Proposition 5.1.**

*Comment*: In this article, the author refered [13]. This proposition is also a corollary of Theorem 1.1 of [23].

•p.428. **Theorem 5.6.** line 2 from the bottom.

*Error*: $d_5(p,q,r)=\mathrm{disc}(C_2)$, $4q-(p+1)^2-4=\mathrm{disc}(C_1)$.

*Correction*: $d_5(p,q,r)=\mathrm{disc}(C_1)$, $4q-(p+1)^2-4=\mathrm{disc}(C_2)$.

•p.429. **Corollary 5.7.**

Original Corollary 5.7 is incorrect. It must be replaced by the following:

**Lemma 5.6b.** (1) *Let* $I \subset A$. *If* $0 \neq f \in \mathcal{P}_I$ *and* $\dim \mathcal{P}_I = 1$, *then* $f$ *is an extremal element of* $\mathcal{P}$.

(2) *Let* $a_1, \ldots, a_r \in A$. *If* $\dim(\mathcal{P}_{a_1} \cap \cdots \cap \mathcal{P}_{a_r}) = 1$ *and* $f \in \mathcal{P}$ *satisfies* $f(a_1) = \cdots = f(a_r) = 0$, *then* $f$ *is an extremal element of* $\mathcal{P}$.

*Proof.* (1) Assume that $f = \alpha g + \beta h$ ($g$, $h \in \mathcal{P}$, $\alpha$, $\beta \in \mathbb{R}_+$). Take $a \in I$. Since $0 = f(a) = \alpha g(a) + \beta h(a)$, $g(a) \geq 0$ and $h(a) \geq 0$, we have $g(a) = h(a) = 0$. Thus $g$, $h \in \mathcal{P}_I = \mathbb{R} \cdot f$. Thus $f$ is an extremal element of $\mathcal{P}$.

(2) Let $I = \{a_1, \ldots, a_r\}$, and apply (1). ∎

**Corollary 5.7.** *Let*

$$\begin{aligned}
&f_p^A(a,b,c) := s_0 + ps_1 - (p+1)s_2 + p^2 s_3,\\
&\ell(t) := 2 - t^2 + t\sqrt{(t-1)(t+2)},\\
&s_m(t) := (1/2)(\ell(t) - \sqrt{\ell(t)^2 - 4},\\
&f_t^B(a,b,c) := s_0 + (1 - 2\ell(t))s_1 + (t^3 + 2t^2 - 2 - 2(t^2-1)\ell(t))s_2\\
&\qquad\qquad - ((t+1)^2(2t+3) - 4(t+1)^2\ell(t))s_3,\\
&g_t(a,b,c) := s_1 + (t^2-1)s_2 - 2(t+1)^2 s_3.
\end{aligned}$$

(1) *For all* $t \geq 0$, $g_t$ *is an extrelal element of* $\mathcal{P}_5^{s0+}$, *and* $g_t \in \mathcal{L}_{t,1}^{s0+} \cap \mathcal{L}_{0,0}^{s0+}$.

(2) *Let* $t \geq 2$, *and put* $s := s_m(t)$. *Then* $0 < s \leq 1$, *and* $f_t^B \in \mathcal{L}_{t,1}^{s0+} \cap \mathcal{L}_{0,s}^{s0+}$. $f_t^B$ *is an extrelal element of* $\mathcal{P}_5^{s0+}$.

(3) *Let* $0 \leq t \leq 2$, *and put* $p := -t-1$. *Then* $f_p^A \in \mathcal{L}_{t,1}^{s0+} \cap \mathcal{L}_{0,1}^{s0+}$, *and* $f_p^A$ *is an extremal element of* $\mathcal{P}_5^{s0+}$.

(4) *All the extremal elements of* $\mathcal{P}_5^{s0+}$ *are positive multiples of* $f_p^A$ $(-3 \leq p \leq -1)$, $f_t^B$ $(t \geq 2)$, $g_t$ $(t \geq 0)$, $s_2 - 2s_3$ *and* $s_3$.

*Proof.* (1) $f \in \mathcal{L}_{0,0}^{s0+}$ implies the cofficient of $s_0$ in $f$ is equal to zero. Since,

$$\begin{aligned}
g_t(s,1,1) &= 2(s-1)^2(s-t)^2,\\
g_t(0,s,1) &= s(s+1)((s-1)^2 + t^2 s),
\end{aligned}$$

we have $g_t \in \mathcal{L}_{t,1}^{s0+}$ by Proposition 5.1.

(2) It is easy exercise to veryfy that $s_m(t)$ varies $(0, 1]$ when $t \geq 2$. Since

$$\begin{aligned}
f_t^B(s,1,1) &= (s-t)^2(s-1)^2(s + 2(t - \sqrt{(t-1)(t+2)})^2),\\
f_t^B(0,s,1) &= (s+1)(s^2 - (2 - t^2 + t\sqrt{(t-1)(t+2)})s + 1)^2,
\end{aligned}$$

we have $f_t^B \in \mathcal{L}_{t,1}^{s0+} \cap \mathcal{L}_{0,s}^{s0+}$.

(3) follow from

$$\begin{aligned}
f_p^A(t,1,1) &= t(t-1)^2(t+p+1)^2,\\
f_p(0,t,1) &= (t+1)(t-1)^2(t^2 + (p+1)t + 1).
\end{aligned}$$

(4) All the extremal elements of $\mathcal{L}_{0,0}^{s0+}$ are positive multiples of $g_t$ $(0 \geq 0)$ and $g_\infty := s_2$. $\mathcal{L}_{0,s}^{s0+} \cap \mathcal{L}_{0,0}^{s0+} = \mathbb{R}_+ \cdot s_3$. Thus we obtain (4). ∎

•p.430. **Lemme 5.8.** line 2.

*Error*: Note that $\dim \mathcal{L}_s^{c0+} = 6 - 2 = 4$ by Proposition 2.7(1).
*Correction*: Note that $\dim \mathcal{L}_s^{c0+} = 6 - 2 \leq 4$ by Proposition 2.7(1).

•p.431. **Theorem 5.9.**

*Comment*: Let $D_n(c_1, \ldots, c_n)$ be the discriminant of $f(x) = x^n + \sum_{i=1}^{n} c_i x^{n-i}$. Then, $\mathrm{disc}_5^{c+}(x, y, z, w) = D_5(x, z, w, y)$.

•p.434 line 14.
*Error*: Assume that $g_0(S + 2, 2S) \leq 0$.
*Correction*: Assume that $g_0(S + 2, 2S + 1) \leq 0$.

•p.437 line 8. **Lemma 6.7.**
*Error*: Note that $\dim \mathcal{L}_s^{c0+} = 9 - 2 = 7$ by Proposition 2.7(1).
*Correction*: Note that $\dim \mathcal{L}_s^{c0+} = 9 - 2 \leq 7$ by Proposition 2.7(1).

•p.437-439. **Theorem 6.8.**

*Comment*: Let $D_n(c_1, \ldots, c_n)$ be the discriminant of $f(x) = x^n + \sum_{i=1}^{n} c_i x^{n-i}$. Then, $\mathrm{disc}_6^{c+}(x, y, z, w, u) = D_6(x, z, u, w, y)$. In general,

**Theorem 6.11.** Take the base of $\mathcal{H}_n^c$ so that $s_0 = S_n$, $s_1 = S_{n-1,1}$, $s_2 = S_{n-2,2}, \ldots$, $s_{n-1} = S_{1,n-1}, \ldots$. Here, if $i \geq n$, then $s_i$ is a multiple of $U$. We represent $f \in \mathcal{H}_n^c$ as $f = \sum p_i s_i$. Then, the edge discriminant of $\mathcal{P}_n^{c+}$ agrees with $D_n(p_1, \ldots, p_{n-1}, 1)$.

*Proof.* Take $f \in \mathcal{L}_{0,t}^{c+} \subset \mathcal{E}_n^{c+}$, where $t > 0$. Then $f(0, t, 1) = 0$. Since $f(0, x, 1) \geq 0$ for all $x > 0$, the equation $f(0, x, 1) = 0$ has a multiple root at $x = t$. Thus, the discriminant of $f$ is equal to 0. Since $S_{i,n-1}(0, x, 1) = x^i$ $(1 \leq i \leq n - 1)$, $S_n(0, x, 1) = x^n + 1$ and $U(0, x, 1) = 0$, we have $f(0, x, 1) = x^n + p_1 x^{n-1} + \cdots + p_{n-1} x + 1$.
Since $D_n$ and $\mathrm{disc}_n^{c+}$ are irreducible, we have the conclusion. □

# Additional Results.

•p.425.

After the end of §4, the followin new result may be added. This will be published somewhere else.

**4.6. The PSD cones $\mathcal{P}_4^{s+}$.**

We choose $s_0 := S_4 - US_1$, $s_1 := T_{3,1} - 2US_1$, $s_2 := S_{2,2} - US_1$, $s_3 := US_1$ as a base of $\mathcal{H}_4^s$.

**Theorem 4.15.** *Take* $f(a_0, a_1, a_2) := s_0 + \sum_{i=1}^{3} p_i s_i = S_4 + p_1 T_{3,1} + p_2 S_{2,2} + (p_3 - 1 - 2p_1 - p_2)US_1 \in \mathcal{H}_4^s$. *Let* $d_4(p_1, p_2, p_3)$ *be the discriminant of the quartic equation* $f(x, 1, 1) = 0$, *and take it's irreducible factor* $\mathrm{disc}_4^s(p_1, p_2, p_3) := d_4(p_1, p_2, p_3)/(16(1+2p_1+p_2+p_3))$. *Then,* $f(a_0, a_1, a_2) \geq 0$ *for all* $a_0$, $a_1$, $a_2 \in \mathbb{R}_+$ *if and only if one of the* (1)—(7) *holds.*
(1) $p_3 = 0$, $p_1 \leq -1$ *and* $p_2 \geq p_1^2 - 1$.
(2) $0 \leq p_3 \leq 3$, $-1 - p_3 \leq p_1$ *and* $p_2 \geq -2 - 2p_1$.
(3) $0 < p_3 \leq 3$, $p_1 \leq -1 - p_3$, $\mathrm{disc}_4^s(p_1, p_2, p_3) \geq 0$ *and* $p_2 \geq p_1^2 - (p_3 + 2\sqrt{3p_3} + 1)$.
(4) $3 \leq p_3$, $-4 \leq p_1$ *and* $p_2 \geq -2 - 2p_1$.
(5) $3 \leq p_3$, $-2\sqrt{p_3/3} - 2 \leq p_1 \leq -4$ *and* $p_2 \geq (8 + p_1^2)/4$.
(6) $3 \leq p_3 \leq 27$, $p_1 \leq -2\sqrt{p_3/3} - 2$, $\mathrm{disc}_4^s(p_1, p_2, p_3) \geq 0$ *and* $p_2 \geq p_1^2 - (p_3 + 2\sqrt{3p_3} + 1)$.
(7) $27 < p_3$, $p_1 \leq -2\sqrt{p_3/3} - 2$, $\mathrm{disc}_4^s(p_1, p_2, p_3) \geq 0$ *and* $p_2 \geq (8 + p_1^2)/4$.

This theorem will be proved at the end of this subsection. $\Phi := \Phi_{\mathcal{H}_4^s} : \mathbb{P}_+^2 \to X := X(A, \mathcal{H}_4^s)$ is decomposed as $\Phi : \mathbb{P}_+^2 \xrightarrow{\pi} \mathbb{P}_+^2/\mathfrak{S}_3 \xrightarrow{\Psi} X$. By Propisition 2.13, 2.14 and §4.5, we conclude that $\Psi : \mathbb{P}_+^2/\mathfrak{S}_3 \longrightarrow X$ is an isomorphism. Let

$$L_{F+}^b := \{(s:1:1) \in \mathbb{P}_{\mathbb{R}}^2 \mid 0 < s < 1 \text{ or } 1 < s < \infty\},$$
$$L_{F+}^0 := \{(0:s:1) \in \mathbb{P}_{\mathbb{R}}^2 \mid 0 < s < 1\}.$$

By Propisition 2.14, we have the following:

**Proposition 4.16.** $\Delta^2(X) = \{X^\circ\}$, $\Delta^1(X) = \{\Phi(L_{F+}^b)), \Phi(L_{F+}^0)\}$, $\Delta^0(X) = \{\Phi(0:0:1), \Phi(0:1:1), \Phi(1:1:1)\}$.

Put $C^b := \Phi(L_{F+}^b)$, $C^0 := \Phi(L_{F+}^0)$, $P_1 := \Phi(0:0:1) = (1:0:0:0)$, $P_2 := \Phi(0:1:1) = (2:2:1:0)$ and $P_3 := \Phi(1:1:1) = (0:0:0:1)$. By Remark 1.21 (3), $\mathrm{disc}(P_1) = x_0$, $\mathrm{disc}(P_2) = 2x_0 + 2x_1 + x_2$ and $\mathrm{disc}(P_3) = x_3$. Thus $\mathcal{F}(P_1)$ is at infinity, and $\mathcal{F}(P_3) = \mathcal{P}_4^{s0+}$. Thus, $\mathcal{F}(C^b)$ and $\mathcal{F}(C^0)$ are essential for $\partial\mathcal{P}_4^{s0+}$.

On $\mathcal{H}_4^s$, $\mathfrak{g}$ becone very simple:

$$\mathfrak{g}_t(a,b,c) := \mathfrak{g}_{t,1}^A(a,b,c) = s_0 - (t+1)s_1 + (t^2+2t)s_2,$$
$$\mathfrak{e}_k^X(a,b,c) := \left(S_2 - 1\mathfrak{k}\mathfrak{S}_{1,1}\right)^2 = \mathfrak{s}_0 - \frac{2}{\mathfrak{k}}\mathfrak{s}_1 + \frac{2\mathfrak{k}^2+1}{\mathfrak{k}^2}\mathfrak{s}_2 + 3\left(\frac{1}{\mathfrak{k}} - 1\right)^2\mathfrak{s}_3,$$
$$\mathfrak{k}(s,t) = \frac{S_{1,1}(s,t,1)}{S_2(s,t,1)} \in [0,\, 1], \quad \mathfrak{e}_{s,t}^A(a,b,c) := \mathfrak{e}_{\mathfrak{k}(s,t)}^X(a,b,c),$$

where $s \in [0, \infty]$ and $k \in [0, 1]$. By the next Proposition 4.17 (1), $\boldsymbol{\mathfrak{F}}(X^\circ)$ is not a face component. Since $\mathfrak{g}_s(s,1,1) = 0$ and $\mathfrak{e}^A_{s,1}(s,1,1) = 0$, we have $\mathfrak{g}_s$, $\mathfrak{e}^A_{s,1} \in \boldsymbol{\mathfrak{F}}(C^b)$. Since $\mathfrak{e}^A_{0,s}(0,s,1) = 0$ and $US_1(0,s,1) = 0$, we have $\mathfrak{e}^A_{0,s}$, $US_1 \in \boldsymbol{\mathfrak{F}}(C^0)$.

**Proposition 4.17.** *Let $\boldsymbol{\mathfrak{L}}^{s+}_{s,t}$ be the local cone of $\boldsymbol{\mathfrak{P}}^{s+}_4$ at $(s : t : 1) \in A = \mathbb{P}^2_+$.*
(1) *If $0 < s \neq 1$, $0 < t \neq 1$ and $s \neq t$, then $\boldsymbol{\mathfrak{L}}^{s+}_{s,t} = \mathbb{R} \cdot \mathfrak{e}^A_{s,t}$.*
(2) *If $0 < t \neq 1$ then $\boldsymbol{\mathfrak{L}}^{s+}_{t,1} = \mathbb{R} \cdot \mathfrak{g}_t + \mathbb{R} \cdot \mathfrak{e}^A_{t,1}$.*
(3) *If $0 < t \neq 1$ then $\boldsymbol{\mathfrak{L}}^{s+}_{0,t} = \mathbb{R} \cdot \mathfrak{e}^A_{0,t} + \mathbb{R} \cdot US_1$.*
(4) $\boldsymbol{\mathfrak{L}}^{s+}_{0,1} = \mathbb{R} \cdot (S_4 + US_1 - 2S_{2,2}) + \mathbb{R} \cdot \mathfrak{e}^A_{0,1} + \mathbb{R} \cdot (T_{3,1} - 2S_{2,2}) + \mathbb{R} \cdot US_1$.

*Proof.* (1) When $0 < s \neq 1$, $t \neq 1$, and $s \neq t$, $\dim \boldsymbol{\mathfrak{L}}^{s+}_{s,t} \leq 3 - 2 = 1$. On the other hand, $\mathfrak{e}^A_{s,t} \in \boldsymbol{\mathfrak{L}}^{s+}_{s,t}$. Thus, (1) holds.

(2) Assume that $0 < t \neq 1$. $\dim \boldsymbol{\mathfrak{L}}^{s+}_{t,1} \leq 3 - 1 = 2$. Since $\mathfrak{g}_t$, $\mathfrak{e}^A_{t,1} \in \boldsymbol{\mathfrak{L}}^{s+}_{t,1}$, any $f \in \boldsymbol{\mathfrak{L}}^{s+}_{t,1}$ can be expressed as $f = \alpha \mathfrak{g}_t + \beta \mathfrak{e}^A_{t,1}$ by certain $\alpha$, $\beta \in \mathbb{R}$. Note that $\mathfrak{g}_t(1,1,1) = 0$. Since $t \neq 1$, $\mathfrak{e}^A_{t,1}(1,1,1) > 0$. Since $0 \leq f(1,1,1) = \beta \mathfrak{e}^A_{t,1}(1,1,1)$, we have $\beta \geq 0$. On the other hand, there exists $a' = (s', t', 1) \in \mathbb{P}^2_+$ suth that $\mathfrak{e}^A_{s',t'} = \mathfrak{e}^A_{s,1}$ and $\pi(a') \in \mathrm{Int}(\mathbb{P}^2_+/\mathfrak{S}_3)$. Then $\mathfrak{g}_s(s', t', 1) > 0$. Since $0 \leq f(s', t', 1) = \alpha \mathfrak{f}^A_t(s', t', 1)$, we have $\alpha \geq 0$.

(3) Assume that $0 < t \neq 1$. $\dim \boldsymbol{\mathfrak{L}}^{s+}_{0,t} \leq 3 - 1 = 2$. Since $\mathfrak{e}^A_{0,t}$, $US_1 \in \boldsymbol{\mathfrak{L}}^{s+}_{0,t}$, any $f \in \boldsymbol{\mathfrak{L}}^{s+}_{0,t}$ can be expressed as $f = \alpha \mathfrak{e}^A_{0,t} + \beta US_1$ by certain $\alpha$, $\beta \in \mathbb{R}$. Since $0 \geq f(0,0,1) = \alpha \mathfrak{e}^A_{0,t}$ and $\mathfrak{e}^A_{0,t} > 0$, we have $\alpha > 0$. There exists $a' = (s', t', 1) \in \mathbb{P}^2_+$ suth that $\mathfrak{e}^A_{s',t'} = \mathfrak{e}^A_{0,t}$ and $\pi(a') \in \mathrm{Int}(\mathbb{P}^2_+/\mathfrak{S}_3)$. Since $0 \leq f(s', t', 1) = \beta s't'(s' + t' + 1)$, we have $\beta \geq 0$.

(4) Note that $\boldsymbol{\mathfrak{L}}^{s+}_{0,1} = \boldsymbol{\mathfrak{F}}(P_2) \subset V(2x_0 + 2x_1 + x_2)$. By Proposition 1.33, $\partial \boldsymbol{\mathfrak{L}}^{s+}_{0,1} \subset \boldsymbol{\mathfrak{F}}(P_2) \cap \big(V(x_0) \cup V(x_4) \cup \boldsymbol{\mathfrak{F}}(C^b) \cup \boldsymbol{\mathfrak{F}}(C^0)\big)$. $\boldsymbol{\mathfrak{F}}(P_2) \cap \boldsymbol{\mathfrak{F}}(C^b) = \lim_{t \to 0} \boldsymbol{\mathfrak{L}}^{s+}_{t,1} = \mathbb{R} \cdot \mathfrak{g}_0 + \mathbb{R} \cdot \mathfrak{e}^A_{0,1} = \mathbb{R} \cdot (S_4 + US_1 - 2S_{2,2}) + \mathbb{R} \cdot \mathfrak{e}^A_{0,1}$. $\boldsymbol{\mathfrak{F}}(P_2) \cap \boldsymbol{\mathfrak{F}}(C^0) = \lim_{t \to 1} \boldsymbol{\mathfrak{L}}^{s+}_{0,t} = \mathbb{R} \cdot \mathfrak{e}^A_{0,1} + \mathbb{R} \cdot US_1$. By Theorem 0.3, we have $\boldsymbol{\mathfrak{F}}(P_2) \cap V(x_3) = \mathbb{R} \cdot \mathfrak{g}_0 + \mathbb{R} \cdot (T_{3,1} - 2S_{2,2})$. Now, it is east to see that $\boldsymbol{\mathfrak{F}}(P_2) \cap V(x_0) = \mathbb{R} \cdot US_1 + \mathbb{R} \cdot (T_{3,1} - 2S_{2,2})$. Thus, we have (4). □

**Corollary 4.18.** *All the extremal elements of $\boldsymbol{\mathfrak{P}}^{s+}_4$ are positive multiple of following polynomials: $\mathfrak{e}^X_k$ $(0 < k \leq 1)$, $\mathfrak{g}_s$ $(s \geq 0)$, $s_0 - 2s_2 = S_4 + US_1 - 2S_{2,2}$, $s_1 - 2s_2 = T_{3,1} - 2S_{2,2}$ and $s_3 = US_1$.*

**Corollary 4.19.** *For* $f := s_0 + \sum_{i=1}^{3} p_i s_i$, $\mathrm{disc}(C^b) = \mathrm{disc}^s_4(p_1, p_2, p_3)$ *and* $\mathrm{disc}(C^0) = 4p_2 - 8 - p_1^2$.

*Proof.* Since $\mathrm{disc}^s_4$ is irredusible, and $\mathfrak{g}_t$, $\mathfrak{e}^A_{t,1} \in \mathrm{disc}^s_4$ for all $t \in \mathbb{R}$, we have $\mathrm{disc}(C^b) = \mathrm{disc}^s_4$. Since $US_1$, $\mathfrak{e}^A_{0,t} \in V(4p_2 - 8 - p_1^2)$ for all $t \in \mathbb{R}$, we have $\mathrm{disc}(C^0) = 4p_2 - 8 - p_1^2$. □

Note that $\mathfrak{e}^X_k \in \boldsymbol{\mathfrak{F}}(C^0) \cap \boldsymbol{\mathfrak{F}}(C^b)$ if and only if $0 \leq k \leq 1/2$. When $1/2 < k \leq 1$, $\mathfrak{e}^X_k \in \boldsymbol{\mathfrak{F}}(C^b) - \boldsymbol{\mathfrak{F}}(C^0)$.

*Proof of Theorem 4.15.* Put $\boldsymbol{\mathfrak{P}} := \boldsymbol{\mathfrak{P}}^{s+}_4$.

(I) We use the same symbols with the proof of Theorem 4.11. There we denote $x := p_1$, $y := p_2$ and $z := p_3$. Fix a constant $v > 0$, and let $H_v$ be the plane $z = v$ in $\breve{\mathfrak{H}}^s_4$. Let $T_v := \mathfrak{P}^s_4 \cap H_v$, $F_v := \mathfrak{F}^s_4 \cap H_v$, and let $C_v$ be the curve defined by $\mathrm{disc}^s_4(x, y, v) = 0$ on $H_v$. Note that $F_v \subset C_v$. Moreover, let $C$ be the curv e defined by $4y - 8 - x^2 = 0$ on $H_v$, and let $L$ be the line defined by $2 + 2x + y = 0$ on $H_v$. $C^b_v$, $C$, $L$ represent the zero loci of $\mathrm{disc}(C^b)$, $\mathrm{disc}(C^0)$, $\mathrm{disc}(P2)$ respectively. When $v = 0$, (1) follows from Theorem 0.3.

(II) Put $L(x \geq c) := \{(x, y) \in L \mid x \geq c\}$. If $x \geq 0$, The point $(x, -2x - 2) \in L$ corresponds to $(S_4 + US_1 - 2S_{2,2}) + x(T_{3,1} - 2S_{2,2}) + vUS_1 \in \partial\mathfrak{P}$. Thus $L(x \geq 0) \subset \partial\mathfrak{P}$.

Note that $L$ tangents to $C_v$ at $(x,\ y) = (-v - 1,\ 2v)$ with the multiplicity 2, and $L$ tangent to $C$ at $(-4,\ 6)$. When $0 \leq v \leq 3$, the point $(-v - 1,\ 2v)$ corresponds to $\dfrac{v+1}{4}\mathfrak{e}^X_{1/2} + \dfrac{3-v}{4}(S_4 + US_1 - 2S_{2,2}) + \dfrac{v+1}{4}US_1 \in \partial\mathfrak{P}$. Thus $L(x \geq -v - 1) \subset \partial\mathfrak{P}$. When $v \geq 3$, the point $(-4, 6)$ corresponds to $\mathfrak{e}^X_{1/2} + (v-3)US_1 \in \partial\mathfrak{P}$. Thus $L(x \geq -4) \subset \partial\mathfrak{P}$. This implies (2) and (4).

(III) When $v > 0$, the curve $C_v$ has a node at

$$P_v : (x, y) = \left(-2\sqrt{\frac{v}{3}} - 2,\ \frac{v + 2\sqrt{3v} + 9}{3}\right).$$

$P_v$ corresponds to extremal polynomials $\mathfrak{e}^X_k$, where $k = \dfrac{1}{\sqrt{v/3} + 1}$, $v = 3(k - 1)^2/k^2$ $(0 \leq k \leq 1)$. Moreover $P_v$, $Q_v \in C \cap C_v$.

When $v \geq 3$, we put $C[P_v, -4] := \{(x, y) \in C \mid -2\sqrt{v/3} - 2 \leq x \leq -4\}$. Consider $f := \mathfrak{e}^X_k + (v - 3(1/k - 1)^2)US_1 \in C$. $f$ corresponds to $(x,\ y) = (-2/k,\ 1/k^2 + 2) \in C$. Since $0 \leq k \leq 1/2$, we have $x \leq -4$. $v - 3(1/k - 1)^2 \geq 0$ is equivalent to $x = -2/k \geq -2(\sqrt{v/3} + 1)$. Thus $C[P_v, -4] \subset \partial\mathfrak{P}$, if $v \geq 3$. This implies (6).

(IV) We consider the cases (3), (6) and (7). Put

$$C'_v := \begin{cases} \{(x, y) \in C_v \mid x \leq -1 - v,\ y \geq x^2 - (v + 2\sqrt{3v} + 1)\} & (0 < v \leq 3) \\ \{(x, y) \in C_v \mid x \leq -2\sqrt{v/3} - 2,\ y \geq x^2 - (v + 2\sqrt{3v} + 1)\} & (2 \leq v \leq 27) \\ \{(x, y) \in C_v \mid x \leq -2\sqrt{v/3} - 2,\ y \geq (8 + x^2)/4\} & (v \geq 27) \end{cases}$$

Theorem 0.3 implies there exists a $f \in \mathfrak{H}^s_4$ of the form $f = \alpha\mathfrak{e}^A_{t,1} + (1 - \alpha)\mathfrak{g}^A_t$ $(\exists\alpha \in [0, 1]$, $\exists t \in \mathbb{R})$. The $x$-coordinate of $f$ is equal to $\alpha\dfrac{-2(t^2 + 2)}{2t + 1} + (1 - \alpha)(-t - 1)$. Since this is negative on $C'_v$, we have $t \geq 0$. Thus $C'_v \in \partial\mathfrak{P}$. $\square$