Some Quintic and Sextic Cyclic Inequalities of Three Variables.

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Abstract. We prove some families of quintic and sextic cyclic inequalities. Each of them belongs to an edge component of a positive semidefinite cone.

Section 1. Introduction.

Let $s > 0$ be a real number and let

$$\mathcal{H}_s^d := \left\{ f(a,b,c) \;\middle|\; \begin{array}{l} f \text{ is a cyclic homogeneous polynomial of } \deg f = d, \\ f(0,s,1) = 0, \text{ and } f(a,a,a) = 0 \end{array} \right\},$$

$$\mathcal{P}_s^d := \left\{ f \in \mathcal{H}_s^d \;\middle|\; f(a,\,b,\,c) \geq 0 \text{ for all } a \geq 0,\, b \geq 0,\, c \geq 0. \right\}.$$

$\mathcal{H}_s^d$ is a $\mathbb{R}$-vector space and $\mathcal{P}_s^d$ is a closed convex cone in $\mathcal{H}_s^d$. In the cases $d \leq 4$, these are studies in [3]. In this article we study the cases $d = 5$ and 6.

It is easy to see that $\dim_{\mathbb{R}} \mathcal{H}_s^5 = 5$ and $\dim_{\mathbb{R}} \mathcal{H}_s^6 = 8$. But it is proved that $\dim_{\mathbb{R}} \mathcal{P}_s^5 = 4$ and $\dim_{\mathbb{R}} \mathcal{P}_s^6 = 7$ in [3]. We denote

$$\sum a^n := a^n + b^n + c^n, \quad \sum a^m b^n := a^m b^n + b^m c^n + c^m a^n,$$
$$\sum a^l b^m c^n := a^l b^m c^n + b^l c^m a^n + c^l a^m b^n.$$

The following two theorems ware announced in [3], but full proof ware not given.

Theorem 1.1. *The following $F_{1,s}, \ldots, F_{4,s}$ are linearly independent elements in $\mathcal{P}_s^5$.*

$$F_{1,s}(a,b,c) := 3s^4 \sum a^5 - (4s^5 - 1) \sum a^4 b + (s^8 - 4s^3) \sum ab^4 - (s^8 - 4s^5 + 3s^4 - 4s^3 + 1) \sum a^3 bc,$$

$$F_{2,s}(a,b,c) := 2 \sum a^4 b - 3s \sum a^3 b^2 + s^3 \sum ab^4 - (s^3 - 3s + 2) \sum a^3 bc,$$

$$F_{3,s}(a,b,c) := \sum a^4 b - 3s^2 \sum a^2 b^3 + 2s^3 \sum ab^4 - (2s^3 - 3s^2 + 1) \sum a^3 bc,$$

$$F_{4,s}(a,b,c) := \sum a^3 bc - \sum a^2 b^2 c.$$

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Theorem 1.2. *The following $G_{1,s}, \ldots, G_{7,s}$ are linearly independent elements in $\mathfrak{P}^6_s$.*

$$\begin{aligned}
G_{1,s} &:= s^4 \sum a^6 - (2s^6 - 1) \sum a^4 b^2 + (s^8 - 2s^2) \sum a^2 b^4 \\
&\qquad - 3(s^8 - 2s^6 + s^4 - 2s^2 + 1) a^2 b^2 c^2, \\
G_{2,s} &:= 2 \sum a^5 b - 3s \sum a^4 b^2 + s^3 \sum a^2 b^4 - 3(s-1)^2 (s+2) a^2 b^2 c^2, \\
G_{3,s} &:= 2s^3 \sum ab^5 + \sum a^4 b^2 - 3s^2 \sum a^2 b^4 - 3(s-1)^2 (2s+1) a^2 b^2 c^2, \\
G_{4,s} &:= \sum a^4 b^2 + s^2 \sum a^2 b^4 - 2s \sum a^3 b^3 - 3(s-1)^2 a^2 b^2 c^2, \\
G_{5,s} &:= \sum a^4 bc - \sum a^3 b^2 c, \\
G_{6,s} &:= \sum a^4 bc - \sum a^2 b^3 c, \\
G_{7,s} &:= \sum a^4 bc - 3a^2 b^2 c^2.
\end{aligned}$$

In [3], proofs of $F_{1,s} \geq 0$, $F_{2,s} \geq 0$ and $G_{2,s} \geq 0$ are omitted. Other part is already proved. We shall prove these in Section 2.

We should explain why Theorem 1.1 and 1.2 are important. Consider cyclic homogeneous inequalities of three variables of degree d. These are simple, but interesting objects to study. There were not so many studies till the 21st century. Cîrtoaje studied in the case $d = 4$ in [7], [8], [9] and [10]. The case $d = 3$ was treated in [1], [2] and [3]. The cases $d = 5$ and 6 were partly treated in [3], [6], [11] and [12]. PSD cones in the classical sense are studied in many articles relating to Hilbert's 17th problem. For example, [4], [5], [13] and [14]. A notion of PSD cones are generalized in [3]. $\mathfrak{P}^d_s$ is one of PSD cones and especially treated as local cones. Please read [3] to understand why $\mathfrak{P}^d_s$ is important. Theorem 1.1 and 1.2 do not determine $\mathfrak{P}^5_s$ and $\mathfrak{P}^6_s$, but determine the edge discriminants of degree 5 and 6.

Section 2. Main Theorems.

Let

$$S_n := a^n + b^n + c^n, \quad S_{m,n} := a^m b^n + b^m c^n + c^m a^n, \quad U := abc.$$

Theorem 2.1. *Let*

$$F_{2,s}(a,b,c) = 2S_{4,1} + s^3 S_{1,4} - 3s S_{3,2} - (s^3 - 3s + 2) U S_2.$$

Then, $F_{2,s}(a,b,c) \geq 0$ holds for $a \geq 0$, $b \geq 0$, $c \geq 0$ and $s \geq 0$. Moreover, $F_{2,s}(0,s,1) = 0$, $F_{2,s}(1,1,1) = 0$, and $F_{2,s}(0,0,1) = 0$.

Proof. $F_{2,s}(1,1,1) = 0$ and $F_{2,s}(0,0,1) = 0$ are trivial. $F_{2,s}(0,s,1) = 0$ can be checked using Mathematica. Let

$$\begin{aligned}
&p := S_{4,1} - US_2, \quad q := S_{3,2} - US_2, \quad r := S_{1,4} - US_2, \\
&f(s) := s^3 F_{2,s}(a,b,c,1/s) = 2s^3 S_{4,1} + S_{1,4} - 3s^2 S_{3,2} - (1 - 3s^2 + 2s^3) U S_2 \\
&\qquad = 2ps^3 - 3qs^2 + r.
\end{aligned}$$

Note that $p \geq 0$ and $r \geq 0$. We shall find the minimum of $f(s)$ in $s \geq 0$. Since $f'(s) = 6s(ps-q)$, if $q \geq 0$ then $\min f = f(q/p) = (p^2r - q^3)/p^2$, and if $q < 0$ then $\min f = f(0) = r \geq 0$. Thus, it is enough to show $p^2r - q^3 \geq 0$ under the assumption $q \geq 0$. Note that

$$\begin{aligned}
p^2r - q^3 &= (S_{4,1} - US_2)^2(S_{1,4} - US_2) - (S_{3,2} - US_2)^3 \\
&= U(S_2 - S_{1,1})\Big\{ S_{9,1} - S_{7,3} + S_{6,4} + 2S_{4,6} \\
&\quad - US_7 + 2US_{6,1} - US_{1,6} - 3US_{5,2} - 2US_{2,5} + US_{4,3} - 2US_{3,4} \\
&\quad - U^2S_4 - U^2S_{3,1} + 5U^2S_{1,3} - 2U^2S_{2,2} + 2U^3S_1 \Big\}.
\end{aligned}$$

Thus, let $g(a, b, c) := \dfrac{p^2r - q^3}{U(S_2 - S_{1,1})}$. Since $g(a,\, b,\, c)$ is a cyclic polynomial, we may assume $0 \leq a \leq 1$, $0 \leq b \leq 1$ and $c = 1$.

Case 1: We shall prove that if $1 \geq a \geq b \geq 0$ then $g \geq 0$.

Let $k := (1 - a)/(1 - b)$. Note that $0 \leq k \leq 1$. Using Mathematica, we obtain

$$g(a, 1 - k(1 - a), 1) = (1 - a)^4 \sum_{i=0}^{6} \varphi_i(k) a^i,$$

here,

$$\begin{aligned}
\varphi_0(k) &:= (1 - k)^4(3 - 4k + 8k^2 - 9k^3 + 5k^4 - k^5), \\
\varphi_1(k) &:= 6 - 24k + 44k^2 - 31k^3 - 42k^4 + 142k^5 - 173k^6 + 111k^7 - 37k^8 + 5k^9, \\
\varphi_2(k) &:= 7 - 13k + 2k^2 + 37k^3 - 57k^4 - 3k^5 + 104k^6 - 122k^7 + 58k^8 - 10k^9, \\
\varphi_3(k) &:= 6 - 5k - 3k^2 + 17k^3 - 3k^4 - 21k^5 - 3k^6 + 50k^7 - 42k^8 + 10k^9, \\
\varphi_4(k) &:= 3 + 3k - 6k^2 + 3k^3 + 18k^4 - 17k^5 - 2k^6 - 3k^7 + 13k^8 - 5k^9, \\
\varphi_5(k) &:= 1 + 2k - k^2 + k^4 + 10k^5 - 6k^6 - k^7 - k^8 + k^9, \\
\varphi_6(k) &:= k(1 - k^2 + k^3 + 2k^5).
\end{aligned}$$

It is elementary excise to check $\varphi_i(k) \geq 0$ for $0 \leq \forall k \leq 1$ ($i = 0,\, 1, \ldots,\, 6$). Thus we have $g(a, b, 1) \geq 0$ if $a \leq b$.

Case 2: We shall prove that if $0 \leq a < b \leq 1$ then $g \geq 0$.

Let $k := (1 - b)/(1 - a)$. Note that $0 \leq k \leq 1$. Using Mathematica, we obtain

$$g(1 - k(1 - b), b, 1) = (1 - b)^4 \sum_{i=0}^{6} \psi_i(k) b^i,$$

here,

$$\begin{aligned}
\psi_0(k) &:= (1 - k)(3 - 11k + 22k^2 - 21k^3 + 10k^4 - 2k^5), \\
\psi_1(k) &:= 6 - 12k + 2k^2 + 23k^3 - 18k^4 - 23k^5 + 43k^6 - 27k^7 + 8k^8 - k^9, \\
\psi_2(k) &:= 7 - 13k + 32k^2 - 68k^3 + 78k^4 - 6k^5 - 79k^6 + 79k^7 - 32k^8 + 5k^9, \\
\psi_3(k) &:= 6 - 9k + 9k^2 + 27k^3 - 83k^4 + 81k^5 + 11k^6 - 74k^7 + 48k^8 - 10k^9, \\
\psi_4(k) &:= 3 - 3k + 3k^2 + 3k^3 + 18k^4 - 47k^5 + 34k^6 + 18k^7 - 32k^8 + 10k^9, \\
\psi_5(k) &:= 1 - k - k^2 + 6k^3 - 2k^4 + 7k^5 - 12k^6 + 5k^7 + 8k^8 - 5k^9, \\
\psi_6(k) &:= k^4(2 + k^2 - k^3 + k^5).
\end{aligned}$$

We can check that if $0 \leq k \leq 1$ then $\psi_i(k) \geq 0$ ($i = 0, 1, \ldots, 6$). Thus we have $g(a, b, 1) \geq 0$ if $a > b$. □

Theorem 2.2. *Let*

$$F_{1,s}(a,b,c) = 3s^4 S_5 - (4s^5 - 1)S_{4,1} + (s^8 - 4s^3)S_{1,4} - (s^8 - 4s^5 + 3s^4 - 4s^3 + 1)US_2.$$

Then, $F_{1,s}(a,b,c) \geq 0$ holds for $a \geq 0$, $b \geq 0$, $c \geq 0$ and $s \geq 0$. Moreover, $F_{1,s}(0,s,1) = 0$ and $F_{1,s}(1,1,1) = 0$.

Proof. Let

$$G_s(a,b,c) := (s^8 - 4s^3)(S_{4,1} - US_2) + (1 - 4s^5)(S_{1,4} - US_2) + 3s^4(S_5 - US_2).$$

Since $s^8 F_{1,1/s}(a,\ b,\ c) = G_s(a,\ b,\ c)$, it is enough to show $G_s \geq 0$. Since $s^8 G_{1/s}(b,\ a,\ c) = G_s(a,\ b,\ c)$, we may assume $0 \leq s \leq 1$. since G_s is cyclic, we may assume $0 \leq a \leq 1$, $0 \leq b \leq 1$, and $c = 1$.

Case 1. We shall show that if $1 \geq a \geq b \geq 0$ then $G_s(a,b,1) \geq 0$.

Let $k := (1-a)/(1-b)$. Then $0 \leq k \leq 1$.

$$G_s(a,b,1) = G_s(a, 1 - k(1-a), 1) = (1-a)^2 \sum_{i=0}^{3} C_i(k,s) a^i,$$

here,

$$\begin{aligned}
C_0(k,s) &:= (1 - s + ks)^2(1 - k + 2s - 4ks + 2k^2 s + 3s^2 - 9ks^2 \\
&\qquad + 9k^2 s^2 - 3k^3 s^2 + 3s^4 + 2s^5 - 2ks^5 + s^6 - 2ks^6 + k^2 s^6), \\
C_1(k,s) &:= 1 - k + 3k^2 - 3k^3 + k^4 - 4s^3 + 12k^2 s^3 - 20k^3 s^3 + 8k^4 s^3 \\
&\qquad + 6s^4 - 3ks^4 - 9k^2 s^4 + 33k^3 s^4 - 30k^4 s^4 + 9k^5 s^4 \\
&\qquad - 4s^5 + 4ks^5 - 12k^2 s^5 + 12k^3 s^5 - 4k^4 s^5 \\
&\qquad + s^8 - 3k^2 s^8 + 5k^3 s^8 - 2k^4 s^8, \\
C_2(k,s) &:= 1 - k + 3k^3 - 2k^4 - 4s^3 + 4k^3 s^3 - 4k^4 s^3 \\
&\qquad + 6s^4 - 3ks^4 - 3k^3 s^4 + 15k^4 s^4 - 9k^5 s^4 \\
&\qquad - 4s^5 + 4ks^5 - 12k^3 s^5 + 8k^4 s^5 + s^8 - k^3 s^8 + k^4 s^8, \\
C_3(k,s) &:= (k - s)^2(k^2 + 2ks + 3s^2 + 3k^3 s^4 + 2k^2 s^5 + ks^6).
\end{aligned}$$

It is enough to show $C_i(k,\ s) \geq 0$ for $0 \leq k \leq 1$ and $0 \leq s \leq 1$. Clearly, $C_3(k,\ s) \geq 0$. Since

$$C_0(1-k, s) = (1 - ks)^2(k + 2k^2 s + 3k^3 s^2 + 3s^4 + 2ks^5 + k^2 s^6) \geq 0,$$

we have $C_0(k,\ s) \geq 0$. Let

$$A_i(k,s) := \sum_{j=0}^{i} C_j(k,s).$$

Since $C_3(k,\ s) \geq 0$, we have $A_2(k,\ s) \leq A_3(k,\ s)$. Since

$$\sum_{i=0}^{3} C_i(k,s) a^i = \sum_{i=0}^{2} A_i(k,s) a^i (1-a) + A_3(k,s) a^3,$$

if we prove $A_1(k,\ s) \geq 0$ and $A_2(k,s) \geq 0$ for $0 \leq k \leq 1$ and $0 \leq s \leq 1$, then we have $G_s(a,b,1) \geq 0$ for $1 \geq a \geq b \geq 0$.

Case 1-1. We shall show that if $k + s \leq 1$, $k \geq 0$ and $0 \geq s < 1$, then $A_1(k,\ s) \geq 0$.

Let $x := k/(1-s)$, then $0 \le x \le 1$.

$$\frac{1}{(1-s)^2} A_1(k,s) = \frac{1}{(1-s)^2} A_1((1-s)x, s) = \sum_{i=0}^{4} a_i(s)(1-x)x^i + a_5(s)x^5,$$

here,

$$\begin{aligned}
a_0(s) &:= 2 + 4s + 6s^2 + 6s^4 + 4s^5 + 2s^6, \\
a_1(s) &:= 2s(1 + 2s + 7s^2 + s^3 + 4s^4 + 3s^5 + 2s^6), \\
a_2(s) &:= 3 + 2s + 4s^2 + 2s^3 + 23s^4 - 4s^5 + 6s^6 + 4s^7 + 3s^8, \\
a_3(s) &:= s(5 + 4s - 2s^2 + 30s^3 + 5s^4 - 6s^5 + 4s^6 + 4s^7 - s^8), \\
a_4(s) &:= 1 + 3s + 5s^2 + 2s^3 + 7s^4 + 35s^5 - 13s^6 + 3s^8 + s^9 - s^{10}, \\
a_5(s) &:= 1 + 3s + 5s^2 + 2s^3 + 13s^4 + 17s^5 + 5s^6 - 6s^7 + 3s^8 + s^9 - s^{10}.
\end{aligned}$$

In general, a polynomial $h(s) = \sum_{i=0}^{n} \alpha_i s^i$ satisfies $\beta_0 \ge 0, \ldots, \beta_n \ge 0$ here $\beta_i := \sum_{j=0}^{i} \alpha_i$, then $h(s) \ge 0$ for all $0 \le s \le 1$. Thus we have $a_i(s) \ge 0$ for $0 \le s \le 1$ ($i = 0, \ldots, 5$). Therefore, if $k + s \le 1$, then $A_1(k,\, s) \ge 0$.

Case 1-2. We shall show that if $k + s > 1$, $k \le 1$ and $s \le 1$, then $A_1(k,\, s) \ge 0$.

Let $x := (1-s)/k$. Then $0 \le x \le 1$. Let $m := 1 - k$.

$$\frac{1}{k^2} A_1(k,s) = \frac{1}{(1-m)^2} A_1(1-m, 1-x+mx) = \sum_{i=0}^{9} b_i(x)(1-m)m^i + b_{10}(x)m^{10},$$

here,

$$\begin{aligned}
b_0(x) &:= 3 - 12x + 30x^2 - 48x^3 + 59x^4 - 52x^5 + 28x^6 - 8x^7 + x^8, \\
b_1(x) &:= 6 + 4x - 44x^2 + 104x^3 - 152x^4 + 164x^5 - 112x^6 + 40x^7 - 6x^8, \\
b_2(x) &:= 9 + 20x - 44x^2 + 93x^4 - 172x^5 + 168x^6 - 80x^7 + 15x^8, \\
b_3(x) &:= 3 + 40x + 26x^2 - 184x^3 + 227x^4 - 112x^5 - 28x^6 + 56x^7 - 17x^8, \\
b_4(x) &:= 3 + 16x + 60x^2 + 32x^3 - 302x^4 + 404x^5 - 252x^6 + 64x^7 - x^8, \\
b_5(x) &:= 3 + 16x + 24x^2 + 24x^3 + 69x^4 - 292x^5 + 336x^6 - 168x^7 + 29x^8, \\
b_6(x) &:= 3 + 16x + 24x^2 + 6x^4 + 60x^5 - 168x^6 + 144x^7 - 39x^8, \\
b_7(x) &:= 3 + 16x + 24x^2 + 28x^6 - 56x^7 + 25x^8, \\
b_8(x) &:= 3 + 16x + 24x^2 + 8x^7 - 8x^8, \\
b_9(x) &:= 3 + 16x + 24x^2 + x^8, \\
b_{10}(x) &:= 3 + 16x + 24x^2.
\end{aligned}$$

We can check that $b_i(x) \ge 0$ for $0 \le x \le 1$ ($i = 0, \ldots, 10$). Thus we have $A_1(k,\, s) \ge 0$ for all $0 \le k \le 1$ and $0 \le s \le 1$.

Case 1-3. We shall show that if $k + s \le 1$, $k \ge 0$ and $s \ge 0$, then $A_2(k,\, s) \ge 0$.

Let $x := k/(1-s)$. Then $0 \le x \le 1$.

$$\frac{1}{(1-s)^2} A_2(k,s) = \frac{1}{(1-s)^2} A_2((1-s)x, s) = \sum_{i=0}^{4} c_i(s)(1-x)x^i + c_5(s)x^i,$$

here,

$$
\begin{aligned}
c_0(s) &:= 3(1 + 2s + 3s^2 + 3s^4 + 2s^5 + s^6),\\
c_1(s) &:= s(3 + 6s + 13s^2 + s^3 + 10s^4 + 7s^5 + 4s^6),\\
c_2(s) &:= 3 + 3s + 6s^2 + s^3 + 22s^4 - 2s^5 + 7s^6 + 4s^7 + 3s^8,\\
c_3(s) &:= c_2(s),\\
c_4(s) &:= 2 + 5s + 5s^2 + s^3 + 22s^4 + 2s^5 - s^6 + 8s^7 + 3s^8,\\
c_5(s) &:= 2 + 5s + 5s^2 + s^3 + 19s^4 + 11s^5 - 10s^6 + 11s^7 + 3s^8.
\end{aligned}
$$

Similarly as Case 1-1, we have $c_i(s) \geq 0$ for $0 \leq s \leq 1$ ($i = 0,\dots, 5$). Thus $A_2(k,\ s) \geq 0$ if $k + s \leq 1$.

Case 1-4. We shall show that if $k + s > 1$, $k \leq 1$ and $s \leq 1$, then $A_2(k,\ s) \geq 0$.

Let $x = (1 - s)/k$ and $m = 1 - k$. Then

$$
\frac{1}{(1-m)^2} A_2(k, s) = \frac{1}{(1-m)^2} A_2(1 - m, 1 - x + mx) = \sum_{i=0}^{7} d_i(x)(1 - m)m^i + d_8(x)m^8,
$$

here,

$$
\begin{aligned}
d_0(x) &:= 3 - 12x + 42x^2 - 84x^3 + 115x^4 - 104x^5 + 56x^6 - 16x^7 + 2x^8,\\
d_1(x) &:= 6 + 12x - 66x^2 + 168x^3 - 281x^4 + 324x^5 - 224x^6 + 80x^7 - 12x^8,\\
d_2(x) &:= 24x + 48x^2 - 240x^3 + 427x^4 - 516x^5 + 420x^6 - 184x^7 + 33x^8,\\
d_3(x) &:= 3 - 4x + 68x^2 + 40x^3 - 311x^4 + 492x^5 - 476x^6 + 256x^7 - 55x^8,\\
d_4(x) &:= 3 + 8x + 18x^2 + 20x^3 + 55x^4 - 216x^5 + 308x^6 - 224x^7 + 60x^8,\\
d_5(x) &:= 3 + 8x + 36x^2 - 12x^3 - 2x^4 + 24x^5 - 84x^6 + 112x^7 - 42x^8,\\
d_6(x) &:= 3 + 8x + 36x^2 - 3x^4 - 4x^5 - 24x^7 + 17x^8,\\
d_7(x) &:= 3 + 8x + 36x^2 - 3x^8,\\
d_8(x) &:= 3 + 8x + 36x^2.
\end{aligned}
$$

We can check that $d_i(x) \geq 0$ for $0 \leq x \leq 1$ ($i = 0,\dots, 8$). Thus we have $A_2(k,\ s) \geq 0$, and we have proved $G_s(a, b, 1) \geq 0$ if $a \geq b$.

Case 2. We shall show that if $0 \leq a < b \leq 1$ then $G_s(a, b, 1) \geq 0$.

Let $k := (1 - b)/(1 - a)$. Note that $0 \leq k < 1$ and $0 \leq s \leq 1$.

$$
G_s(a, b, 1) = G_s(1 - k(1 - b), b, 1) = (1 - b)^2 \left(\sum_{i=0}^{2} B_i(k, s)b^i(1 - b) + B_3(k, s)b^3 \right),
$$

here,

$$
\begin{aligned}
B_0(1 - k, s) &= (k - s)^2(k^2 + 2ks + 3s^2 + 3k^3s^4 + 2k^2s^5 + ks^6) \geq 0,\\
B_1(k, s) &= 2 - 4k + 3k^2 + k^3 - k^4 - 8s^3 + 8ks^3 - 12k^2s^3 + 12k^3s^3 - 4k^4s^3\\
&\quad + 12s^4 - 18ks^4 + 21k^2s^4 + 3k^3s^4 - 15k^4s^4\\
&\quad + 6k^5s^4 - 8s^5 + 16ks^5 - 12k^2s^5 - 4k^3s^5 + 4k^4s^5\\
&\quad + 2s^8 - 2ks^8 + 3k^2s^8 - 3k^3s^8 + k^4s^8,\\
B_2(k, s) &= 3 - 4k + 3k^2 - 12s^3 + 12ks^3 - 12k^2s^3 + 4k^4s^3\\
&\quad + 18s^4 - 21ks^4 + 21k^2s^4 - 3k^5s^4 - 12s^5 + 16ks^5 - 12k^2s^5\\
&\quad + 3s^8 - 3ks^8 + 3k^2s^8 - k^4s^8,\\
B_3(k, s) - B_2(k, s) &= (1 - ks)^2(k + 2k^2s + 3k^3s^2 + 3s^4 + 2ks^5 + k^2s^6) \geq 0.
\end{aligned}
$$

Thus it is enough to show that $B_1(k,\ s) \geq 0$ and $B_2(k,\ s) \geq 0$ for all $0 \leq k < 1$ and $0 \leq s \leq 1$.

Case 2-1. We shall show that if $k + s \leq 1$, $k \geq 0$ and $s \geq 0$, then $B_1(k,\ s) \geq 0$.

Let $x := k/(1-s)$. Then $0 \leq x \leq 1$.

$$\frac{1}{(1-s)^2} B_1(k,s) = \frac{1}{(1-s)^2} B_1((1-s)x, s) = \sum_{i=0}^{10} e_i(x) s^i,$$

here,

$$\begin{aligned}
e_0(x) &:= 2 - 4x + 3x^2 + x^3 - x^4, \\
e_1(x) &:= 4 - 4x - x^3 + 2x^4, \\
e_2(x) &:= 6 - 4x - x^4, \\
e_3(x) &:= 4x - 12x^2 + 12x^3 - 4x^4, \\
e_4(x) &:= 6 - 14x + 21x^2 - 9x^3 - 7x^4 + 6x^5, \\
e_5(x) &:= 4 + 2x - 12x^2 - 7x^3 + 30x^4 - 18x^5, \\
e_6(x) &:= 2 + 2x + 4x^3 - 23x^4 + 18x^5, \\
e_7(x) &:= 2x + 4x^4 - 6x^5, \\
e_8(x) &:= 3x^2 - 3x^3 + x^4, \\
e_9(x) &:= 3x^3 - 2x^4, \\
e_{10}(x) &:= x^4.
\end{aligned}$$

If $i \neq 5$, $e_i(x) \geq 0$ for all $0 \leq x \leq 1$. $e_5(x)$ may be negative, but we can prove that $e_0(x) + e_5(x) \geq 0$ for $0 \leq x \leq 1$. Thus we have $B_1(k,\ s) \geq 0$ if $k + s \leq 1$.

Case 2-2. We shall show that if $k + s > 1$, $k \leq 1$ and $s \leq 1$, then $B_1(k,\ s) \geq 0$.

Let $x := (1-s)/k$ and $m = 1 - k$. Then

$$\begin{aligned}
\frac{1}{(1-m)^2} B_1(k,s) &= \frac{1}{(1-m)^2} B_1((1-m), 1-(1-m)x) \\
&= \sum_{i=0}^{9} f_i(x)(1-m)m^i + f_{10}(x)m^{10},
\end{aligned}$$

here,

$$\begin{aligned}
f_0(x) &:= 3 - 12x + 30x^2 - 48x^3 + 59x^4 - 52x^5 + 28x^6 - 8x^7 + x^8, \\
f_1(x) &:= 6 - 28x + 68x^2 - 80x^3 + 28x^4 + 44x^5 - 56x^6 + 24x^7 - 4x^8, \\
f_2(x) &:= 9 - 44x + 68x^2 - 87x^4 + 68x^5 - 16x^7 + 5x^8, \\
f_3(x) &:= 3 + 8x - 86x^2 + 184x^3 - 133x^4 + 8x^5 + 28x^6 - 8x^7 - x^8, \\
f_4(x) &:= 3 - 16x + 60x^2 - 152x^3 + 238x^4 - 196x^5 + 84x^6 - 16x^7 + x^8, \\
f_5(x) &:= 3 - 16x + 24x^2 + 24x^3 - 111x^4 + 188x^5 - 168x^6 + 72x^7 - 11x^8, \\
f_6(x) &:= 3 - 16x + 24x^2 + 6x^4 - 60x^5 + 112x^6 - 80x^7 + 19x^8, \\
f_7(x) &:= 3 - 16x + 24x^2 - 28x^6 + 40x^7 - 15x^8, \\
f_8(x) &:= 3 - 16x + 24x^2 - 8x^7 + 6x^8, \\
f_9(x) &:= 3 - 16x + 24x^2 - x^8, \\
f_{10}(x) &:= 3 - 16x + 24x^2.
\end{aligned}$$

We can check $f_i(x) \geq 0$ for $0 \leq x \leq 1$. Thus we have $B_1(k,\, s) \geq 0$.

Case 2-3. We shall show that if $k+s \leq 1$, $k \geq 0$ and $s \geq 0$, then $B_2(k,\, s) \geq 0$.

Let $x := k/(1-s)$. Then $0 \leq x \leq 1$.

$$\frac{1}{(1-s)^2}B_2(k,s) = \frac{1}{(1-s)^2}B_2((1-s)x,s) = \sum_{i=0}^{10} g_i(x)s^i,$$

here,

$$\begin{aligned}
g_0(x) &:= 3-4x+3x^2 \geq 0,\\
g_1(x) &:= 6-4x \geq 0,\\
g_2(x) &:= 9-4x \geq 0,\\
g_3(x) &:= 8x-12x^2+4x^4 \geq 0,\\
g_4(x) &:= 9-13x+21x^2-8x^4-3x^5 \geq 0,\\
g_5(x) &:= 6+3x-12x^2+4x^4+9x^5 \geq 0,\\
g_6(x) &:= 3+3x-9x^5,\\
g_7(x) &:= 3x+3x^5 \geq 0,\\
g_8(x) &:= 3x^2-x^4 \geq 0,\\
g_9(x) &:= 2x^4 \geq 0,\\
g_{10}(x) &:= -x^4.
\end{aligned}$$

$g_6(x)$ and $g_{10}(x)$ may be negative, but $g_5(x)+g_6(x) \geq 0$ and $g_9(x)+g_{10}(x) \geq 0$ for all $0 \leq x \leq 1$. Thus we have $B_2(k,\, s) \geq 0$ if $k+s \leq 1$.

Case 2-4. We shall show that if $k+s > 1$, $k \leq 1$ and $s \leq 1$, then $B_2(k,\, s) \geq 0$.

Let $x = (1-s)/k$ and $m = 1-k$.

$$\begin{aligned}
\frac{1}{(1-m)^2}B_2(k,s) &= \frac{1}{(1-m)^2}B_2(1-m, 1-x+mx)\\
&= \sum_{i=0}^{9} h_i(x)(1-m)m^i + h_{10}(x)m^{10},
\end{aligned}$$

here,

$$\begin{aligned}
h_0(x) &:= 3-12x+42x^2-84x^3+115x^4-104x^5+56x^6-16x^7+2x^8,\\
h_1(x) &:= 6-36x+102x^2-108x^3-11x^4+144x^5-140x^6+56x^7-9x^8,\\
h_2(x) &:= x(24-120x+312x^2-383x^3+204x^4-40x^6+12x^7),\\
h_3(x) &:= 3-20x+124x^2-328x^3+589x^4-648x^5+392x^6-112x^7+9x^8,\\
h_4(x) &:= 3-8x+18x^2+112x^3-395x^4+684x^5-644x^6+296x^7-51x^8,\\
h_5(x) &:= 3-8x+36x^2-12x^3+88x^4-336x^5+504x^6-328x^7+78x^8,\\
h_6(x) &:= 3-8x+36x^2-3x^4+56x^5-196x^6+200x^7-66x^8,\\
h_7(x) &:= 3-8x+36x^2+28x^6-64x^7+33x^8,\\
h_8(x) &:= 3-8x+36x^2+8x^7-9x^8,\\
h_9(x) &:= 3-8x+36x^2+x^8,\\
h_{10}(x) &:= 3-8x+36x^2.
\end{aligned}$$

We can check that $h_i(x) \geq 0$ for $0 \leq x \leq 1$ $(i = 0, \ldots, 10)$. Thus we have $B_2(k,\, s) \geq 0$, and complete the proof. $\square$

Theorem 2.3. *Let*

$$G_{2,s}(a,b,c) = 2S_{5,1} - 3sS_{4,2} + s^3S_{2,4} - 3(s-1)^2(s+2)U^2.$$

Then $G_{2,s}(a,b,c) \geq 0$ *for* $a \geq 0$, $b \geq 0$, $c \geq 0$ *and* $s \geq 0$. *Moreover,* $G_{2,s}(0,s,1) = 0$, $G_{2,s}(1,1,1) = 0$, *and* $G_{2,s}(0,0,1) = 0$.

Proof. We can check $G_{2,s}(0,s,1) = 0$ using Mathematica. Let

$$\begin{aligned}
&p := S_{5,1} - 3U^2, \quad q := S_{4,2} - 3U^2, \quad r := S_{2,4} - 3U^2,\\
&f(s) := s^3G_{2,1/s}(a,b,c) = 2s^3S_{5,1} - 3s^2S_{4,2} + S_{2,4} - 3(1-s)^2(1+2s)U^2\\
&\qquad = 2s^3p - 3s^2q + r.
\end{aligned}$$

Note that $p \geq 0$, $q \geq 0$ and $r \geq 0$. Since $f'(s) = 6s(sp - q)$, we have $f(s) \geq f(q/p) = (p^2r - q^3)/p^2$ if $s \geq 0$. Thus, it is enough to show that $p^2r - q^3 \geq 0$. Let

$$\begin{aligned}
g(a,b,c) &:= (p^2r - q^3)/U\\
&= 2S_{6,9} + US_{12} - 3US_{10,2} + 9US_{8,4} - 6US_{7,5} - 3US_{6,6}\\
&\quad + 2U^2S_{6,3} - 6U^2S_{4,5} - 6U^2S_{1,8} - 2U^3S_6 + 18U^3S_{5,1}\\
&\quad - 27U^3S_{4,2} + 27U^3S_{2,4} + 2U^4S_3 - 6U^4S_{1,2} - 6U^5.
\end{aligned}$$

Since g is cyclic, we may assume $0 \leq a \leq 1$, $0 \leq b \leq 1$ and $c = 1$.

Case 1: We shall prove that if $1 \geq a \geq b \geq 0$ then $g(a,b,1) \geq 0$.

Let $k := (1-a)/(1-b)$. Note that $0 \leq k \leq 1$.

$$g(1-k(1-b),b,1) = (1-b)^6 \sum_{i=0}^{9} \varphi_i(k)b^i,$$

here,

$$\begin{aligned}
\varphi_0(k) &:= 2(1-k)^9,\\
\varphi_1(k) &:= (1-k)\big(11 - 72k + 222k^2 - 430k^3 + 639k^4 - 810k^5 + 879k^6 - 762k^7\\
&\qquad + 489k^8 - 220k^9 + 66k^{10} - 12k^{11} + k^{12}\big),\\
\varphi_2(k) &:= 26 - 143k + 368k^2 - 542k^3 + 426k^4 + 187k^5 - 1302k^6 + 2439k^7\\
&\qquad - 2826k^8 + 2199k^9 - 1150k^{10} + 390k^{11} - 78k^{12} + 7k^{13},\\
\varphi_3(k) &:= 36 - 122k + 152k^2 + 80k^3 - 631k^4 + 1285k^5 - 1365k^6 + 309k^7\\
&\qquad + 1314k^8 - 2156k^9 + 1707k^{10} - 777k^{11} + 195k^{12} - 21k^{13},\\
\varphi_4(k) &:= 32 - 50k - 28k^2 + 248k^3 - 446k^4 + 271k^5 + 512k^6 - 1181k^7\\
&\qquad + 774k^8 + 394k^9 - 1048k^{10} + 765k^{11} - 260k^{12} + 35k^{13},\\
\varphi_5(k) &:= 18 + 2k - 48k^2 + 96k^3 + 32k^4 - 402k^5 + 623k^6 - 167k^7\\
&\qquad - 466k^8 + 432k^9 + 112k^{10} - 360k^{11} + 195k^{12} - 35k^{13}.\\
\varphi_6(k) &:= 6 + 12k - 18k^2 + 20k^3 + 60k^4 - 60k^5 - 168k^6 + 361k^7\\
&\qquad - 54k^8 - 140k^9 + 126k^{10} + 48k^{11} - 78k^{12} + 21k^{13},\\
\varphi_7(k) &:= 1 + 5k - 6k^3 + 30k^4 + 12k^5 - 27k^6 - 45k^7\\
&\qquad + 95k^8 - 27k^9 - 33k^{10} + 15k^{11} + 13k^{12} - 7k^{13},\\
\varphi_8(k) &:= k(1 + 12k^4 - 3k^6 - 6k^7 + 9k^8 - 3k^{10} + k^{12}),\\
\varphi_9(k) &:= 2k^6.
\end{aligned}$$

We can check that if $0 \le k \le 1$ then $\varphi_i(k) \ge 0$ ($i = 0, 1, \ldots, 9$). Thus we have $g(a, b, 1) \ge 0$ if $a \ge b$.

Case 2: We shall prove that if $0 \le a < b \le 1$ then $g \ge 0$.

Let $k := (1-b)/(1-a)$. Then $0 \le k \le 1$.

$$q(a, 1-k(1-a), 1) = (1-a)^6 \sum_{i=0}^{9} \psi_i(k) a^i,$$

here,

$$\begin{aligned}
\psi_0(k) &:= 2(1-k)^6, \\
\psi_1(k) &:= (1-k)\big(11 - 42k + 84k^2 - 142k^3 + 252k^4 - 396k^5 + 501k^6 - 498k^7\big), \\
&\qquad + 369k^8 - 190k^9 + 63k^{10} - 12k^{11} + k^{12}, \\
\psi_2(k) &:= 26 - 107k + 272k^2 - 544k^3 + 808k^4 - 724k^5 + 50k^6 + 983k^7 \\
&\qquad - 1732k^8 + 1677k^9 - 1012k^{10} + 375k^{11} - 78k^{12} + 7k^{13}, \\
\psi_3(k) &:= 36 - 128k + 311k^2 - 437k^3 + 160k^4 + 572k^5 - 1113k^6 + 691k^7 \\
&\qquad + 593k^8 - 1615k^9 + 1512k^{10} - 750k^{11} + 195k^{12} - 21k^{13}, \\
\psi_4(k) &:= 32 - 86k + 134k^2 + 59k^3 - 446k^4 + 536k^5 + 14k^6 - 677k^7 \\
&\qquad + 542k^8 + 373k^9 - 988k^{10} + 750k^{11} - 260k^{12} + 35k^{13}, \\
\psi_5(k) &:= 18 - 22k - 30k^2 + 206k^3 - 227k^4 + 11k^5 + 183k^6 - 25k^7 \\
&\qquad - 270k^8 + 186k^9 + 217k^{10} - 375k^{11} + 195k^{12} - 35k^{13}, \\
\psi_6(k) &:= 6 + 6k - 42k^2 + 70k^3 + 28k^4 - 115k^5 + 126k^6 - 39k^7 \\
&\qquad + 18k^8 - 64k^9 + 24k^{10} + 75k^{11} - 78k^{12} + 21k^{13}, \\
\psi_7(k) &:= 1 + 5k - 9k^2 - 9k^3 + 45k^4 - 25k^5 - 21k^6 + 69k^7 \\
&\qquad - 36k^8 + 6k^9 - 6k^{10} + 13k^{12} - 7k^{13}, \\
\psi_8(k) &:= k\big(1 - 3k^2 + 9k^4 - 6k^5 - 3k^6 + 18k^7 - 6k^8 + k^{12}\big), \\
\psi_9(k) &:= 2k^9.
\end{aligned}$$

We can check that if $0 \le k \le 1$ then $\psi_i(k) \ge 0$ ($i = 0, 1, \ldots, 9$). Thus we have $g(a, b, 1) \ge 0$ if $a < b$. $\square$

References

[1] T. Ando, Some homogeneous cyclic inequalities of three variables of degree three and four, Aust. J. Math. Anal. Appl. **7** (2) Art. 12 (2010)

[2] T. Ando, Cubic and quartic cyclic homogeneous inequalities of Three Variables, Math. Inequal. Appl., **16**, 127-142 (2013)

[3] T. Ando, Discriminants of Cyclic Homogeneous Inequalities of Three Variables, J. Alg., **514**, 383-441 (2018).

[4] G. Blekherman & P. A. Parrilo & R. Thomas, ed., Semidefinite Optimization and Convex Algebraic Geometry, Cambridge University Press, 2013.

[5] M.D. Choi & T.Y. Lam, Extremal Positive Semidefinite Forms, Math. Ann., **231**, 1-18 (1977)

[6] M.D. Choi & T.Y. Lam & B. Reznik, Even Symmetric Sextics, Math. Z., **195**, 559-580 (1987)

[7] V. Cîrtoaje, On the cyclic homogeneous polynomial inequalities of degree four, J. Inequal. Pure and Appl. Math., **10**(3) Art. 67 (2009)

[8] V. Cîrtoaje, Necessary and sufficient conditions for symmetric homogeneous polynomial inequalities of degree four and six in real variables, J. Nonlinear Sci. Appl., **5**, 307-320 (2012)

[9] V. Cîrtoaje, Necessary and sufficient conditions for cyclic homogeneous polynomial inequalities of degree four in real variables, Aust. J. Math. Anal. Appl. **9** (1) (2012)

[10] V. Cîrtoaje & Y. Zhou, Some strong sufficient conditions for cyclic homogeneous polynomial inequalities of degree four in real variables, Nonlinear Ann. Appl., jnaa-00151 (2012)

[11] V. Cîrtoaje, Necessary and Sufficient Conditions for Symmetric Homogeneous Polynomial Inequalities in Nonnegative Real Variables, Math. Ineq. Appl., **16**, 413-438 (2013).

[12] V. Cîrtoaje, A Strong Method for Symmetric Homogeneous Polynomial Inequalities of Degree Six in Nonnegative Real Variables, Brit. J. Math. & Comp. Sci., **4**, 685-703 (2014)

[13] J. Nie, Discriminant and nonnegative polynimials, J. Symbolic Comput, **47**, 167-191 (2012).

[14] B. Reznick, Forms Derived from Arithmetic-Geometric Inequality, Math. Ann., **238** 431-464 (1989)