# Semialgebraic Variety

Tetsuya Ando

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**Abstract.**
In this article, we give basic concepts of semialgebraic varieties which are improvement of real algebraic varieties. A semialgebraic variety is a locally ringed space whose topological space is a certain semialgebraic subset of a real algebraic variety and whose structure sheaf is defined similarly as the case of real algebraic varieties. We show that the category of semialgebraic varieties has more natural properties than the category of real algebraic varieties in the sense of Bochnak, Coste and Roy, which is not $\mathbb{R}$-scheme. For example, images of semialgebraic varieties by regular maps are also semialgebraic varieties. We also study relations with semialgebraic varieties and complex algebraic varieties. Some properties are similar, but some are completely different. For example, all semialgebraic varieties are affine, and higher cohomologies of their quasi-coherent sheaves vanish. The real projective surface $\mathbb{P}^2_{\mathbb{R}}$ is not minimal as semialgebraic varieties, but semialgebraic surfaces with non-negative Kodaira dimensions have minimal models.

## Section 0. Introduction.

For example, as is well known, $\mathbb{P}^n_{\mathbb{C}}/\mathfrak{S}_{n+1} \cong \mathbb{P}_{\mathbb{C}}(1,2,3,\ldots,n+1)$, here the right hand side is a weighted projective space. But if $n \geq 2$, the set $\mathbb{P}^n_{\mathbb{R}}/\mathfrak{S}_{n+1}$ is a proper closed semialgebraic subset of real weighted projective space $\mathbb{P}_{\mathbb{R}}(1,2,3,\ldots,n+1)$, and is not a real algebraic variety. We want to extend a theory of real algebraic varieties so that $\mathbb{P}^n_{\mathbb{R}}/\mathfrak{S}_{n+1}$ and many other useful semialgebraic sets can be treated as certain kind of generalized abstract algebraic variety with a coherent structure sheaf. Otherwise, we can't study even the singularities of $\mathbb{P}^n_{\mathbb{R}}/\mathfrak{S}_{n+1}$. One of such ideas is theory of semialgebraic varieties.

An algebraic variety $X$ over $\mathbb{R}$ often means a $\mathbb{R}$-scheme which can be identified with a self-conjugate complex algebraic variety. On the set of $\mathbb{R}$-valued points $V := X(\mathbb{R})$, we can define a sheaf of rings $\mathcal{R}_V$ such that every maximal ideal of $\mathcal{R}_V$ corresponds to a point of $V$, and that $\mathcal{R}_{V,P} = \mathcal{O}_{X,P}(\mathbb{R})$ for all $P \in V$. This locally ringed space $(V,\, \mathcal{R}_V)$ is called a real algebraic variety (see §1 and [4]). Note that for a given real algebraic variety $(V,\, \mathcal{R}_V)$, there exists infinitely many algebraic varieties $X$ over $\mathbb{R}$ such that $X(\mathbb{R}) = V$, if $\dim V \geq 2$. So, the notion of a $\mathbb{R}$-scheme $X$ is not always useful to study the set $V$ itself.

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T. Ando
Department of Mathematics and Informatics, Chiba University,
Yayoi-cho 1-33, Inage-ku, Chiba 263-8522, JAPAN
e-mail ando@math.s.chiba-u.ac.jp
Phone: +81-43-290-3675, Fax: +81-43-290-2828
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In this article, algebraic curves or algebraic surfaces always mean real algebraic varieties with $\mathcal{R}_V$, not $\mathbb{R}$-schemes. Please do not confuse with many results in other articles in which real algebraic varieties are $\mathbb{R}$-schemes.

The word 'semialgebraic variety' was used in many articles without an exact definition. Most of them are semialgebraic sets of $\mathbb{R}^n$ with $C^\infty$-manifold structure which have boundaries. In this article, we define a semialgebraic variety $A$ as an irreducible reduced semialgebraic subset of a real algebraic variety $V$ with a structure sheaf $\mathcal{R}_A$ such that every maximal ideal of $\mathcal{R}_A$ corresponds to a point of $A$. If $\dim A = \dim V$, then $\mathcal{R}_{A,P} = \mathcal{R}_{V,P}$ for all $P \in A$. Of course, a real algebraic variety is a semialgebraic variety. Note that for a given semialgebraic variety $A$, there may exist infinitely many complete real algebraic varieties $V$ such that $A \subset V$ and $\dim A = \dim V$, if $\dim A \geq 2$.

As the notions of 'algebraic variety over $\mathbb{R}$' and 'real algebraic varieties' are completely different, the notions of 'semialgebraic sets' and 'semialgebraic varieties' are different. In the category of semialgebraic sets, a morphism is a semialgebraic map. On the other hand, in the category of semialgebraic varieties, a morphism is a regular map.

There are some merits, to consider semialgebraic varieties. We shall explain an example. It is well known that $\mathbb{P}^n_{\mathbb{R}}$ is affine as a real algebraic variety. Similarly, we can prove that any semialgebraic variety $A$ is affine as a semialgebraic variety (Theorem 1.12). This implies that $H^n(A, \mathcal{F}) = 0$ for any $n \geq 1$ and for any quasi-coherent $\mathcal{R}_A$-module $\mathcal{F}$ (Proposition 4.1). As a collorary, for any real algebraic variety $(V. \mathcal{R}_V)$, we conclude that the structure sheaf $\mathcal{R}_V$ is the sheafification of the coordinate ring $\mathcal{R}_V(V)$.

Originally, the notion of semialgebraic varieties are introduced in [1], to study PSD (=Positive SemiDefinite) cones on $\mathbb{P}^n_{\mathbb{R}}$ or $\mathbb{P}^n_+$ (see also [2]). By virtue of notion of semialgebraic varieties, we could apply theorems and ideas of algebraic geometry to theory of algebraic inequalities. In this sense, the notion of semialgebraic varieties already provided useful results.

On the other hand, we also aware the notion of semialgebraic varieties is also useful for studies of real algebraic varieties. As is well known, an image of a real algebraic variety by a regular map is not always a real algebraic variety. But it is a semialgebraic variety. When a finite group $G$ acts on a real algebraic variety $V$, the quotient $V/G$ is not always a real algebraic variety, but $V/G$ is a semialgebraic variety (see Proposition 2.11). As is mentioned in the above, any real semialgebraic variety is an affine semialgebraic variety, though it is not always an affine real algebraic variety.

In some special cases, complex algebraic geometry is useful to study semialgebraic varieties. For example, consider non-singular semialgebraic surfaces $A$ whose Kodaira dimension $\kappa(A)$ is non-negative. Then, there exists a unique non-singular complete self-conjugate complex algebraic surface $X \supset A$ which is 'relatively minimal'. Using this $X$, we can define a sheaf $\mathcal{O}_A = \mathcal{O}_X|_A$, $\mathcal{O}_A$-modules, divisors, and the intersection number on $A$. $H^i(A, \mathcal{F}) \otimes_{\mathbb{R}} \mathbb{C} \cong H^i(X, \mathcal{F} \otimes_{\mathcal{O}_A} \mathcal{O}_X)$ holds for a coherent $\mathcal{O}_A$ module $\mathcal{F}$. We can also define Weil divisors and their intersection numbers in this case.

We didn't yet find so many applications of theory of semialgebraic varieties, but the notion of semialgebraic varieties must be useful for study of real algebraic geometry. Note that the category of semialgebraic varieties has more natural properties than the category of real algebraic varieties.

**Section 1. Basic definitions.**

A standard definition of real algebraic varieties is given in [4]. Real algebraic varieties defined in [4]are reduced but may not be irreducible. Moreover, the separability condition is omitted as is commented in [4]. We need nore exact definition. Here we give a generalized definition of real algebraic varieties including non-reduced case. In this article, $\mathbb{R}$ is any real closed field and $\mathbb{C} = \mathbb{R}[T]/(T^2+1)$. $\mathbb{R}$ or its elements are called 'real', and $\mathbb{C}$ or its elements are called 'complex'. A complex algebraic variety is an integral separated scheme of finite type over $\mathrm{Spec}(\mathbb{C})$, and a *complex algebraic quasi-variety* is a separated scheme of finite type over $\mathrm{Spec}(\mathbb{C})$ which is possibly reducible or non-reduced.

**Definition 1.1.** (Real algebraic quasi-variety) (I) A locally ringed space $(X,\, \mathcal{R}_X)$ is called a *real algebraic quasi-variety*, if there exists a separated scheme $(Y,\, \mathcal{O}_Y)$ of finite type over $\mathrm{Spec}\,\mathbb{R}$ which satisfies the following:

(1) There exists an injective morphism $\iota\colon (X,\, \mathcal{R}_X) \longrightarrow (Y,\, \mathcal{O}_Y)$ as locally ringed spaces, and $\iota$ induces a homeomorphism $X \to Y(\mathbb{R})$ as toplogical spaces.

(2) Take any affine open subset $V \subset Y$. Let $\mathfrak{n}_P$ be the maximal ideal of $\mathcal{O}_Y(V)$ corresponding to a closed point $P \in Y$. For an arbitral non-empty subset $U \subset V \cap \iota(X)$, we put

$$S_U := \bigcap_{P \in U} \big(\mathcal{O}_Y(V) - \mathfrak{n}_P\big).$$

Then, $\iota^* : S_U^{-1}\mathcal{O}_Y(V) \longrightarrow \mathcal{R}_X(\iota^{-1}(U))$ is an isomorphism of $\mathbb{R}$-algebra. Thus, each maximal ideal $\mathfrak{m} \subset \mathcal{R}_X(\iota^{-1}(V))$ corresponds to a pount $P \in \iota^{-1}(V) \subset X$.

(3) Take an arbitral affine open subset $V \subset Y$. Then

$$\big\{f \in \mathcal{O}_Y(V) \;\big|\; f(P) = 0 \text{ for all } P \in V(\mathbb{R})\big\}$$

is a nilpotent ideal of $\mathcal{O}_Y(V)$.

In this case, $Y$ is sayed to be a $\mathbb{R}$-*scheme represents* $X$.

(II) Let $X$ and $Y$ be the same as the above. $U \subset X$ is called an *affine open subset* of $X$, if there exists a affine open subset $U_Y \subset Y$ such that $U = U_Y(\mathbb{R})$. Zariski open (resp. closed) subsets are defied similarly. The *Eucledean topology* of $X$ is the topology induded from the analytic topology of $Y_{\mathbb{C}}$. $Y(\mathbb{R})$ is also denoted as $Y_{\mathbb{C}}(\mathbb{R})$. When $V \subset Y$ is an affine open subset and $B \subset V(\mathbb{R})$ is any subset, we put

$$S_B := \bigcap_{P \in B} \big(\mathcal{O}_Y(V) - \mathfrak{n}_P\big),$$

and $\mathcal{R}_X(\iota^{-1}(B)) := \iota^*\big(S_B^{-1}\mathcal{O}_Y(V)\big)$. By this definition, $(X,\, \mathcal{R}_X)$ can be also regarded as a locally ringed space with respect to the Euclidean topology.

A *morphism* between real algebraic quasi-varieties is defined as a morphism of locally ringed spaces. A morphism is also called a *holomorphic map* or a *regular map.* A *rational map* and so on between real algebraic quasi-varieties are defined similarly with complex schemes.

If $X$ is irreducible and reduced, $X$ is called a *real algebraic variety.*

(III) We regard $\mathbb{R}^n$ and $\mathbb{P}^n_{\mathbb{R}}$ to be real algebraic varieties as natural way. Let $X$ be a real algebraic quasi-variety. If there exists $n \in \mathbb{N}$ and a closed immersion $\varphi\colon X \to \mathbb{R}^n$, then $X$ is sayed to be an *affine.*

**Note 1.2.** (1) $Y$ is not unique for a given $X$.

(2) Let $R = \mathbb{R}[X_1,\dots, X_n]/(f_1^2 + \cdots + f_n^2)$ and $Y := \operatorname{Spec} R$ where $f_1,\dots, f_n$ are algebraically independent polynomials with $n \geq 2$. Then $X := Y(\mathbb{R})$ and $Y$ does not satisfy the above definition.

(3) In (3) of (I), $S_U^{-1}\mathfrak{O}_Y(V)$ is a Noetherian ring. Thus, $\mathfrak{R}_X(\iota^{-1}(U))$ is a Noetherian ring.

(4) It is well known that there exists an immersion $\iota\colon \mathbb{P}^n_{\mathbb{R}} \to \mathbb{R}^{2n+1}$ as real algebraic varieties. Thus every projective real algebraic quasi-variety is affine.

(5) Hironake gave a complete complex non-sigular algebraic variety $Z$ which is not projective. But $Z(\mathbb{R})$ is affine. The auther knows no real complete algebraic variety which is not affine.

**Definition 1.3.**(Semialgebraic quasi-variety) A locally ringed space $(A, \mathfrak{R}_A)$ is called *semialgebraic quasi-variety*, if there exists a real algebraic quase-variety $(X, \mathfrak{R}_X)$ and a finite affine open covering $\{V_i\}_{i=1}^r$ of $X$ which satisfies the following:

(1) There exists an injective morphism $\iota\colon (A, \mathfrak{R}_A) \longrightarrow (X, \mathfrak{R}_X)$ as locally ringed spaces, and $\iota$ induces a homeomorphism $A \to \iota(A)$ as toplogical spaces. Moreover, $\iota(A)$ is a semialgebraic subset of $X$, i.e. $\iota(A) \cap V_i$ is a semialgebraic suvset of $V_i$ for each $i = 1,\dots, r$.

(2) $\operatorname{Zar}_X(A) = X$. Here $\operatorname{Zar}_X(A)$ is the minimal closed Zariski subset of $X$ which contains $A$, and is called the Zariski closure of $A$.

(3) Take an arbitral $i \in \{1, 2,\dots, r\}$, and take any Euclidian open subset $U \subset \iota^{-1}(V_i)$. Put $R_i := \mathfrak{R}_{V_i}(V_i)$. For a point $P \in \iota(U)$, let $\mathfrak{m}_P$ be the maximal ideal of $R_i$ corresponding to $P$, and let
$$S_U := \bigcap_{P\in U} \big(R_i - \mathfrak{m}_P\big) \subset R_i.$$
Then $\iota^* : S_U^{-1}R_i \longrightarrow \mathfrak{R}_A(U)$ is an isomorphism of $\mathbb{R}$-algebra.

Moreover, if $X$ is a real algebraic variety, then $A$ is said to be an *semialgebraic variety.* In this case, the field of fractions $Q(\mathfrak{R}_A(U_i))$ is called the *field of rational functions*, and is denoted by $\operatorname{Rat}(A) := Q\big(\mathfrak{R}_A(U_i)\big)$.

The Zariski topology and the Euclidean topology on $A$ are defined naturally. A semialgebraic quasi-variety $A$ is called *irreducible* if it is irreducible with respect to the Zariski topology. $A$ is said to be *reduced* if $\mathfrak{R}_{A,P}$ has no nilpotent elements except 0 for each $P \in A$. $\dim A$ is defined by $\dim A = \max_{P\in A} \operatorname{Krull}\dim \mathfrak{R}_{A,P}$. $A$ is called *connected* if it is connected with respect to Eucledean topology. Note that $A$ may not be connected even if $A$ is irreducible. $A$ is called *affine*, if we can choose $X$ to be affine.

A semialgebraic variety $A$ is called *normal*, if $\mathfrak{R}_{A,P}$ is an integrally closed for each $P \in A$. A semialgebraic variety $A$ is called *non-singular*, if $\mathfrak{R}_{A,P}$ is a regular local ring for each $P \in A$.

A *regular map* or *holomorphic map* (resp. *isomorphism*) between semialgebraic quasi-varieties is defined as a morphism (resp. *isomorphism*) of locally ringed space.

For a subset $B \subset A$, the minimum Zariski closed subset of $A$ which includes $B$ is called the *Zariski closure* of $B$ in $A$ and is denoted by $\operatorname{Zar}_A(B)$ or $\operatorname{Zar}(B)$.

We can choose $X$ to be complete. Then the interior of $B$ with respect to the Euclidian topology of $X$ is denoted as $\operatorname{Int}(B)$. $\partial B := B - \operatorname{Int}(B)$ is called the *absolute boundary* of $B$ (see also Definition 2.7). Note that $\operatorname{Int}(B)$ and $\partial B$ do not depend on the choice of a complete $X$ (see Proposition 2.3). $A$ is called *open* if $\partial A = \emptyset$. $A$ is called *closed* if $A$ is compact with respect to the Eucledean topology.

**Definition 1.4.**(Image of a regular map) Let $A$, $B$ be semialgebraic quasi-varieties, and $\varphi\colon A \to B$ be a regular map. Let $C := \varphi(B)$. By Tarski-Seidenberg theorem, $C$ is a semialgebraic subset of $B$. We define $\mathfrak{R}_C$ as the following:

We may assume $A$ and $B$ are affine, since definition of $\mathfrak{R}_C$ is local. Let $R_A := \mathfrak{R}_A(A)$, $R_B := \mathfrak{R}_B(B)$, and $\varphi^*\colon R_B \to R_A$ be the homomorphism induced by $\varphi$. We put $R := R_B/\operatorname{Ker}\varphi^*$. (Note that $R$ defines $\operatorname{Zar}_B(C)$.) For a point $P \in C$, there exists the unique maximal ideal $\mathfrak{m}_P \subset R$ corresponding to $P$. Put $S := \bigcap_{P\in C}(R - \mathfrak{m}_P)$, and $R_C := S^{-1}R$. Note that $R_C$ is a $R_B$-module. The structure sheaf of $C$ is defined by $\mathfrak{R}_C := \widetilde{R_C}$ which is the coherent $\mathfrak{R}_B$-module defined by $R_C$.

$(C, \mathfrak{R}_C)$ is called the *image* of $\varphi$, and simply denoted by $C = \varphi(A)$.

**Definition 1.5.**(Semialgebraic quasi-subvariety) Let $A$, $B$ be semialgebraic quasi-varieties. A morphism $\varphi : (B, \mathfrak{R}_B) \longrightarrow (A, \mathfrak{R}_A)$ is called an *immersion*, if $\varphi$ induces an isomorphism $B \longrightarrow \varphi(B)$.

If $B$ is a semialgebraic subset of $A$, and the inclusion map $B \to A$ is an immersion, then $B$ is called a semialgebraic *quasi-subvariety* of $A$.

**Definition 1.6.**(Fibre product) Let $A$, $B$, $C$ be semialgebraic quasi-varieties, and $f\colon A \to C$, $g\colon B \to C$ be regular maps. The *fiber product* $A \times_C B$ is a semialgebraic set

$$A \times_C B = \{(a, b) \in A \times B \mid f(a) = g(b)\}$$

with a structure sheaf $\mathfrak{R}_A \otimes_{\mathfrak{R}_C} \mathfrak{R}_B$.

**Definition 1.7.**(Inverse image) Let $A$, $B$ be semialgebraic quasi-varieties, and $\varphi\colon A \to B$ be a regular map. Let $C \subset B$ be a semialgebraic quasi-subvariety. The *inverse image* $\varphi^{-1}(C)$ is defined as the fiber product $\varphi^{-1}(C) := A \times_B C$.

**Definition 1.8.**(Birational map) Let $A$, $B$ be semialgebraic quasi-varieties. If there exists dense Zariski open subsets $U \subset A$ and $W \subset B$, and there exists a regular map $\varphi\colon U \to W$, then we say that there exists a *rational map* $\varphi\colon A \cdots \to B$. Moreover, if $\varphi\colon U \to W$ is an isomorphism, we say that $\varphi\colon A \cdots \to B$ is a *birational map*, and $A$ and $B$ are *birational equivalent*.

**Definition 1.9.** Let $A$ be a semialgebraic quasi-variety. A point $P \in A$ is said to be a *non-singular point* of $A$ if $\mathfrak{R}_{A,P}$ is a regular local ring. We denote

$$\begin{aligned}\operatorname{Sing}(A) &:= \{P \in A \mid P \text{ is a singular point of } A.\},\\ \operatorname{Reg}(A) &:= \operatorname{Int}(A) - \operatorname{Sing}(A).\end{aligned}$$

**Remark 1.10.** (1) $\operatorname{Reg}(A) \neq \emptyset$ if $A$ is reduced.

(2) $\operatorname{Reg}(A)$ is not always dense in $A$ with respect to the Euclidean topology. For example, consider the case that $A$ has an isolated singularity as a connected component.

(3) If $P \in \operatorname{Reg}(A) \cap \operatorname{Int}(A)$ and $\dim A = n$, then there exists an Euclidean open neighborhood $P \in U \subset A$ such that $U$ is homeomorphic to an open subset of $\mathbb{R}^n$.

(4) By our definition, an isolated singular locus of $A$ is included in $\operatorname{Int}(A)$. If you want to exclude such points from $\operatorname{Int}(A)$, you have better to discuss $\operatorname{Int}(A) \cap \operatorname{Reg}(A)$. $\operatorname{Sing}(A)$ sometimes acts as if it is a boundary.

**Definition 1.11.** Let $A$ be a non-singular semialgebraic variety with $\dim A = n$, and $P \in A$. Then, there exists an Euclidean open set $P \in U \subset A$ such that $U$ has an algebraic coordinate system $(x_1,\ldots, x_n)$. $n$-formes $f_1\ dx_1 + \cdots + f_n\ dx_n$ ($f_k \in \mathcal{R}_A(U)$) determine a locally free $\mathcal{R}_A$-module $\Omega_A^1$. Wedge this $n$ times, we obtain $\mathcal{R}_A$-module $\Omega_A^n$. This can be identified with the real line bundle $\det T_A^\vee$.

Note that $\mathcal{R}_A(U)$ can be defined for any (may bo non-open) subset $U \subset A$ as a localization.

**Theorem 1.12.** Every semialgebraic quasi-variety is affine.

*Proof.* Let $A$ be a semialgebraic quasi-variety. There exists a real algebraic quasi-variety $X \supset A$ as Definition 1.3. Take an affine opne covering $\{V_1,\ldots, V_r\}$ of $X$. Fix a $1 \leq j \leq r$. We may assume $V_j$ is a closed subset of $\mathbb{R}^n$. Let $(x_1,\ldots, x_n)$ be the coordinate system of $\mathbb{R}^n$, and $s_i := 1/(x_i^2+1)$, $t_i := x_i/(x_i^2+1)$. For $P \in X - V_j$, we put $s_i(P) = 0$ and $t_i(P) = 0$. Then $s_i$ and $t_i$ are regular functions on $X$. The set of functions $F_j := \{s_i,\ t_i \mid 1 \leq i \leq n\}$ defines a map $\Phi_j \colon X \longrightarrow \mathbb{R}^{2n}$. This $\Phi_j$ is a regular map as semialgebraic quasi-varieties, and $\Phi_j|_{V_j} \colon V_j \longrightarrow \mathbb{R}^{2n}$ is an immersion. Note that $\Phi_j(X)$ is a semialgebraic quasi-variety but is not always algebraic quasi-variety. Put $F := F_1 \cup \cdots \cup F_r$ and $N := \#F$. $F$ defines a regular map $\Phi \colon X \to \mathbb{R}^N$, and $F$ is an immersion as semialgebraic quasi-varieties. □

**Remark 1.13.** (1) The above proof does not prove that a real algebraic quasi-variety is a real affine quasi-variety, because $\mathrm{Zar}_{\mathbb{R}^N}(\Phi(X)) \neq \Phi(X)$.

(2) The structure sheaf $\mathcal{R}_A$ can be obttained from the coordinate ring $\mathcal{R}_A(A)$ as it's sheafification.

In a complete complex algebraic varieties, exceptional subsets are special subsets. This is not true for complete semialgebraic varieties. (See also Proposition 6.2.)

**Theorem 1.14.** *Let $A$ be a semialgebraic quasi-variety, $E \subset A$ be a closed semialgebraic subset such that $E = \mathrm{Zar}_A(E) \subsetneqq A$. Then there exists a semialgebraic quasi-variety $B$ and a regurar surjective morphism $\varphi \colon A \to B$ such that $P := \varphi(E)$ is a point and that $\varphi|_{A-E} : (A - E) \longrightarrow (B - P)$ is an isomorphism, e.g. $\varphi$ is a contraction of $E$ to a point $P$.*

*Proof.* We may assume $A \subset \mathbb{R}^n$. Let $f_1,\ldots, f_r$ be defining polynomials of $\mathrm{Zar}_{\mathbb{R}^n}(E)$ in $\mathbb{R}[x_1,\ldots, x_n]$. Consider a map $\Phi \colon \mathbb{R}^n \to \mathbb{R}^{rn}$ defined by linear system with the base $\{x_i f_j \mid 1 \leq i \leq n,\ 1 \leq j \leq r\}$. $\Phi$ is a regular map. Put $B := \Phi(A)$ and $\varphi := \Phi|_A \colon A \to B$. Then, $B$ and $\varphi$ satisfy the conclusion of the Proposition. □

## Section 2. Complex envelope.

A semialgebraic quasi-variety is a subset of certain complex complete quasi-varieties. In this section, we study such complex quasi-varieties.

**Definition 2.1.**(Conjugate) (1) Let $X$ be a complex algebraic quasi-variety. If there exists a $\mathbb{R}$-scheme $Y$ such that $X \cong Y \times_{\mathrm{Spec}\,\mathbb{R}} \mathrm{Spec}\,\mathbb{C}$ as $\mathbb{R}$-schemes, then $X$ is called *self-conjugate*. In this case, $Y(\mathbb{R})$ is also denoted by $X(\mathbb{R})$. The anti-holomorphic involution map $J \colon X \to X$ with $J|_Y = \mathrm{id}_Y$ is also called the *conjugate map*.

(2) Let $X$ and $Z$ be self-conjugate complex algebraic quasi-varieties, and $Y_X$, $Y_Z$ be $\mathbb{R}$-schemes such that $X = Y_X \times_{\operatorname{Spec}\mathbb{R}} \operatorname{Spec}\mathbb{C}$ and $Z = Y_Z \times_{\operatorname{Spec}\mathbb{R}} \operatorname{Spec}\mathbb{C}$. If a $\mathbb{C}$-morphism $\varphi\colon X \to Z$ is induced from a $\mathbb{R}$-morphism $\varphi_{\mathbb{R}}\colon Y_Z \longrightarrow Y_Z$, then we say that $\varphi\colon X \to Z$ is *real.* Similarly, we define a real rational map $\varphi\colon X \cdots \to Z$.

**Definition 2.2.**(Faithful embedding) Let $A$ be a semialgebraic quasi-variety, and $Z$ be a self-conjugate complex algebraic quasi-variety. A map $\varphi\colon A \to Z$ is called a *faithful embedding* if:

(1) There exists a real algebraic quasi-variety $X$, and an immesion $i\colon A \to X$ such that $\operatorname{Zar}_X(i(A)) = X$.
(2) There exists a $\mathbb{R}$-scheme $Y$ which represents $X$.
(3) There exists a $\mathbb{R}$-isomorphism $\psi\colon Y \times_{\operatorname{Spec}\mathbb{R}} \operatorname{Spec}\mathbb{C} \longrightarrow Z$ which induces a continious map $j\colon X = Y(\mathbb{R}) \longrightarrow Z$.
(4) $\varphi = j \circ i$.

Note that $Z(\mathbb{R}) \subset Z(\mathbb{C})$ is not always faithful embedding. For example, when $Z = \operatorname{Spec}\mathbb{C}[X,Y]/(X^2+Y^2)$, $Z(\mathbb{R}) = \{(0,0)\} \subset Z$ is not faithful embedding.

**Proposition 2.3.** *Let $A$ be a semialgebraic quasi-variety, $\iota_X\colon A \to X$ and $\iota_Y\colon A \to Y$ be faithful embeddings to complex algebraic quasi-varieties. Then, $X$ and $Y$ are birational equivalent.*

*Proof.* We may assume that $X$ and $Y$ are complete. Define $\delta\colon A \to X \times Y$ by $\delta(P) = \big(\iota_X(P), \iota_Y(P)\big)$ $(P \in A)$, and put $\Delta := \operatorname{Zar}_{X\times Y}(\delta(A)) \subset X \times Y$. Clearly, $\delta$ is an immersion. $\pi_X\colon \Delta \to X$ indeuces an isomorphism $\tau\colon \delta(A) \to A$. Since $\tau^* : \mathfrak{R}_{A,P} \longrightarrow \mathfrak{R}_{\delta(A),\delta(P)}$ is an isomorphism, $\pi_X^* : \mathfrak{O}_{X,P} \longrightarrow \mathfrak{O}_{\Delta,\delta(P)}$ is also an isomorphism. Thus the projections $\pi_X\colon \Delta \longrightarrow X$ is a birational regular map. Similarly, $\pi_Y\colon \Delta \longrightarrow Y$ is also a birational regular map. Thus, we have the conclusion. □

**Definition 2.4.** Let $A$ be a semialgebraic quasi-variety, and $X$ be a self-conjugate complete complex algebraic quasi-variety with a faithful embedding $\iota\colon A \to X$. Then, we say $X$ is a *complex envelope* of $A$. If $A$ and $X$ are normal (resp. non-singular), we say $X$ is a normal (resp. non-singular) complex envelope of $A$.

**Proposition 2.5.** *Let $A$ be a semialgebraic quasi-variety. Then, there exists a complex envelope $X$ of $A$. If $A$ is norml (resp. non-singular), we can chooes $X$ to be normal (non-singular).*

*Proof.* Clear □

**Definition 2.6.** Let $A$ be a normal semialgebraic variety, and $X$, $Y$ be normal complex envelopes of $A$. Since $\dim_{\mathbb{C}} H^i(X,\, \mathfrak{O}_X) = \dim_{\mathbb{C}} H^i(Y,\, \mathfrak{O}_Y)$, we can define $h^i(A) := \dim_{\mathbb{C}} H^i(X,\, \mathfrak{O}_X)$. When $\dim A = 1$, $g(A) := h^1(A)$ is called the *genus* of $A$. When $\dim A = 2$, $p_g(A) := h^2(A)$ is called the *geometric genus* of $A$, and $q(A) := h^1(A)$ is called the *irregularity* of $A$.

Since $\dim_{\mathbb{C}} H^0(X,\, \mathfrak{O}_X(mK_X)) = \dim_{\mathbb{C}} H^i(Y,\, \mathfrak{O}_Y(mK_Y))$ for $m \in \mathbb{N}$, we can define $P_m(A) := \dim_{\mathbb{C}} H^0(X,\, \mathfrak{O}_X(mK_X))$. $P_m(A)$ is called the *$m$-genus* of $A$,

Using $P_m(A)$, we can define the *Kodaira dimension* $\kappa(A)$. Note that $\kappa(A) = \kappa(X)$.

**Definition 2.7.** Let $A$ be a closed semialgebraic quasi-variety, and let $X$ be a real algebraic quasi-variety with a faithful embedding $A \subset X$. Put $B := \partial A$. For an arbitral affine open subset $U \subset X$ such that $U \cap B \neq \emptyset$, let

$$I_U := \{ f \in \mathfrak{R}_X(U) \mid f(P) = 0 \text{ for all } P \in B \},$$

and put $\mathfrak{R}_B(U \cap B) := \mathfrak{R}_X(U)/I_U$. This defines a coherent sheaf $\mathfrak{R}_B$ of commutative rings on $B$. We call the semialgebraic quasi-variety $(B, \mathfrak{R}_B)$ to be the *boundary* of $A$.

Similarly, we can define $\mathfrak{R}_{\mathrm{Sing}(A)}$. Thus, $\mathrm{Sing}(A)$ can be regarded as a semialgebraic quasi-variety.

**Remark 2.8.** (1) In the above definition, $\dim B \leq \dim A - 1$, but it may happen that $\dim B < \dim A - 1$ if $B \subset \mathrm{Sing}(X)$.

(2) Using the above dfenition, we can define the *critical decomposition* of $A$, similarly as Definition 1.5 of [1].

**Definition 2.9.**(Blowing up) Let $A$ be a semialgebraic variety, $X$ be a complex envelope of $A$, and $I$ be an ideal sheaf of $\mathfrak{R}_A$. There exists an ideal $J$ of $\mathfrak{O}_X$ such that $J\mathfrak{O}_X|_A \otimes_{\mathfrak{O}_X|_A} \mathfrak{R}_A = I$. Let $\mathfrak{J}$ be the set of all the ideals $J$ of $\mathfrak{O}_X$ which satisfy the above condition. If $J_1$, $J_2 \in \mathfrak{J}$, then $J_1 + J_2 \in \mathfrak{J}$. Thus, there exists the unique maximal element of $\mathfrak{J}$. Take such the maximal $J \in \mathfrak{J}$. Let $\pi\colon Y \to X$ be the blowing up of $X$ by the ideal $J$, and $B = \pi^{-1}(A) \subset Y$. $B$ is a semialgebraic quasi-variety as Definition 1.6. We say that $\pi|_B\colon B \to A$ is the *blowing up* of $A$ by $I$.

If $I$ is a defining ideal of a closed semialgebraic subvariety $C \subset A$, then $\pi|_B\colon B \to A$ is also called to be the *blowing up* of $A$ with/along/at the *center* $C$. By Proposition 2.3, $B$ does not depend on choice of $X$. Let $E \subset Y$ be the exceptional set of $\pi$. Then,

$$B_0 := \mathrm{Cls}_Y \left( \pi^{-1}(A - (\pi(E) \cap \partial A)) \right) \subset B$$

is called the *strict transform* of $A$.

**Proposition 2.10.** *Let $A$, $B$ be semialgebraic varieties, and $\varphi\colon A \to B$ be a regular map. Then there exists complex envelopes $i_X\colon A \to X$, $i_Y\colon B \to Y$ and a regular map $\Phi\colon X \to Y$ such that $i_Y \circ \varphi = \Phi \circ i_X$.*

*Proof.* Take complete real algebraic varieties $X$, $Y$ and complex envelopes $X_{\mathbb{C}}$, $Y_{\mathbb{C}}$ with faithful embeddings $A \xrightarrow{i_X} X \subset X_{\mathbb{C}}$, $B \xrightarrow{i_Y} Y \subset Y_{\mathbb{C}}$. Take a point $P \in \mathrm{Int}(A)$ such that $Q := \varphi(P) \in \mathrm{Int}(B)$, and take an affine open subset $W \subset Y_{\mathbb{C}}$ such that $Q \subset W \subset Y_{\mathbb{C}}$. Since $\mathfrak{R}_{X,P} = \mathfrak{R}_{A,P}$ and $\mathfrak{R}_{Y,Q} = \mathfrak{R}_{B,Q}$, the homomorphism $\varphi_P^*\colon \mathfrak{R}_{B,Q} \longrightarrow \mathfrak{R}_{A,P}$ induces $\psi_W\colon \mathfrak{O}_{Y_{\mathbb{C}}}|_Y(W) \longrightarrow \mathfrak{R}_{A,P}$.

Since $\mathfrak{O}_{Y_{\mathbb{C}}}|_Y(W)$ is a finitely generated $\mathbb{R}$-algebra, we can choose $f_1, \ldots, f_r \in \mathfrak{R}_{Y,Q} \subset \mathrm{Rat}(B) = \mathrm{Rat}(Y) \subset \mathrm{Rat}(Y_{\mathbb{C}})$ such that $\mathfrak{O}_{Y_{\mathbb{C}}}|_Y(W) = \mathbb{R}[f_1, \ldots, f_r]$. Put $g_j := \psi_W(f_j) \in \mathfrak{R}_{A,P} \subset \mathrm{Rat}(A) = \mathrm{Rat}(X) \subset \mathrm{Rat}(X_{\mathbb{C}})$. We can find an affine open subset $U \subset X_{\mathbb{C}}$ such that $g_1, \ldots, g_r$ are holomorphic (regular) on $U$, and that $U \cap X$ is dense in $X$ and $U \cap A$ is dense in $A$. Then, $\psi_W$ induces $\Psi_W\colon \mathfrak{O}_{Y_{\mathbb{C}}}(W) \longrightarrow \mathfrak{O}_{X_{\mathbb{C}}}(U)$. $\Psi_W$ induces regular maps $\Phi_U\colon U \to W$, $\Phi_{U \cap X} : (U \cap X) \to (W \cap Y)$, and rational maps $\Phi_{\mathbb{C}}\colon X_{\mathbb{C}} \cdots \to Y_{\mathbb{C}}$, $\Phi\colon X \cdots \to Y$.

By the Hironaka's theorem of a resolution of the indeterminacy, there exists a composit of blowing ups $\pi\colon \widetilde{X} \to X$ and a regular map $\widetilde{\Phi}\colon \widetilde{X} \to Y$ such that $\widetilde{\Phi} = \Phi \circ \pi$. Since there exists no indeterminacy of $\Psi$ on $A$, we can choose $\pi$ so that the exceptional set $E$ of $\pi$ satisfies $\pi(E) \cap A = \emptyset$. Thus $\widetilde{X}$ and $\widetilde{\Phi}$ satisfy the conditions of $X$ and $\Phi$ in the theorem. $\square$

**Proposition 2.11.** *Let $A$ be a semialgebraic variety, and assume that a finite group $G$ acts on $A$ faithfully. Then, $A/G$ is a semialgebraic variety.*

*Proof.* Let $X$ be a complex envelope of $A$. Since $G$ acts on $\mathfrak{R}_A$ and $\mathfrak{O}_X$ is a subring of $\mathfrak{R}_A \otimes_{\mathbb{R}} \mathbb{C}$, the action of $G$ on $A$ can be extended to $X$. It is well known that $X/G$ is complex algebraic variety, Let $\pi\colon X \to X/G$ be the natural surjection. Then $A/G \cong \pi(A)$. Thus, $A/G$ is a semialgebraic subset of $X/G$. The invariant sheaf $\mathfrak{R}_A^G$ is the structure sheaf of $X/A$. Thus, $X/G$ is a semialgebraic variety. □

As mentioned in the Introduction, A quotient of real algebraic variety may not be real algebraic variety. If $X$ is an algebraic variety over $\mathbb{R}$ which is a $\mathbb{R}$-scheme, $X(\mathbb{R})/G$ may not agree with $(X/G)(\mathbb{R})$. We shall explain the example which we presented the first part of the Introduction.

**Example 2.12.**(Discriminant of cubic equation) Let $A = \mathbb{P}^2_+ := \{(x_0 : x_1 : x_2) \in \mathbb{P}^2_{\mathbb{R}} \mid x_i x_j \geq 0 \text{ for all } 0 \leq i < j \leq 2\}$. Consider functions $\sigma_1 := x_0 + x_1 + x_2$, $\sigma_2 := x_0x_1 + x_1x_2 + x_2x_0$ and $\sigma_3 = x_0x_1x_2$ on $A$. Define a regular map $\varphi\colon A \to \mathbb{P}_{\mathbb{R}}(1,2,3)$ by $\varphi(P) = (\sigma_1(P),\ \sigma_2(P),\ \sigma_3(P))$, where $\mathbb{P}_{\mathbb{R}}(1,2,3)$ is defined as the set of real points of the weighted projective space $\mathbb{P}_{\mathbb{C}}(1,2,3)$. Then $B := \varphi(A)$ is just $\mathbb{P}^2_+/\mathfrak{S}_3$. We can choose a fundamental domains of $\varphi$ as $A_0 := \{(s : t : 1) \in \mathbb{P}^2_+ \mid 0 \leq s \leq t \leq 1\}$. Since $\varphi\colon A_0 \to B$ is birational, we know that $B$ is the semialgebraic subset of $\mathbb{P}_{\mathbb{R}}(1,2,3)$ defined by

$$27a_3^2 - 18a_1a_2a_3 + 4a_1^3a_3 + 4a_2^3 - a_1^2a_2^2 \leq 0, \quad a_2 \geq 0, \quad a_3 \geq 0,$$

where $(a_1,\ a_2,\ a_3)$ is the homogeneous coordinate system of $\mathbb{P}_{\mathbb{R}}(1,2,3)$. Now, consider the cubic polynomial $f(t) = t^3 - a_1t^2 + a_2t - a_3$, where $a_1,\ a_2,\ a_3 \in \mathbb{R}$. The condition that the all roots of $f(t) = 0$ are non-negative real numbers, is equivalent to $(a_1,\ a_2,\ a_3) \in B$.

Please try this for an algebraic equation of higher degree.

**Section 3. Maximal extension and proper extension.**

**Definition 3.1.** (1) Let $A$ be a semialgebraic quasi-variety, $X$ be a complex envelope of $A$, and $U \subset A$ be a Zariski open subset. Let $\mathfrak{W}$ be the set of all the Zariski open subsets $W$ of $X$ such that $W \cap A = U$. Put $U_X^m := \bigcup_{W \in \mathfrak{W}} W$. Then, $U_X^m$ is the maximal Zariski open subset of $X$ which satisfies $U_X^m \cap A = U$. We call this $U_X^m$ to be *the maximal extension* of $U$ to $X$.

(2) Let $A$ be a semialgebraic variety, and $X$ be a complex envelope of $A$. For $f \in \operatorname{Rat}(A)$ (or for a homogeneous function $f$), we denote

$$V_A(f) := \operatorname{Zar}_A\left(\{P \in A \mid f(P) = 0\}\right),$$
$$V_X(f) := \operatorname{Zar}_X\left(\{P \in X \mid f(P) = 0\}\right),$$

$D_A(f) := A - V_A(f)$, and $D_X(f) := X - V_X(f)$.

(3) Let $A$ be a normal semialgebraic variety and $X$ be a normal complex envelope of $A$. Take an affine open subset $U \subset A$. Then $\mathfrak{U}_A^p := \{D_A(f) \mid f \in \mathfrak{R}(A)\}$ is a base of the Zariski topology on $A$. For $D_A(f) \in \mathfrak{U}_A^p$, we put

$$F(f) := \{g \in \mathfrak{R}(A) \mid D_A(g) = D_A(f)\}, \quad (D_A(f))_X^p := \bigcup_{g \in F(f)} D_X(g).$$

For an arbitral Zariski open subset $U = D_A(f_1) \cup \cdots \cup D_A(f_r)$ of $A$, we put $U_X^p := (D_A(f_1))_X^p \cup \cdots \cup (D_A(f_r))_X^p$. We say $U_X^p$ is the *proper extension* of $U$ to $X$.

**Remark 3.2.** (1) $U_X^m$ and $U_X^p$ are self-conjugate.

(2) $U_X^m$ and $U_X^p$ may not be affine, even if $U$ is an affine open subset of $A$.

(3) $(X(\mathbb{R}) - A) \subset U_X^p \subset U_X^m$.

(4) Let $B \subset A$ be normal semialgebraic varieties, $B \subset Y$, $A \subset X$ be complex envelopes with $Y \subset X$. Take a Zariski open subset $U \subset X$. It is easy to see that $(U \cap B)_Y^p \supset U_X^p \cap Y$ and $(U \cap B)_Y^m \supset U_X^m \cap Y$. But they will not agree.

For example, let $X = \mathbb{P}^2_{\mathbb{C}}$ with the coordinate system $(x, y, z)$. Let $Y = \mathbb{P}^1_{\mathbb{C}} \subset X$ be the line defined by $y = 0$. Take $B = \mathbb{P}^1_{\mathbb{R}} \subset Y$ and $A = \mathbb{P}^2_{\mathbb{R}} \subset X$ as the natural way. Consider an affine open subset $U = D_A\big(y^2 z - x(x^2 + z^2)\big) \subset A$. Then $U_X^p = U_X^m = D_X\big(y^2 z - x(x^2 + z^2)\big)$. But, since $U \cap B = D_B(x)$, we have $(U \cap B)_Y^p = (U \cap B)_Y^m = D_Y$. Thus $(U \cap B)_Y^p \neq U_X^p \cap Y$ and $(U \cap B)_Y^m \neq U_X^m \cap Y$.

**Proposition 3.3.** *Let $A$ be a normal semialgebraic variety, and $X$ be a normal complex envelope of $A$.*

(1) *Let $f \in \mathrm{Rat}(A) - \mathbb{C}$, and $U = D_A(f)$. If $V_X(f)$ is an irreducible subvariety of $X$ and $\dim(V_X(f) \cap A) = \dim A - 1$, then $U_X^p = D_X(f)$.*

(2) *Let $f_1, \ldots, f_r \in \mathrm{Rat}(A) - \mathbb{C}$, and $U = D_A(f_1 \cdots f_r)$. If $V_X(f_i)$ are irreducible subvarieties of $X$ and $\dim(V_X(f_i) \cap A) = \dim A - 1$ for all $i = 1, \ldots, r$, then $U_X^p = D_X(f_1 \cdots f_r)$.*

*Proof.* (1) Take $g \in F(f)$. Then $V_A(g) = V_A(f) = V_X(f) \cap A$. Since $V_X(f)$ is irreducible, we have $V_X(g) \supset V_X(f)$. This implies $U_X^p \subset D_X(f)$. Since $D_X(f) \subset U_X^p$, we have $U_X^p = D_X(f)$.

(2) follows from (1). □

**Example 3.4.** We present an example such that $U_X^p \neq U_X^m$. Let $A = \mathbb{P}^2_{\mathbb{R}} \subset X_0 = \mathbb{P}^2_{\mathbb{C}}$, $P_1 = (0 : 1 : \sqrt{-1})$, $P_2 = (0 : 1 : -\sqrt{-1}) \in X_0$, and $\varphi: X \to X_0$ be the blowing up at $P_1$ and $P_2$. Then, $A \subset X$ is also a complex envelope. Let $(x_0 : x_1 : x_2)$ be the homogeneous coordinate system of $X_0$. Consider $U := D_A(x_0)$. Put $L_0 := V_{X_0}(x_0)$, $E_i := \varphi^{-1}(P_i)$, and let $L$ be the strict transform of $L_0$ by $\varphi$. Note that $\varphi^{-1}(L_0) = L \cup E_1 \cup E_2$. By the above proposition, we have $U_X^p = X - (L \cup E_1 \cup E_2)$. On the other hand, $U_X^m = X - L$.

**Remark 3.5.** $\mathcal{O}_X(U_X^m)$ and $\mathcal{O}_X(U_X^p)$ depend on the complex envelope $X$. We shall give an example.

(1) Let $X := \mathbb{P}^2_{\mathbb{C}} \supset A := \mathbb{P}^2_{\mathbb{R}}$ with the homogeneous coordinate system $(x_0 : x_1 : x_2)$, and $U = D_A(x_0) \subset A$. For $z \in \mathbb{C} - \mathbb{R}$, put $f_z := x_0^2/(x_1^2 + z\bar{z}x_2^2)$. Note that $f_z \in \mathcal{R}_A(A)$ and $U = D_A(f_z)$. Put $L := V_X(x_0) \subset X$, $P_1 := (0 : 1 : \sqrt{-1}) \in X$, and $P_2 := (0 : 1 : -\sqrt{-1}) \in X$. Let $\varphi: Y \to X$ be the blowing up at $P_1$ and $P_2$. Put $E_i = \varphi^{-1}(P_i) \subset Y$, and $L' \subset Y$ be the strict transform of $L$. Then $U_X^p = U_X^m = X - L$ and $U_Y^p = U_Y^m = Y - L'$. For example, $x_1/x_0 \in \mathcal{O}_X(U_X^m)$ but $x_1/x_0 \notin \mathcal{O}_Y(U_Y^m)$. Thus $\mathcal{O}_X(U_X^m) = \mathcal{O}_X(U_X^p) \supsetneqq \mathcal{O}_Y(U_Y^m) = \mathcal{O}_Y(U_Y^p)$.

(2) Let $X := \mathbb{P}^2_{\mathbb{C}} \supset A := \mathbb{P}^2_{\mathbb{R}}$, $z \in \mathbb{C} - \mathbb{R}$, $P_1 := (1 : 1 : z)$, $P_2 := (1 : 1 : \bar{z})$, $P_3 := (1 : 2 : z)$, $P_4 := (1 : 2 : \bar{z})$. and $U := D_A(x_0) \subset A$. Let $\varphi: Y \to X$ be the blowing up at $P_1$, $P_2$, $P_3$, $P_4$, and let $L_{ij} \subset Y$ be the strict transform of the line $P_iP_j$ in $X$. Note that $L_{ij}$ are $(-1)$-curves $(1 \leq i < j \leq 4)$. Let $\psi_z: Y \to V_z$ be the contraction of $L_{13}$ and $L_{24}$. Then $X$, $Y$, $V_z$ are complex envelopes of $A$. Note that $\psi_z^{-1}(D_{V_z}) = D_Y(U_Y^m) - (L_{13} \cup L_{24})$, and

$\bigcup_{z \in \mathbb{C}-\mathbb{R}} \varphi(\psi_z^{-1}(D_{V_z})) = U \cup \{P_1, \ldots, P_4\}$. This implies that we cannot find the 'standard' $\mathcal{O}_{V_z}(U^m_{V_z})(\mathbb{R})$ or $\mathcal{O}_{V_x}(U^p_{V_z})(\mathbb{R})$.

**Definition 3.6.** Let $A$ be a semialgebraic quasi-variety, $X$ be a complex envelope of $A$, and $Y$ be a $\mathbb{R}$-scheme such that $X = Y \times_{\operatorname{Spec}\mathbb{R}} \operatorname{Spec}\mathbb{C}$. When we discuss about $U^p_X$, we assume that $A$ and $X$ are normal. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module.

(1) We say $\mathcal{F}$ is a *real*, if there exists $\mathcal{O}_Y$-mofule $\mathcal{G}$ such that $\mathcal{F} = \mathcal{G} \otimes_{\mathbb{R}} \mathbb{C}$.

(2) Assume that $\mathcal{F}$ is a real quasi-coherent $\mathcal{O}_X$-module. For an affine open subset $U \subset A$, let $U^m_X$ and $U^p_X$ be the maximal and the proper extension of $U$ to $X$. We define the sheaf $\mathcal{F}|^m_A$ and $\mathcal{F}|^p_A$ on $A$ by

$$(\mathcal{F}|^m_A)(U) = \mathcal{F}(U^m_X)(\mathbb{R}) := \{f \in \mathcal{F}(U^m_X) \mid J(f) = f\},$$
$$(\mathcal{F}|^p_A)(U) = \mathcal{F}(U^p_X)(\mathbb{R}) := \{f \in \mathcal{F}(U^p_X) \mid J(f) = f\},$$

here $J(f) = \overline{f}$ is the complex conjulate. Then, $\mathcal{F}|^m_A$ is a $\mathcal{O}_X|^m_A$-module. Similarly, $\mathcal{F}|^p_A$ is a $\mathcal{O}_X|^p_A$-module.

(3) Let $(U_X, |_A) = (U^m_X, |^m_A)$ or $(U^p_X, |^p_A)$. Assume that $\mathcal{F}$ is a real quasi-coherent $\mathcal{O}_X$-module. If $(\mathcal{F}|_A) \otimes_{\mathcal{O}_X|_A} \mathcal{O}_X \cong \mathcal{F}$, then we say $\mathcal{F}$ is an *$A$-sheaf* with respect to $|_A$.

(4) $\mathcal{F}$ is called an *$A$-pure locally free sheaf* with respect to $|_A$, if $\mathcal{F}$ is a locally free $\mathcal{O}_X$-module of rank $r$, and if for any point $P \in A$, there exists an affine open neighborhood $P \in U \subset A$ and $e_1, \ldots, e_r \in \mathcal{F}(U_X)(\mathbb{R})$ such that $\mathcal{F}|_{U_X} = \mathcal{O}_X|_{U_X} \cdot e_1 \oplus \cdots \oplus \mathcal{O}_X|_{U_X} \cdot e_r$.

**Assumption 3.7.** The maximal extension and the proper extension have some similar properties. Thus we shall discuss them together. We use one of the following assumptions:

(1) $A$ is a semialgebraic quasi-variety, and $X$ is a complex envelope of $A$. For a Zariski open subset $U \subset A$, $U_X := U^m_X$ is the maximal extension of $U$ to $X$. For a real quasi-coherent $\mathcal{O}_X$-module $\mathcal{F}$, $\mathcal{F}|_A := \mathcal{F}|^m_A$.

(2) $A$ is a normal semialgebraic variety and $X$ is a normal complex envelope of $A$, For a Zariski open subset $U \subset A$, $U_X := U^p_X$ is the proper extension of $U$ to $X$. For a real quasi-coherent $\mathcal{O}_X$-module $\mathcal{F}$, $\mathcal{F}|_A := \mathcal{F}|^p_A$.

**Remark 3.8.** We use the same notation with Definition 3.6 and Assumption 3.7. Then:

(1) $\mathcal{R}_A$ is a $\mathcal{O}_X|^m_A$-algebra, and is a $\mathcal{O}_X|^p_A$-algebra.

(2) $H^0(U, \mathcal{F}|_A) \otimes_{\mathbb{R}} \mathbb{C} = H^0(U_X, \mathcal{F})$ for $(U_X, |_A) = (U^m_X, |^m_A)$ and $(U^p_X, |^p_A)$, if $\mathcal{F}$ is a real quasi-coherent $\mathcal{O}_X$-module.

(3) $\mathcal{F}|_A$ may not be a locally free $\mathcal{O}_X|_A$-module, even if $\mathcal{F}$ is a real locally free $\mathcal{O}_X$-module.

(4) Even if $\mathcal{F}|_A = \mathcal{O}_X|_A$, it may happen that $\mathcal{F} \not\cong \mathcal{O}_X$.

(5) Let $\mathcal{F}$ and $\mathcal{G}$ be real quasi-coherent $\mathcal{O}_X$-modules. It may happen $\mathcal{F}(U_X) \otimes_{\mathcal{O}_X(U_X)} \mathcal{G}(U_X) \subsetneqq (\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G})(U_X)$. Thus, $\mathcal{F}|_A \otimes_{\mathcal{O}_X|_A} \mathcal{G}|_A$ may not agree with $(\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G})|_A$. Similarly, $\mathcal{H}om_{\mathcal{O}_X|_A}(\mathcal{F}|_A, \mathcal{G}|_A)$ may not agree with $(\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G}))|_A$.

(6) The stalk $(\mathcal{O}_X|_A)_P$ is not always a local ring for $P \in A$.

**Proposition 3.9.** *Under each of assumption (1) and (2) in Assumption 3.7, let $(U_X, |_A) = (U^m_X, |^m_A)$ or $(U^p_X, |^p_A)$. Assume that $\mathcal{F}$ is an $A$-pure locally free $\mathcal{O}_X$-modules of the rank $r$, and $\mathcal{G}$ is a real quasi-coherent $\mathcal{O}_X$-module. Then,*

(1) *An $A$-pure locally free sheaf $\mathcal{F}$ is an $A$-sheaf.*

(2) *$\mathcal{F}|_A \otimes_{\mathcal{O}_X|_A} \mathcal{G}|_A \cong (\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G})|_A$.*

(3) *$\mathcal{H}om_{\mathcal{O}_X|_A}(\mathcal{F}|_A, \mathcal{G}|_A) \cong \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})|_A$.*

(4) *If* $\mathfrak{F} \cong \mathfrak{O}_X$ *and* $\mathfrak{F}|_A = \mathfrak{O}_X|_A$, *then* $\mathfrak{F} = \mathfrak{O}_X$.

(5) *If* $0 \to \mathfrak{L} \to \mathfrak{M} \to \mathfrak{N}$ *is an exact sequence of real quasi-coherent* $\mathfrak{O}_X$*-modules. Then* $0 \to \mathfrak{L}|_A \to \mathfrak{M}|_A \to \mathfrak{N}|_A$ *is an exact sequence of* $\mathfrak{O}_X|_A$*-modules, and* $0 \to \mathfrak{L}|_A(U) \to \mathfrak{M}|_A(U) \to \mathfrak{N}|_A(U)$ *is an exact sequence of* $\mathfrak{O}_X|_A(U)$*-modules.*

*Proof.* Let $U$ be an affine open subset of $A$. Assume that $\mathfrak{F}|_{U_X} = \oplus_{i=1}^r \mathfrak{O}_X|_{U_X} \cdot e_i$.

(1) Take any affine open subset $W \subset U_X$. Then

$$\begin{aligned}\mathfrak{F}|_A(U) \otimes_{\mathfrak{O}_X|_A(U)} \mathfrak{O}_X(W) &\cong \left(\oplus_{i=1}^r \mathfrak{O}_X(U_X) \cdot e_i\right) \otimes_{\mathfrak{O}_X(U_X)} \mathfrak{O}_X(W) \\ &\cong \oplus_{i=1}^r \mathfrak{O}_X(W) \cdot e_i = \mathfrak{F}(W).\end{aligned}$$

Thus, $(\mathfrak{F}|_A) \otimes_{\mathfrak{O}_X|_A} \mathfrak{O}_X \cong \mathfrak{F}$.

(2) Let $\mathfrak{H}$ be the presheaf defined by

$$\mathfrak{H}(U) = \mathfrak{F}|_A(U) \otimes_{\mathfrak{O}_X|_A(U)} \mathfrak{G}|_A(U) = \left(\mathfrak{F}(U_X) \otimes_{\mathfrak{O}_X(U_X)} \mathfrak{G}(U_X)\right)(\mathbb{R})$$

for each Zariski open subset $U \subset A$. Then, $\mathfrak{F}|_A \otimes_{\mathfrak{O}_X|_A} \mathfrak{G}|_A$ is the sheafication of $\mathfrak{H}$. For any sufficiently small $U \subset A$,

$$\begin{aligned}&\mathfrak{F}(U_X) \otimes_{\mathfrak{O}_X(U_X)} \mathfrak{G}(U_X) \cong \left(\oplus_{i=1}^r \mathfrak{O}_X(U_X) \cdot e_i\right) \otimes_{\mathfrak{O}_X(U_X)} \mathfrak{G}(U_X) \\ &\cong \oplus_{i=1}^r \mathfrak{G}(U_X) \cdot e_i \cong \left(\left(\oplus_{i=1}^r \mathfrak{O}_X \cdot e_i\right) \otimes_{\mathfrak{O}_X} \mathfrak{G}\right)(U_X) \cong (\mathfrak{F} \otimes_{\mathfrak{O}_X} \mathfrak{G})(U_X).\end{aligned}$$

Thus, we have (2).

(3) Let $\mathfrak{H}$ be the presheaf defined by

$$\mathfrak{H}(U) = \mathrm{Hom}_{\mathfrak{O}_X|_{U_X}}\left(\mathfrak{F}|_{U_X},\, \mathfrak{G}|_{U_X}\right)(\mathbb{R})$$

for each Zariski open subset $U \subset A$. Then $\mathcal{H}om_{\mathfrak{O}_X|_A}\left(\mathfrak{F}|_A,\, \mathfrak{G}|_A\right)$ is the sheafication of $\mathfrak{H}$. Since $\mathfrak{F}$ is $A$-pure locally free, for any sufficiently small $U \subset A$, $\mathfrak{F}|_{U_X} = \mathfrak{O}_X|_{U_X} \cdot e_1 \oplus \cdots \oplus \mathfrak{O}_X|_{U_X} \cdot e_r$. Then,

$$\begin{aligned}\mathfrak{H}(U) &= \oplus_{i=1}^r \mathrm{Hom}_{\mathfrak{O}_X|_{U_X}}\left(\mathfrak{O}_X|_{U_X} \cdot e_i,\, \mathfrak{G}|_{U_X}\right)(\mathbb{R}) \\ &= \oplus_{i=1}^r \mathrm{Hom}_{\mathfrak{O}_X|_{U_X}}\left(\mathfrak{O}_X|_{U_X},\, \mathfrak{G}|_{U_X} \cdot (1/e_i)\right)(\mathbb{R}) \\ &= \oplus_{i=1}^r \mathfrak{G}|_{U_X}(\mathbb{R}) \cdot (1/e_i) \\ &= \oplus_{i=1}^r \mathcal{H}om_{\mathfrak{O}_X(U_X)}\left(\mathfrak{O}_X(U_X) \cdot e_i,\, \mathfrak{G}(U_X)\right)(\mathbb{R}) \\ &= \mathcal{H}om_{\mathfrak{O}_X(U_X)}\left(\mathfrak{F}(U_X),\, \mathfrak{G}(U_X)\right)(\mathbb{R}) \\ &= \mathcal{H}om_{\mathfrak{O}_X}(\mathfrak{F},\, \mathfrak{G})|_A(U)\end{aligned}$$

Thus we have (3).

(4) Let $h\colon \mathfrak{O}_X \to \mathfrak{F}$ be an isomorphism. Put $e = h(1) \in \mathfrak{F}(X)$, here $1 \in \mathfrak{O}_X(X)$. Then $\mathfrak{F} = \mathfrak{O}_X \cdot e$. $\mathfrak{F}|_A = \mathfrak{O}_X|_A$ implies $e$ and $1/e$ are regular on $X$. This implies $e$ is a non-zero constant function. Thus $\mathfrak{F} = \mathfrak{O}_X$.

(5) is clear. □

**Section 4. Cohomology.**

To begin with, we confirm that $\mathfrak{R}_A$-modules do not have higher cohomologies.

**Proposition 4.1.** *Let* $A$ *be a semialgebraic quasi-variety and* $\mathfrak{F}$ *be a quasi-coherent* $\mathfrak{R}_A$*-module. Then,* $H^i(A,\, \mathfrak{F}) = 0$ *for all* $i > 0$.

*Proof.* There exists a faithful embedding $\tau\colon A \to X$ into an affine real algebraic quasi-variety $X$. There also exists a closed immersion $\iota\colon X \to \mathbb{R}^m$ as real algebraic varieties for a certain $m \in \mathbb{N}$. Let $R_X := \mathfrak{R}_X(X)$ and $R_A := \mathfrak{R}_A(A)$. We can present as $R_A = S_A^{-1} R_X$. Since $R_A$ is an $R_X$-module, $\mathfrak{F}$ is an quasi-coherent $\mathfrak{R}_X$-module. Thus, $\mathfrak{F}$ is an quasi-coherent $\mathfrak{R}_{\mathbb{R}^m}$-module. Thus we have

$$H^i(A, \mathfrak{F}) \cong H^i\big(\mathbb{R}^m, \mathfrak{F}\big) = 0$$

(see [6] Chap.III, Theorem 3.5). □

If $\mathfrak{F}$ is a sheaf of Abelian group on $A$, $H^i(A, \mathfrak{F}) \neq 0$ may happen.

**Proposition 4.2.**(Grothendieck) *Let $A$ be a semialgebraic quasi-variety, and $\mathfrak{F}$ be any sheaf of Abelian group on $A$. Then, $H^i(A, \mathfrak{F}) = 0$ for all $i > \dim A$.*

*Proof.* See [6] Chap.III, Theorem 2.7. □

Next, we consider $\mathfrak{O}_X|_A^m$-modules and $\mathfrak{O}_X|_A^p$-modules. These have similar cohomological properties. For example,

$$H^i(A, \mathfrak{O}_X|_A^m) \otimes_{\mathbb{R}} \mathbb{C} \cong H^i(A, \mathfrak{O}_X|_A^p) \otimes_{\mathbb{R}} \mathbb{C} \cong H^i(X, \mathfrak{O}_X).$$

Thus, throughout the left part of this section, we use the any of assumptions (1) or (2) in Assumption 3.7. Let $(U_X, |_A) = (U_X^m, |_A^m)$ or $(U_X^p, |_A^p)$.

**Theorem 4.3.** *Let $\mathfrak{F}$ be a quasi-coherent $\mathfrak{O}_X|_A$-module and $U \subset A$ be a Zariski open subset. Then,*

$$H^i\big(U_X, \mathfrak{F} \otimes_{\mathfrak{O}_X|_A} \mathfrak{O}_X\big) \cong H^i(U, \mathfrak{F}) \otimes_{\mathbb{R}} \mathbb{C} \quad \textit{for all } i \in \mathbb{Z}.$$

*Proof.* (1) Temporary, put $\mathfrak{O} := \mathfrak{O}_X|_A$. Since $\mathfrak{O}_X(U_X) = \mathfrak{O}(U) \otimes_{\mathbb{R}} \mathbb{C}$, $\mathfrak{O}_X(U_X)$ is a free $\mathfrak{O}(U)$-module of rank 2. Thus $\mathfrak{O}_X$ is a locally free $\mathfrak{O}$-module of rank 2. Its dual sheaf $\mathfrak{H} := \mathcal{H}om_{\mathfrak{O}}(\mathfrak{O}_X, \mathfrak{O})$ is also a locally free $\mathfrak{O}$-module of rank 2. This implies that $\mathfrak{H}$ is an invertible $\mathfrak{O}_X$-module. Especially, $\mathfrak{H}$ is a flat $\mathfrak{O}_X$-module.

(2) Let $R$ be a commutative ring, $S$ be a $R$-commutative algebra, and $E$ be an injective $R$-module. If $P$ is a $R$-module and a $S$-flat module, then $\mathrm{Hom}_R(P, E)$ is an injective $S$-module by injective producing lemma. Moreover, if $M$ is a $R$-module and $S$ is finite representative $R$-module, then $L \otimes_R \mathrm{Hom}_R(M, N) \cong \mathrm{Hom}_R(\mathrm{Hom}_R(L, M), N)$. Thus, if $\mathfrak{I}$ is an injective $\mathfrak{O}$-module, then $\mathfrak{O}_X \otimes_{\mathfrak{O}} \mathfrak{I} \cong \mathfrak{O}_X \otimes_{\mathfrak{O}} \mathcal{H}om_{\mathfrak{O}}(\mathfrak{O}, \mathfrak{I}) \cong \mathcal{H}om_{\mathfrak{O}}(\mathfrak{H}, \mathfrak{I})$ is an injective $\mathfrak{O}_X$-module.

(3) Take an injective resolution $0 \to \mathfrak{F} \to \mathfrak{I}^0 \to \mathfrak{I}^1 \to \cdots$ as $\mathfrak{O}$-modules. Let $\mathfrak{I}_X^i := \mathfrak{I}^i \otimes_{\mathfrak{O}} \mathfrak{O}_X$. Since $\mathfrak{O}_X$ is a flat $\mathfrak{O}$-module, $0 \to \mathfrak{F} \otimes_{\mathfrak{O}} \mathfrak{O}_X \to \mathfrak{I}_X^0 \to \mathfrak{I}_X^1 \to \cdots$ is exact. Since $\mathfrak{I}_X^i$ is an injective $\mathfrak{O}_X$-module, this is an injective resolution of $\mathfrak{F} \otimes_{\mathfrak{O}} \mathfrak{O}_X$. Since $H^0(U_X, \mathfrak{F}_X) = H^0(U, \mathfrak{F}) \otimes_{\mathbb{R}} \mathbb{C}$, we have the conclusion. □

**Corollary 4.4.** *Let $U \subset A$ be a Zariski open subset, and $\mathfrak{F}$ be a quasi-coherent $\mathfrak{O}_X$-module which is an $A$-sheaf with respect to $|_A$. Then,*

$$H^i\big(U_X, \mathfrak{F}\big) \cong H^i(U, \mathfrak{F}|_A) \otimes_{\mathbb{R}} \mathbb{C} \quad \textit{for all } i \in \mathbb{Z}.$$

**Proposition 4.5.** (Serre duality) *Let $A$ be a non-singular real algebraic variety with $\dim A = n$. Assume that there exists a complex envelope $X$ of $A$ such that $X$ is projective. Put $\mathcal{O}_A = \mathcal{O}_X|_A$ and $\omega_A = \mathcal{O}_X(K_X)|_A$. Let $\mathcal{L}$ be an invertible $\mathcal{O}_A$-module. Then*

$$\dim_{\mathbb{R}} H^i(A,\, \mathcal{L}) = \dim_{\mathbb{R}} H^{n-i}(A,\, \omega_A \otimes_{\mathcal{O}_A} \mathcal{L}^{-1})$$

*for all $i \in \mathbb{Z}$. Here $\mathcal{L}^{-1} := \mathcal{H}om_{\mathcal{O}_A}(\mathcal{L},\, \mathcal{O}_A)$,*

*Proof.* This follows $H^i(A,\, \mathcal{L}) \otimes_{\mathbb{R}} \mathbb{C} \cong H^i(X,\, \mathcal{L}_X)$ and $H^{n-i}(A,\, \omega_A \otimes_{\mathcal{O}_A} \mathcal{L}^{-1}) \otimes_{\mathbb{R}} \mathbb{C} \cong H^{n-i}(X,\, \omega_X \otimes_{\mathcal{O}_X} \mathcal{L}_X^{-1})$. □

**Remark 4.6.** Čech cohomology may not work well for semialgebraic varieties, because $H^i(U,\, \mathcal{F}|_A^p)$ may not be 0 for $i \geq 1$ and for an affine open subset $U \subset A$. Note that $U_X^p$ may include an imaginal complete subvariety of a higher genus.

## Section 5. Semialgebraic curves.

As is well known, a complete real algebraic curve can be identified with a self-conjugate complete complex algebraic curve. This is similar for semialgebraic curves.

**Definition 5.1.** (1) A semialgebraic variety $A$ with $\dim A = 1$ is called a *semialgebraic curve*.

(2) If $A$ is a non-singular semialgebraic curve, then a non-singular complex envelope $X$ of $A$ is unique up to isomorphisms. Thus, we denote the complex envelope $X$ by $E_{\mathbb{C}}(A)$. The set of real points of $E_{\mathbb{C}}(A)$ is denoted by $E_{\mathbb{R}}(A)$. We regard $E_{\mathbb{R}}(A)$ to be a real algebraic variety.

(3) If $A$ is a non-singular semialgebraic curve, and $X := E_{\mathbb{C}}(A)$. We define a sheaf $\mathcal{O}_A$ by $\mathcal{O}_A := \mathcal{O}_X|_A^m = \mathcal{O}_X|_A^p$. The *dualizing sheaf* $\omega_A$ is defined by $\omega_A := \mathcal{O}_X(K_X)|_A^m = \mathcal{O}_X(K_X)|_A^p$.

Note that if $A$ is a non-singular semialgebraic curve, then

$$g(A) = \dim_{\mathbb{R}} H^0(A,\, \omega_A) = \dim_{\mathbb{R}} H^1(A,\, \mathcal{O}_A).$$

Since $E_{\mathbb{C}}(A)$ is unique, many konwn results for algebraic curves over $\mathbb{R}$ are valid in our theory. For example, see [7], [11] and its references.

**Proposition 5.2.** *Let $A$ be a non-singular semialgebraic curve, and $X = E_{\mathbb{C}}(A)$.*

(1) *Take points $P_1,\ldots,\, P_r \in \mathrm{Int}(A)$, and let $U := A - \{P_1,\ldots,\, P_r\}$. Then, $U_X^p = U_X^m = X - \{P_1,\ldots,\, P_r\}$.*

(2) *Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module which is real on $A$. Then, $\mathcal{L}|_A$ is an invertible $\mathcal{O}_A$-module.*

*Proof.* (1) Since $U_X := U_X^m = X - \{P_1,\ldots,\, P_r\}$ is the maximal open Zariski subset of $X$ such that $U_X \cap A = U$, we have $U_X^m = U_X$. Since there exists a $f \in \mathrm{Rat}(A)$ whose zeros are $\{P_1,\ldots,\, P_r\}$ and whose poles are in $X - A$. Thus, we have $U_X^p = U_X$.

(2) For any point $P \in A$, there exists an affine open neighborhood $P \in U \subset A$ and $e \in \mathcal{L}(U_X)(\mathbb{R})$ such that $\mathcal{L}|_{U_X} = \mathcal{O}_X|_{U_X} \cdot e = \mathcal{O}_A \cdot e$, where $U_X := U_X^p = U_X^m$. Thus $\mathcal{L}|_A$ is an invertible $\mathcal{O}_A$-module. □

**Proposition 5.3.** *Let $A$ be a non-singular semialgebraic curve, and $X = E_{\mathbb{C}}(A)$. Take points $P_1,\ldots, P_r \in \mathrm{Int}(A)$, and $k_1,\ldots, k_r \in \mathbb{Z}$. Put $D := \sum k_i P_1$ as a divisor on $X$. Then, there exists the unique invertible $\mathcal{O}_A$-module $\mathcal{L}$ such that $\mathcal{L} \otimes_{\mathcal{O}_A} \mathcal{O}_X = \mathcal{O}_X(D)$.*

We say $D$ is a *Weil divisor* on $A$, and denote $\mathcal{L} = \mathcal{O}_A(D)$. We denote $\deg D := \sum k_i \in \mathbb{Z}$.

*Proof.* $\mathcal{L} := \mathcal{O}_X(D)|_A$ satisfies the above conditions. □

**Proposition 5.4.** *Let $A$ is a non-singular semialgebraic curve, and $D$, $D'$ be Weil divisors on $A$. If $\mathcal{O}_A(D) \cong \mathcal{O}_A(D')$, then $\deg D = \deg D'$.*

When $\mathcal{O}_A(D) \cong \mathcal{O}_A(D')$, we denote $D \sim D'$ and say that $D$ and $D'$ are *linearly equivalent.*

*Proof.* We regards $D$ and $D'$ to be divisors on $X = E_{\mathbb{C}}(A)$. Note that all components of $D$ and $D'$ are points in $\mathrm{Int}(A)$, and $D - D' = \mathrm{div}(f)$ by a certain $f \in \mathrm{Rat}_A^{\times}(X)$. Thus, the result is trivial. □

**Proposition 5.5.** *Let $A$ be a non-singular semialgebraic curve. Then $\omega_A$ is an invertible $\mathcal{O}_A$-module.*

*Proof.* Take $X = E_{\mathbb{C}}(A)$ and $K_X$. There exists a real divisor $D$ on $X$ such that $K_X \sim D$. Then $\omega_A \cong \mathcal{O}_A(D)$. □

**Proposition 5.6.** *Let $A$ be a non-singular semialgebraic curve.*

(1) *If $g(A) = 0$, then $A$ is isomorphic to a certain semialgebraic subset of $\mathbb{P}^1_{\mathbb{R}}$. Moreover, if $A$ is connected, then $A$ is isomorphic to one of $[0, 1]$, $(0, 1)$, $(0, 1]$, $\mathbb{R}$ or $\mathbb{P}^1_{\mathbb{R}}$.*
(2) *If $g(A) = 1$, then $E_{\mathbb{C}}(A)$ is isomorphic to a certain non-singular cubic curve in $\mathbb{P}^2_{\mathbb{C}}$, and $A$ is isomorphic to a certain semialgebraic subset of a real cubic curve $E_{\mathbb{R}}(A) \subset \mathbb{P}^2_{\mathbb{R}}$.*

*Proof.* Trivial. □

**Definition 5.7.**(Normalization) Let $A$ be a semialgebraic curve and $X$ be a complex closure of $A$. Let $\Pi\colon Y \to X$ be the normalization of $X$. For any self-conjugate affine open subset $U \subset X$, $\mathcal{O}_Y(\Pi^{-1}(U))$ is the integral closure of $\mathcal{O}_X(U)$ in $\mathrm{Rat}(X)$. The complex conjugate map $J\colon \mathcal{O}_X(U) \longrightarrow \mathcal{O}_X(U)$ can be extended to $J\colon \mathcal{O}_Y(\Pi^{-1}(U)) \longrightarrow \mathcal{O}_Y(\Pi^{-1}(U))$. Thus, $Y$ is also self-conjugate. Let $\Pi(\mathbb{R})\colon Y(\mathbb{R}) \longrightarrow X(\mathbb{R})$ be the restriction of $\Pi$. For $y \in Y(\mathbb{R})$, $\mathcal{R}_{Y(\mathbb{R}),y} \otimes_{\mathbb{R}} \mathbb{C} \cong \mathcal{O}_{Y,y}$. Thus $Y(\mathbb{R})$ is also normal. Let $B := (\Pi(\mathbb{R}))^{-1}(A)$ as semialgebraic variety, and $\pi\colon B \to A$ be the restriction of $\Pi(\mathbb{R})$. For $y \in B$, $\mathcal{R}_{B,y} = \mathcal{R}_{Y(\mathbb{R}),y}$ is a normal ring. We say $\pi\colon B \to A$ is the *normalization* of $A$. Since $Y$ is non-singular complex algebraic curve, $B$ is non-singular semialgebraic curve and $Y = E_{\mathbb{C}}(B)$.

## Section 6. Semialgebraic surfaces.

**Definition 6.1.** (1) Let $A$ be a non-singular semialgebraic surface, and $C$ be a closed semialgebraic curve $C \subset A$. If there exists a non-singular semialgebraic surface $B$, and a surjective regular map $\pi\colon A \to B$ such that $\pi(C)$ is a point and that $\pi\colon (A - C) \longrightarrow (B - \pi(C))$ is an isomorphism, then $\pi\colon A \to B$ is called a *smooth contraction* of $C$.

(2) Let $A$ be a non-singular semialgebraic surface. $A$ is called *minimal*, if there does not exist any closed semialgebraic curve $C \subset A$ which has a smooth contraction $\pi: A \to B$ of $C$.

(3) Let $A$ be a non-singular semialgebraic surface, and $X$ be a non-singular complex envelope of $A$. $X$ is called a *minimal complex envelope* of $A$, if for any non-singular complex envelope $Y$ of $A$ there exists a regular birational map $\varphi: Y \to X$ such that $\varphi|_A = \mathrm{id}_A$, then $\varphi$ is an isomorphism.

**Proposition 6.2.** *Let $A$ be a non-singular semialgebraic surface, and $X$ be a non-singular complex envelope of $A$. Take a closed semialgebraic curve $C \subset A$ with $\mathrm{Zar}_A(C) = C$, and let $C_X := \mathrm{Zar}_X(C)$. If $C_X \cong \mathbb{P}^1_{\mathbb{C}}$ and $(C_X^2)_X$ is a positive odd integer, then $C$ has a smooth contraction $\pi: A \to B$.*

*Proof.* Put $(C_X^2)_X = 2n - 1$ ($n \in \mathbb{N}$). Take imaginal points $P_1, \ldots, P_n \in C_X$ such that $P_1, \ldots, P_n, \overline{P_1}, \ldots, \overline{P_n}$ are distinct points, here $\overline{P_k}$ is the complex conjugate point of $P_k$. Let $\pi: Y \to X$ be the blowing up at $P_1, \ldots, P_n, \overline{P_1}, \ldots, \overline{P_n}$. $Y$ is also a non-singular complex envelope of $A$. Put $C_Y := \mathrm{Zar}_Y(C)$. Then $(C_Y^2)_Y = -1$. Since $C_Y$ is a $(-1)$-curve, $C_Y$ has a smooth contraction. □

**Corollary 6.3.** *If $A$ is a non-singular semialgebraic surface with $\kappa(A) = -\infty$, then $A$ is not minimal.*

This result does not conflict with [8], [9], [10] and so on, because definitions of real algebraic surface are completely different. Note that any rational semialgebraic surface contains a curve which is isomorphic to $\mathbb{P}^1_{\mathbb{R}}$. In fact, $A$ contains a domain which is isomorphic to a disc, and a disc contains circles.

**Lemma 6.4.** *Let $A$ be a non-singular semialgebraic surface, and $X$ be a non-singular complex envelope of $A$. Assume that $E \subset X$ be a $(-1)$-curve. Let $\overline{E}$ be the complex conjugate of $E$. Moreover, we assume that $(E \cdot \overline{E})_X \geq 1$. Then, $\kappa(X) = -\infty$.*

*Proof.* Let $\pi: X \to Y$ be the contraction of $E$, $m := (E \cdot \overline{E})_X$, $C_X := \overline{E}$, and $C_Y := \pi(C_X) \subset Y$.

(1) We consider the case $m = 1$. Then $C_Y$ is a smooth rational curve with $(C_Y^2)_Y = 0$. Then, $Y$ is a ruled surface. Thus $\kappa(X) = \kappa(Y) = -\infty$. (See [3] Cap. V, Prop. 4.3.)

(2) We consider the case $m \geq 2$. Then $C_Y$ is a singular rational curve with $(C_Y^2)_Y = m - 1 \geq 1$. Note that $(C_Y \cdot K_Y)_Y = (C_X \cdot \pi^* K_Y)_X = (C_X \cdot (K_X - E))_X = -1 - m$. Take non-singular points $P_1, \ldots, P_{m-1} \in L$. Let $\varphi: Z \to Y$ be the blowing up at $P_1, \ldots, P_{m-1}$, and let $C_Z \subset Z$ be the proper transform of $C_Y$. Then $(C_Z^2)_Z = 0$. Put $E_i = \varphi^{-1}(P_i)$. Then

$$\begin{aligned}(C_Z \cdot K_Z)_Z &= (C_Z \cdot \pi^* K_Y)_Z + \sum_{i=1}^{m-1} (C_Z \cdot E_i)_Z \\ &= (C_Y \cdot K_Y)_Y + (m-1) = (-1-m) + (m-1) = -2.\end{aligned}$$

(2-1) Consider the case $H^1(Z, \mathcal{O}_Z) = 0$. Then, since $0 \longrightarrow H^0(\mathcal{O}_Z) \longrightarrow H^0(\mathcal{O}_Z(C_Z)) \longrightarrow H^0(\mathcal{O}_{C_Z}(C_Z)) \longrightarrow 0$ is exact, we have $h^0(\mathcal{O}_Z(C_Z)) = 2$. The divisor $C_Z$ define a regular map $\Phi: Z \to \mathbb{P}^1_{\mathbb{C}}$.

Since $Z$ is non-singular, general fibres of $\Phi$ are smooth curves. Let $F := \Phi^{-1}(Q)$ be a smooth fibre. If $F \cong \mathbb{P}^1$, then $Z$ is rational surface and $\kappa(X) = \kappa(Z) = -\infty$. Assume that $g(F) \geq 1$. Since $F \sim C_Z$, we have $-2 = (C_Z \cdot K_Z)_Z = (F \cdot K_Z)_Z = (F \cdot (K_Z + F))_Z = \deg K_F \geq 0$. A contradiction.

(2-2) Consider the case $q := h^1(Z, \mathcal{O}_Z) \geq 1$. Let $S$ be the Albanese variety of $Z$ and $\alpha\colon Z \to S$ be the Albanese map. Take the Stein factorization $\alpha\colon Z \xrightarrow{f} T \xrightarrow{g} S$. Note that $q(T) = q(S) = q$. Since any complex torus does not include a rational curve, and since $g\colon T \to S$ is a finite map, we have $f(C_Z)$ is a point. If $T$ is a surface, then $(C_Z^2)_Z < 0$. Thus, $\dim T = 1$. Since general fibres of $f\colon Z \to T$ are smooth curves, $C_Z$ is included in a singular fibre $f^{-1}(P)$. Put $F := f^*P$ as a divisor. Since $(C_Z \cdot F)_Z = 0$ and $(C_Z^2)_Z = 0$, $F$ must be irreducible. Thus $F = rC_Z$ for a certain $r \in \mathbb{N}$. Take a general fibre $F_1 := f^{-1}(Q)$. Then

$$-2r = r(C_Z \cdot K_Z)_Z = (F \cdot K_Z)_Z = (F_1 \cdot K_Z)_Z = \deg K_{F_1} \geq -2.$$

Thus, $r = 1$ and $F_1 \cong \mathbb{P}^1_{\mathbb{C}}$. Since $f\colon Z \to T$ is a ruled surface, we have $\kappa(X) = \kappa(Z) = -\infty$. □

**Proposition 6.5.** *Let $A$ be a non-singular semialgebraic surface with $\kappa(A) \geq 0$, and $X$ be a non-singular complex envelope of $A$. Then, there exists a non-singular self-conjugate surface $Y$ and a real birational morphism $\varphi\colon X \to Y$ such that $Y$ is a minimal surface. In this case, $B := \varphi(A)$ is minimal semialgebraic surface.*

*Proof.* Assume that $C \subset X$ be a $(-1)$-curve, and $\overline{C}$ be the complex conjugate of $C$. Then (1) $C \cap \overline{C} = \emptyset$, or (2) $C = \overline{C}$, by Lemma 6.4.

(1) Assume that $C \cap \overline{C} = \emptyset$. Then, there exists the smooth contraction $\pi\colon X \to X_1$ of $C$ and $\overline{C}$. $X_1$ is also a non-singular envelope of $A$. Note that $\pi$ is real morphism.

(2) Assume that $C = \overline{C}$. Let $\pi\colon X \to X_1$ be the smooth contraction of $C$. Note that $\pi$ is real morphism, and $B := \pi(A)$ is a non-singular semialgebraic surface such that $X_1$ is a complex envelope of $B$.

We can obtain a minimal model of $X$ by a composite of contractions of types (1) and (2). □

**Proposition 6.6.** *Let $A$ be a non-singular semialgebraic surface with $\kappa(A) \geq 0$. Then there exists a minimal complex envelope $X$ of $A$. Moreover $X$ is unique up to isomorphism.*

*Proof.* Let $X_0$ be a non-singular complex envelope of $A$. Assume that $X_0$ includes a $(-1)$-curve $C$ such that $\dim(C \cap A) \leq 0$. If $C = \overline{C}$, then let $\pi_0\colon X_0 \to X_1$ be the smooth contraction of $C$. If $C \neq \overline{C}$, then let $\pi_0\colon X_0 \to X_1$ be the smooth contraction of $C$ and $\overline{C}$. Inductively, we repeat this process. After some contractions, we obtain a minimal complex envelope $X$ of $A$.

We shall show that $X$ is unique. Let $\varphi\colon X \to Y$ be the minimal model of $X$. It is well known that $Y$ is unique. $\varphi$ can be decomposed as a composite of contractions of a $(-1)$-curve: $X = X_0 \xrightarrow{f_1} X_1 \xrightarrow{f_2} X_2 \xrightarrow{f_3} \cdots \xrightarrow{f_n} X_n = Y$. Consider $f_1\colon X_0 \to X_1$ which is the contraction of $C_0$. Note that $\dim(C_0 \cap A) = 1$. Let $C_1 \subset X_1$ be a $(-1)$-curve, and let $C_1'$ be the strict transform of $C_1$ to $X_0$. If $C_1 \cap A = \emptyset$, then $C_1'$ is a $(-1)$-curve such that $\dim(C_1' \cap A) \leq 0$. This is impossible for $X_0$ is a minimal complex envelope $X$ of $A$. Thus $C_1 \cap A \neq \emptyset$. Then $C_1' = \overline{C_1'}$ by Lemma 6.4.

So, every $f_i: X_{i-1} \to X_i$ is the contraction of a $(-1)$-curve $C_{i-1} \subset X_{i-1}$ such that $C_{i-1} = \overline{C_{i-1}}$ and $\dim(C_{i-1} \cap A) = 1$.

This implies that $X$ is unique. $\square$

**Definition 6.7.** Let $A$ be a non-singular semialgebraic surface with $\kappa(A) \geq 0$, and let $X$ be the minimal complex envelope of $A$. We define $\mathcal{O}_A := \mathcal{O}_X|_A^p$ and $\omega_A := \mathcal{O}_X(K_X)|_A^p$. $\omega_A$ is called the *dualizing sheaf* of $A$.

**Remark 6.8.** Let $A$ be a non-singular semialgebraic surface with $\kappa(A) \geq 0$, and let $X$ be the minimal complex envelope of $A$.

(1) It is easy to see that $\omega_A \otimes_{\mathcal{O}_A} \mathcal{R}_A \cong \Omega_A^2$, here $\Omega_A^2$ is defined in Definition 1.11.

(2) Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_A$-module, and let $\mathcal{F}_{\mathbb{C}} := \mathcal{F} \otimes_{\mathcal{O}_A} \mathcal{O}_X$. Then

$$H^i(U, \mathcal{F}) \otimes_{\mathbb{R}} \mathbb{C} \cong H^i(U_X^p, \mathcal{F}_{\mathbb{C}})$$

for $i \in \mathbb{Z}$ and any Zariski open subset $U \subset A$.

(3) $X$ is real K3 surface in the sense of [8], if and only if $V := X(\mathbb{R})$ is minimal with $\kappa(V) = 0$, $\omega_V \cong \mathcal{O}_V$ and $q(V) = 0$. Thus, the traditional definition of real K3 surfaces and real Enriques surfaces (see [5]) are valid for real algebraic surfaces $(V, \mathcal{R}_V)$ in our sense, under the assumption $\dim X(\mathbb{R}) = 2$.

Remember that even if $\mathcal{L}$ is a real invertible $\mathcal{O}_X$-module, $\mathcal{O}_A$-module $\mathcal{L}|_A$ may not be invertible.

**Proposition 6.9.** *Let $A$ be a non-singular semialgebraic surface with $\kappa(A) = 2$. Then $\omega_A$ is an invertible $\mathcal{O}_A$-module.*

*Proof.* Let $X$ be the minimal complex envelope of $A$, and $\varphi: X \to Y$ be the minimal model as the above proposition. Put $B = \varphi(A)$.

We shall show that $\omega_B = \mathcal{O}_B(K_Y)|_B$ is an invertible $\mathcal{O}_B$-module. Put $P_m := \dim_{\mathbb{C}} H^0($ $Y, \mathcal{O}_Y(mK_Y))$. There exists $m \in \mathbb{N}$ such that $P_m \geq 2$, $P_{m+1} \geq 2$, $\mathrm{Bs}\,|mK_Y| = \emptyset$, and $\mathrm{Bs}\,|(m+1)K_Y| = \emptyset$. Put $\mathcal{L}_m := \mathcal{O}_Y(mK_Y)|_B$, We may assume $mK_Y \geq 0$. Let $N := h^0(Y,$ $\mathcal{L}_m) - 1 \geq 1$, and $\Phi: Y \to \mathbb{P}_{\mathbb{C}}^N$ be a regular map defined by $|mK_Y|$. There exists a linear function $f_1$ on $\mathbb{P}_{\mathbb{C}}^N$ such that $mK_Y = \Phi^* V_{\mathbb{P}_{\mathbb{C}}^N}(f_1)$. For any Zariski open subset $\emptyset \neq U \subset B$. $\mathcal{L}_m|_{U_Y^p} \cong \mathcal{O}_Y|_{U_Y^p} \cdot \Phi^*(f_1)$. Thus, $\mathcal{L}_m$ is an invertible $\mathcal{O}_B$-module.

Similarly, $\mathcal{L}_{m+1}$ is an invertible $\mathcal{O}_B$-module. Then $\omega_B \cong \mathcal{H}om_{\mathcal{O}_B}(\mathcal{L}_m, \mathcal{L}_{m+1})$ is also an invertible $\mathcal{O}_B$-module.

Take an arbitral point $P \in A$. There exists an affine open set $\varphi(P) \in U \subset B$ such that $\omega_B|_U = \mathcal{O}_B|_U \cdot e$ for a certain $e \in \mathrm{Rat}(B)$. Take $P \in W \subset \varphi^{-1}(U)$. Then $\omega_A|_W = \mathcal{O}_A|_U \cdot \varphi^*(e)$. Thus $\omega_A$ is an invertible $\mathcal{O}_A$-module. $\square$

**Definition 6.10.**(Intersection number) Let $A$ be a non-singular semialgebraic surface with $\kappa(A) \geq 0$, $X$ be the minimal complex envelope of $A$, and $Y$ be any non-singular complex envelope of $A$,

(1) Take closed semialgebraic curves $C_1, C_2 \subset A$. Put $(C_1 \cdot C_2)_Y := \big(\mathrm{Zar}_Y(C_1) \cdot \mathrm{Zar}_Y(C_2)\big)_Y$. Then $(C_1 \cdot C_2)_Y \leq (C_1 \cdot C_2)_X$. Thus, we define the *intersection number* of $C_1$ and $C_2$ on $A$ by $(C_1 \cdot C_2)_A := (C_1 \cdot C_2)_X$.

(2) When all $C_k$ are closed semialgebraic curves on $A$, $D = \sum_{k=1} m_k C_k$ ($m_k \in \mathbb{Z}$) is called a *Weil divisor* on $A$. Using $D_X = \sum_{k=1} m_k \operatorname{Zar}_X(C_k)$, some theories can be constructed for Weil divisors on $A$.

On the other hand, we cannot define the intersection number on rational semialgebraic surfaces. As is mentioned in [8], on a non-singular complete rational real algebraic surface $A$, $(C_1 \cdot C_2)_A$ has a sense only in $\mathbb{Z}/2\mathbb{Z}$. If $A$ is non-complete, this parity is not an invariant.

**Example 6.11.**

(1) Let $A = \mathbb{P}^2_{\mathbb{R}}$ and $L \subset A$ be a line. Then, for any $n \in \mathbb{Z}$, there exists a non-singular complex envelope $X$ of $A$ such that $\big(\operatorname{Zar}_X(L)^2\big)_X = 2n+1$.

(2) Let $A \subset \mathbb{P}^2_{\mathbb{R}}$ be a non-complete semialgebraic surface, and $L \subset A$ be a non-complete line. Then, for any $n \in \mathbb{Z}$, there exists a non-singular complex envelope $X$ of $A$ such that $\big(\operatorname{Zar}_X(L)^2\big)_X = n$.

*Proof.* Consider blowing ups and smooth contractions. ∎

If $\dim(C_1 \cap C_2) \leq 1$, we can define the local intersection numver $I_P(C_1,\, C_2)$ at $P \in C_1 \cap C_2$. But, as the above example, we cannot define the self intersection number $(C_1^2)_A$.

## Section 7. Aspects of theory of semialgebraic varieties.

We intoduced a basic concept of semialgebraic quasi-varieties, and studied their basic properties. In view of theory of algebraic inequalities, the following concepts are also important:

(1) The critical decomposition of semialgebraic variety.

(2) Signed linear systems.

About these topics, please see [1] and [2]. The following proposition is also one of basic theorems about semialgebraic varieties.

**Proposition 7.1.** Let $V$, $W$ be complete real algebraic varieties and $\varphi\colon V \to W$ be a regular map. Let $A \subset V$ be a closed semialgebraic variety with $\operatorname{Zar}_V(A) = V$. Then,

$$\partial\big(\varphi(A)\big) \subset \varphi\big(\operatorname{Sing}(\varphi) \cup \operatorname{Sing}(A) \cup \partial A\big).$$

This is proved in § 2 of [2].

Someone may say they "There are not enough applications of semialgebraic varieties. So, I think it is uncertain, unreliable object." But please consider the fact that since $V/G$ is not always real algebraic variety, moduli or many other important notions in theory of complex algebraic varieties cannot be introduced to real algebraic varieties.

## References


[1] T. Ando, Discriminants of Cyclic Homogeneous Inequalities of Three Variables, J. Alg., to appear.

[2] T. Ando, Theory of PSD Cones on Semialgebraic Varieties, Preprint.

[3] W. Barth & H. Hulek & C. Peters & A. van de Ven, Compact Complex Surfaces (2nd ed.), Springer (2003).

[4] J. Bochnak & M. Coste & M.F. Roy, Real Algebraic Geometry, Springer (1998).

[5] A. Degtyarev, I. Itenberg, V. Kharlamov, Real Enriques Surfaces, Lect. Notes Math. 1746 (2000)

[6] R. Hartshorne, Algebraic Geometry, Springer (1977).

[7] J. Huisman, Non-special divisors on real algebraic curves and embeddings into real projective spaces, Annali di Mat. **182**, (2003), 21-35.

[8] J. Kollár, Real Algebraic Surfaces, anXiv:alg-geom/9712003, (1997).

[9] J. Kollár, The Topology of Real Algebraic Varieties, Current Devel. in Math., (2000), 197-231.

[10] V. Nikulin & S. Saito, Real K3 Surfaces with Non-symplectic Involutions and Applications, Proc. London Math. Soc. (3) **90**, (2005) 591-654.

[11] R. Silhol, Real Algebraic Surfaces, Lect. Notes in Math. 1392, Springer