Some operator monotone functions related to
Petz-Hasegawa’s functions

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Abstract
Let \( f \) be an operator monotone function on \([0, \infty)\) with \( f(t) \geq 0 \) and \( f(1) = 1 \). If \( f(t) \) is neither the constant function 1 nor the identity function \( t \), then
\[
h(t) = \frac{(t - a)(t - b)}{(f(t) - f(a))(f^2(t) - f^2(b))} \quad t \geq 0
\]
is also operator monotone on \([0, \infty)\), where \( a, b \geq 0 \) and \( f^\#(t) = t f(t) \geq 0 \).

1 Introduction

We call a real continuous function \( f(t) \) on an interval \( I \) operator monotone on \( I \) (in short, \( f \in \mathcal{P}(I) \)), if \( A \preceq B \) implies \( f(A) \preceq f(B) \) for any self-adjoint matrices \( A, B \) with their spectrum contained in \( I \). In this paper, we consider only the case \( I = [0, \infty) \) or \( I = (0, \infty) \). We denote \( f \in \mathcal{P}^+(I) \) if \( f \in \mathcal{P}(I) \) satisfies \( f(t) \geq 0 \) for any \( t \in I \).

Let \( \mathbb{H}_+ \) be the upper half-plane of \( \mathbb{C} \), that is,
\[
\mathbb{H}_+ = \{ z \in \mathbb{C} \mid \text{Im} z > 0 \} = \{ z \in \mathbb{C} \mid |z| > 0, \ 0 < \arg z < \pi \},
\]
where \( \text{Im} z \) (resp. \( \arg z \)) means the imaginary part (resp. the argument) of \( z \). As Loewner’s theorem, it is known that \( f \) is operator monotone on \( I \) if and only if \( f \) has an analytic continuation to that maps \( \mathbb{H}_+ \) into itself and also has an analytic continuation to the lower half-plane \( \mathbb{H}_-(= -\mathbb{H}_+) \), obtained by the reflection across \( I \) (see [1],[2] ).

D. Petz [5] proved that an operator monotone function \( f : [0, \infty) \rightarrow [0, \infty) \) satisfying the functional equation
\[
f(t) = tf(t^{-1}) \quad t \geq 0
\]
is related to a Morozova-Chentsov function which gives a monotone metric on the manifold of \( n \times n \) density matrices. In the work [6], the concrete functions (Petz-Hasegawa’s functions)
\[
f_a(t) = a(1-a)\frac{(t-1)^2}{(t^a-1)(t^{1-a}-1)} \quad (-1 < a < 2)
\]
appeared and their operator monotonicity was proved. V.E.S. Szabo introduced an interesting idea for checking their operator monotonicity in [7]. We use a
similar idea as Szabo’s in our argument. M. Uchiyama [8] proved the operator monotonicity of the following extended functions:

\[
\frac{(t - a)(t - b)}{(t^p - a^p)(t^{1-p} - b^{1-p})}
\]

for \(0 < p < 1\) and \(a, b > 0\). It is well known that the function \(t^p (0 \leq p \leq 1)\) is operator monotone as Loewner-Heinz’s inequality. The main result of this paper is as follows:

**Theorem 1.** Let \(a\) and \(b\) be non-negative real. If \(f \in \mathbb{P}_+[0, \infty)\) and both \(f\) and \(f^\sharp\) are not constant, then

\[
h(t) = \frac{(t - a)(t - b)}{(f(t) - f(a))(f^\sharp(t) - f^\sharp(b))}
\]

is operator monotone on \([0, \infty)\), where

\[
f^\sharp(t) = \frac{t}{f(t)} \quad t \geq 0.
\]

## 2 Proof of Main result

The following statement was proved by M. Uchiyama [8]. Here we prove it based on the fact that any operator monotone function is a Pick function, but this is essentially same as his proof.

**Proposition 2.** Let \(f \in \mathbb{P}(0, \infty)\) be not constant and \(a\) be positive real. Then we have that

\[
g(t) = \frac{t - a}{f(t) - f(a)}
\]

is operator monotone on \([0, \infty)\).

**Proof.** Since \(f(\mathbb{H}_+) \subset \mathbb{H}_+\), \(\text{Im}(f(z) - f(a)) = \text{Im}(f(z)) > 0\) for all \(z \in \mathbb{H}_+\). Therefore \(g(z) = \frac{z - a}{f(z) - f(a)}\) is holomorphic on \(\mathbb{H}_+\).

Since \(f\) is not constant, \(f'(a) \neq 0\). We also have

\[
\lim_{t \to 0^+} g(t) = \begin{cases} \frac{a}{f(a) - f(0)} & (> 0) \quad \text{if } f(0) \text{ exists} \\ 0 & \text{otherwise} \end{cases}.
\]

This means \(g(z)\) is continuous on \(\mathbb{H}_+ \cup [0, \infty)\) and we have \(g([0, \infty)) \subset [0, \infty)\).

By Loewner’s theorem, we have the following integral representation of \(f\): for \(z \in \mathbb{H}_+ \cup (0, \infty)\)

\[
f(z) = \alpha + \beta z + \int_0^\alpha \left(-\frac{1}{x + z} + \frac{x}{x^2 + 1}\right) d\nu(x) \quad (\alpha \in \mathbb{R}, \beta \geq 0),
\]

where \(\nu\) is a positive measure on \((0, \infty)\) such that

\[
\int_0^\infty \frac{1}{x^2 + 1} d\nu(x) < \infty.
\]
Using this relation, we have

\[
g(z) = \frac{1}{\beta + \int_0^\infty \frac{1}{(x + z)(x + a)} d\nu(x)}.
\]

If we show that \(\text{Im}(g(z)) \geq 0\) for any \(z \in \mathbb{H}_+\), then \(g \in P_{+}[0,\infty)\). We remark that \(z \in \mathbb{H}_+ \Rightarrow 0 < \arg z < \pi \Rightarrow 0 < \arg(x + z) < \pi \quad (x \in [0,\infty))\)

\[
\Rightarrow \pi < \arg \frac{1}{(x + z)(x + a)} < 2\pi \quad (x \in [0,\infty))
\]

\[
\Rightarrow \text{Im} \frac{1}{(x + z)(x + a)} < 0 \quad (x \in [0,\infty)).
\]

So we have

\[
\text{Im}(\beta + \int_0^\infty \frac{1}{(x + z)(x + a)} d\nu(x)) = \int_0^\infty \text{Im} \frac{1}{(x + z)(x + a)} d\nu(x) < 0.
\]

This shows that \(\text{Im}(g(z)) \geq 0\).

For \(f \in \mathbb{P}_{+}[0,\infty)\), we have the following integral representation:

\[
f(z) = f(0) + \beta z + \int_0^\infty \frac{\lambda}{z + \lambda} d\nu(\lambda) \quad (z \in \mathbb{H}_+ \cup [0,\infty)),
\]

where \(\beta \geq 0\) and

\[
\int_0^\infty \frac{\lambda}{1 + \lambda} d\nu(\lambda) < \infty.
\]

When \(f(0) \geq 0\) (i.e., \(f \in \mathbb{P}_{+}[0,\infty)\)), \(f(z)\) can be approximated by

\[
\sum_{i=1}^n f_i(z),
\]

where each \(f_i(z)\) satisfies that

\[
0 < \arg f_i(z) \leq \arg z \text{ whenever } 0 < \arg z < \pi.
\]

So we have \(0 < \arg f(z) \leq \arg z \text{ whenever } 0 < \arg z < \pi\).

By using elementary geometry, it easily holds that

\[
\arg(z - |z|) = \frac{\pi + \arg z}{2}
\]

for any \(z \in \mathbb{H}_+\). So we can get the following statement:

**Lemma 3.** For any \(z \in \mathbb{H}_+\) and \(l > 0\), we have

\[
\arg z < \arg(z - l) < \frac{\pi + \arg z}{2} \quad \text{if } |z| > l.
\]
Now we can prove the following theorem and remark that Theorem 1 easily follows from this:

**Theorem 4.** Let \( f, g \in \mathbb{P}_+[0, \infty) \) and both \( f \) and \( g \) be non-constant. If \( \frac{f(t)g(t)}{t} \) is operator monotone on \([0, \infty)\), then

\[
h(t) = \frac{(t-a)(t-b)}{(f(t)-f(a))(g(t)-g(b))}
\]

is also operator monotone on \([0, \infty)\) for any \( a, b \geq 0 \).

**Proof.** By the assumption we can consider the function

\[
h(z) = \frac{(z-a)(z-b)}{(f(z)-f(a))(g(z)-g(b))}, \quad z \in \mathbb{H}_+.
\]

It is clear that \( h(z) \) is holomorphic on \( \mathbb{H}_+ \). We may consider values, by taking the limit,

\[
h(a) = \frac{a-b}{f'(a)(g(a)-g(b))} \quad \text{and} \quad h(b) = \frac{b-a}{g'(b)(f(b)-f(a))}.
\]

So we have \( h([0, \infty)) \subset [0, \infty) \).

We assume that \( f(z) \) and \( g(z) \) are continuous on the closure \( \overline{\mathbb{H}_+} \) of \( \mathbb{H}_+ \) and

\[
f(t) - f(a) \neq 0 \quad \text{and} \quad g(t) - g(b) \neq 0 \quad \text{for any} \quad t \in (-\infty, 0).
\]

Then \( h(z) \) is continuous on \( \overline{\mathbb{H}_+} \).

Since \( \frac{z-a}{f(z)-f(a)} \) and \( \frac{z-b}{g(z)-g(b)} \) belong to \( \mathbb{P}_+[0, \infty) \) by Proposition 2, it is clear that \( \arg h(z) \geq 0 \) if \( 0 \leq \arg z \leq \pi \).

In the case \( z \in (-\infty, 0) \), i.e., \( |z| > 0 \) and \( \arg z = \pi \), we have

\[
\arg h(z) = \arg(z-a) - \arg(f(z) - f(a)) + \arg(z-b) - \arg(g(z) - g(b))
\]

\[
\leq \pi - \arg f(z) + \pi - \arg g(z)
\]

\[
\leq 2\pi - \arg z = \pi \quad (\text{since } \arg f(z) + \arg g(z) - \arg z \geq 0).
\]

So \( 0 \leq \arg h(z) \leq \pi \).

In the case \( z \in \mathbb{H}_+ \) satisfying \( |z| > \max\{a, b\} \), it holds that

\[
\arg z < \arg(z-a), \arg(z-b) < \frac{\pi + \arg z}{2}
\]

by above lemma. Since

\[
\arg h(z) = \arg(z-a) - \arg(f(z) - f(a)) + \arg(z-b) - \arg(g(z) - g(b))
\]

\[
\leq \frac{\pi + \arg z}{2} - \arg f(z) + \frac{\pi + \arg z}{2} - \arg g(z)
\]

\[
= \pi + \arg z - \arg f(z) - \arg g(z) \leq \pi,
\]

we have \( 0 < \arg h(z) < \pi \).
For $r > 0$, we define $H(r) = \{ z \in \mathbb{C} \mid |z| \leq r, \text{Im} z \geq 0 \}$. Whenever $r > l = \max\{a, b\}$, we can get

$$0 \leq \arg h(z) \leq \pi$$
on the boundary of $H(r)$. Since $h(z)$ is holomorphic on $H(r)$, Im$h(z)$ is harmonic on $H(r)$. Because $\text{Im} h(z) \geq 0$ on the boundary of $H(r)$, we have $h(H(r)) \subset \mathbb{H}_+$ by the minimum principle of the harmonic function. This implies

$$h(\mathbb{H}_+) = h(\bigcup_{r > l} H(r)) \subset \bigcup_{r > l} h(H(r)) \subset \mathbb{H}_+,$$

and $h \in \mathbb{P}_+[0, \infty)$.  

In general case, we set

$$\frac{f(t)g(t)}{t} = F(t)$$

and $\tilde{f} \in \mathbb{P}_+[0, \infty)$. We define the function $f_p$, $\tilde{f}_p$ and $g_p$ $(0 < p < 1)$ as follows:

$$f_p(z) = f(z^p), \quad \tilde{f}_p(z) = \tilde{f}(z^p), \quad \text{and} \quad g_p(z) = (\tilde{f}_p)^2(z) = \frac{z}{\tilde{f}_p(z)} = \frac{zF(z^p)}{f(z^p)} = z^{1-p}g(z^p)$$

for $z \in \mathbb{H}_+$. Then we have $f_p$, $g_p \in \mathbb{P}_+[0, \infty)$ and

$$h_p(z) = \frac{(z - a)(z - b)}{(f_p(z) - f_p(a))(g_p(z) - g_p(b))}$$

is holomorphic on $\mathbb{H}_+$ and continuous on $\mathbb{H}_+$. By the fact $\frac{f_p(t)g_p(t)}{t} = F(t^p)$ is operator monotone on $[0, \infty)$, $h_p(t)$ becomes operator monotone on $[0, \infty)$. Since

$$h_p(t) = \frac{(t - a)(t - b)}{(f_p(t) - f_p(a))(g_p(t) - g_p(b))} \quad \text{for} \quad t \geq 0,$$

we have

$$\lim_{p \to 1^-} h_p(t) = h(t).$$

So we can get the operator monotonicity of $h(t)$.

We can generalize this result for many operator monotone functions under some assumption([4]).

We remark that, for $f \in \mathbb{P}[0, \infty)$, $f_0$ belongs to $\mathbb{P}_+[0, \infty)$, where we put $f_0(t) = f(t) - f(0)$. Let $g \in \mathbb{P}_+[0, \infty)$ and $\frac{f_0(t)g(t)}{t}$ be operator monotone on
\[0, \infty). \] Under the assumption that \( f \) and \( g \) are not constant, we have
\[
\frac{(t-a)(t-b)}{(f_0(t) - f_0(a))(g(t) - g(b))} = \frac{(t-a)(t-b)}{(f(t) - f(a))(g(t) - g(b))} \in \mathbb{P}_+ [0, \infty)
\]
for any \( a, b \geq 0 \).

**Corollary 5.** Let \( f \in \mathbb{P}_+ (0, \infty) \) and both \( f \) and \( f^2 \) be not constant. For any \( a > 0 \), we define
\[
h_a(t) = \frac{(t-a)(t-a^{-1})}{(f(t) - f(a))(f^2(t) - f^2(a^{-1}))} \quad t \in (0, \infty).
\]
Then we have

1. \( h_a \) is operator monotone on \((0, \infty)\).
2. \( f(t) = t \cdot f(t^{-1}) \) implies \( h_a(t) = t \cdot h_a(t^{-1}) \).
3. \( a = 1 \) and \( f(t^{-1}) = f(t)^{-1} \) implies \( h_1(t) = t \cdot h_1(t^{-1}) \).

**Proof.** We can directly prove (1) from theorem 3. Because
\[
t \cdot h_a(t^{-1}) = \frac{t(t^{-1} - a)(t^{-1} - a^{-1})}{(f(t^{-1}) - f(a))(f^2(t^{-1}) - f^2(a^{-1}))} = \frac{(t-a)(t-a^{-1})}{t(f(t^{-1}) - f(a))(f^2(t^{-1}) - f^2(a^{-1}))},
\]
we can compute
\[
t(f(t^{-1}) - f(a))(f^2(t^{-1}) - f^2(a^{-1})) - (f(t) - f(a))(f^2(t) - f^2(a^{-1})) = (f(t^{-1}) - f(a))(1/f(t^{-1}) - t/a f(a^{-1})) - (f(t) - f(a))(t/f(t) - 1/a f(a^{-1})) = 0
\]
if it holds \( f(t) = t \cdot f(t^{-1}) \) or \( a = 1, f(t^{-1}) = f(t)^{-1} \). So we have (2) and (3).

**Example 6.** Using this corollary, we can repeatedly construct an operator monotone function \( h(t) \) on \([0, \infty)\) satisfying the relation
\[
h(t) = t \cdot h(t^{-1}) \quad t > 0.
\] (*)

If we choose \( p \) \((0 < p < 1)\) as \( f(t) \) in Corollary 5(3),
\[
h(t) = \frac{(t-1)^2}{(p-1)(t^{1-p} - 1)}.
\]
If we choose \(\frac{(t-1)^2}{(t^p-1)(t^{1-p}-1)}\) as \(f(t)\) in Corollary 5(2),

\[
h(t) = \frac{t - a}{(t - 1)^2} - \frac{(a - 1)^2}{(t^p - 1)(t^{1-p} - 1)}
\times \frac{t - a^{-1}}{t(t^p - 1)(t^{1-p} - 1)} - \frac{a(a - p - 1)(a^p - 1)}{(a - 1)^2}
\]

for \(a > 0\). If we choose \(t^p + t^{1-p}\) \((0 < p < 1)\) as \(f(t)\) in Corollary 5(2),

\[
h(t) = \frac{t - a}{t^p + t^{1-p} - a^p - a^{1-p}} \times \frac{1}{t^{p-1} + t^{-p} - a^p + a^{1-p}}
\]

\[
= \sqrt{t} (\cosh(\log t) - \cosh(\log a)) / \left(\cosh(\log \sqrt{t}) - \cosh(\log t + \log(t^p + t^{1-p}) - \log(a^p + a^{1-p}))\right).
\]

These functions, \(h \in \mathbb{P} \times [0, \infty)\), satisfy the relation (*)

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