A variant of Jacobi type formula for Picard curves

Keiji Matsumoto (Department of Mathematics, Hokkaido Univ.)
and Hironori Shiga (Graduate School of Mathematics and Physics, Chiba Univ.)

Sept. 21,2008

1 Introduction

Start from the family of elliptic curves

$$E(\lambda) : w^2 = z(z-1)(z-\lambda) \quad \lambda(\lambda-1) \neq 0.$$

For a real parameter λ in the interval $(0,1)$, take the ratio of the periods

$$\tau = \int_1^\infty \frac{dz}{\sqrt{z(z-1)(z-\lambda)}} \Bigg/ \int_{-\infty}^0 \frac{dz}{\sqrt{z(z-1)(z-\lambda)}} \quad \text{with } \frac{\tau}{i} > 0.$$

We have the theta representation of the λ-invariant

$$\lambda(\tau) = \frac{\vartheta_{01}^4(\tau)}{\vartheta_{00}^4(\tau)}. \tag{1.1}$$

Here, ϑ_{jk} indicates the Jacobi theta constant

$$\vartheta_{jk}(\tau) = \sum_{n \in \mathbb{Z}} \exp[\pi i (n + \frac{j}{2})^2 \tau + 2\pi i (n + \frac{j}{2})\frac{k}{2}] \quad \text{for } \tau \in \boldsymbol{H} = \{\mathrm{Im}\ \tau > 0\}.$$

The classical theorem of arithmetic geometric mean by Gauss says

$$\frac{1}{M(1,x)} = F\left(\frac{1}{2}, \frac{1}{2}, 1; 1-x^2\right). \tag{1.2}$$

Here $M(a,b)$ means the arithmetic geometric mean with the initial positive values a, b, and $F(\alpha, \beta, \gamma; x)$ indicates the Gauss hypergeometric function. Recall the Jacobi formula relating the elliptic integral and the theta constant:

Theorem 1.1. (see [J] p.235) *Under the relation (1.1) we have*

$$\vartheta_{00}^2(\tau) = F\left(\frac{1}{2}, \frac{1}{2}, 1; 1-\lambda\right) = \frac{1}{\pi} \int_1^\infty \frac{dz}{\sqrt{z(z-1)(z-(1-\lambda))}}. \tag{1.3}$$

Using the duplication formula

$$\begin{cases} \vartheta_{00}^2(2\tau) = \frac{1}{2}\left(\vartheta_0^2(\tau) + \vartheta_{01}^2(\tau)\right), \\ \vartheta_{01}^2(2\tau) = \vartheta_{00}(\tau)\vartheta_{01}(\tau), \end{cases}$$

and by putting $x = \vartheta_{01}^2(\tau)/\vartheta_{00}^2(\tau)$, we can derive the Gauss AGM theorem (1.2) from the above Jacobi formula. In fact, we have

$$M(\vartheta_{00}^2(\tau), \vartheta_{01}^2(\tau)) = \lim_{n\to\infty} \frac{1}{2}\left(\vartheta_{00}^2(2^n\tau) + \vartheta_{01}^2(2^n\tau)\right) = \lim_{\tau \to i\infty} \vartheta_{00}^2(\tau) = 1.$$

So we have $\vartheta_{00}^2(\tau)\, M(1,x) = 1$.

In this article we show a variant of this Jacobi formula for the Picard curves (2.1).

The Jacobi formula shows that the modular form $\vartheta_{00}^4(\tau)$ with respect to the principal congruence subgroup $\Gamma(2)$ of $PSL(2,\mathbb{Z})$ has an expression by the Gauss hypergeometric function $F(\frac{1}{2},\frac{1}{2},1;1-\lambda)$ of the algebraic parameter λ via the inverse of the period map (1.1) for the family of elliptic curves $E(\lambda)$.

Our result is a two dimensional exact analogy of this context. We use the Picard curves with two algebraic parameters λ_1, λ_2. The inverse of the period map is given by (3.1). Our modular form $\vartheta_0^3(u,v)$ is defined on a two dimensional complex ball $\mathscr{D} = \{2\mathrm{Re\ v} + |\mathrm{u}|^2 < 1\}$, that can be realized as a Shimura variety in the Siegel upper half space of degree 3 by the map (2.4). It is expressed in terms of the Appell hepergeometric function $F_1(\frac{1}{3},\frac{1}{3},\frac{1}{3},1;1-\lambda_1,1-\lambda_2)$.

As a consequence of our main theorem, we can give a new proof of the three terms AGM theorem already discovered in [K-S] (Theorem 2.2). Still, as a byproduct we show a one variable variant of the Jacobi formula (Theorem 5.1) for the Borweins curves (5.1).

2 Jacobi type formula for the Picard curves

2.1 The Picard modular form revisited

We express the Picard curve with the projective parameters:

$$C(\xi) : y^3 = x(x-\xi_0)(x-\xi_1)(x-\xi_2), \tag{2.1}$$

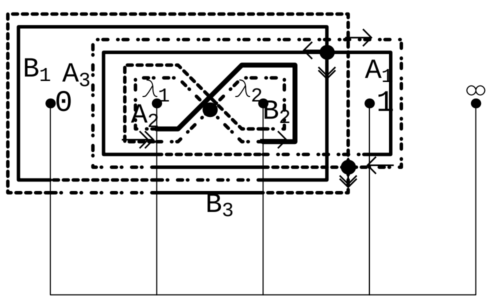
where

$$\xi \in \Xi = \{[\xi_0 : \xi_1 : \xi_2] \in \mathbb{P}^2(\mathbb{C}) : \xi_0\xi_1\xi_2(\xi_0-\xi_1)(\xi_1-\xi_2)(\xi_2-\xi_0) \neq 0\}.$$

It is a curve of genus three, The Jacobian variety $Jac(C(\xi))$ of $C(\xi)$ has a generalized complex multiplication by $\sqrt{-3}$ of type $(2,1)$. In fact we have a basis of holomorphic differentials

$$\varphi = \varphi_1 = \frac{dz}{w},\ \varphi_2 = \frac{dz}{w^2},\ \varphi_3 = \frac{zdz}{w^2}.$$

Put $\lambda_1 = \xi_1/\xi_0, \lambda_2 = \xi_2/\xi_0$. And we assume $0 < \lambda_1 < \lambda_2 < 1$. Under this condition, we choose the following basis of $H_1(C,\mathbb{Z})$ already used in [S]. Here we put cut lines starting from branch points in the lower half z-plane to get simply connected sheets. The real line (resp. the dotted line, the chained line) indicates a path on the first sheet (resp. the second sheet, the third sheet).



homology basis

Setting $\rho(z,w) = (z,\omega w)$, we have

$$B_3 = \rho(B_1),\ A_3 = -\rho^2(A_1),\ B_2 = -\rho^2(A_2),$$

here ω stands for $\exp[2\pi i/3]$. We have $A_iB_j=\delta_{ij}$. Put

$$\eta_0=\int_{A_1}\varphi,\ \eta_1=-\int_{B_3}\varphi,\ \eta_2=\int_{A_2}\varphi. \tag{2.2}$$

By the analytic continuation, they are multivalued analytic functions on the (λ_1,λ_2)-space $\mathbb{P}^2(\mathbb{C})$. It holds

$$\begin{pmatrix}\eta_0\\ \eta_1\\ \eta_2\end{pmatrix}=\begin{pmatrix}\int_{A_1}\varphi_1\\ -\omega^2\int_{B_1}\varphi_1\\ \int_{A_2}\varphi_1\end{pmatrix}=\begin{pmatrix}-\omega^2\int_{A_3}\varphi_i\\ -\int_{B_3}\varphi_i\\ -\omega^2\int_{B_2}\varphi_i\end{pmatrix},\ \begin{pmatrix}\int_{A_1}\varphi_i\\ -\omega\int_{B_1}\varphi_i\\ \int_{A_2}\varphi_i\end{pmatrix}=\begin{pmatrix}-\omega\int_{A_3}\varphi_i\\ -\int_{B_3}\varphi_i\\ -\omega\int_{B_2}\varphi_i\end{pmatrix}\quad \text{for } i=2,3. \tag{2.3}$$

Set

$$\Omega_1=(\int_{A_j}\varphi_i),\ \Omega_2=(\int_{B_j}\varphi_i),\quad (1\le i,j\le 3).$$

The normalized period matrix of $C(\xi)$ is given by $\Omega=\Omega_1^{-1}\Omega_2$. By the relations of periods (2.3) together with the symmetricity ${}^t\Omega=\Omega$, we can rewrite

$$\Omega=\Omega_1^{-1}\Omega_2=\begin{pmatrix}\frac{u^2+2\omega^2v}{1-\omega} & \omega^2u & \frac{\omega u^2-\omega^2 v}{1-\omega}\\ \omega^2 u & -\omega^2 & u\\ \frac{\omega u^2-\omega^2v}{1-\omega} & u & \frac{\omega^2u^2+2\omega^2v}{1-\omega}\end{pmatrix}, \tag{2.4}$$

here we put $u=\frac{\eta_2}{\eta_0},\ v=\frac{\eta_1}{\eta_0}$. So we set $\Omega=\Omega(u,v)$. The Riemann period relation Im $\Omega>0$ induces the inequality $2\mathrm{Re}\ (\mathrm{v})+|\mathrm{u}|^2<0$. We set

$$\mathscr{D}=\{\eta=[\eta_0:\eta_1:\eta_2]\in\mathbb{P}^2:{}^t\eta H\overline{\eta}<0\}=\{(u,v)\in\mathbb{C}^2:2\mathrm{Re}\ (\mathrm{v})+|\mathrm{u}|^2<0\},$$

here we put $H=\begin{pmatrix}0&1&0\\1&0&0\\0&0&1\end{pmatrix}$. We define our period map $\Phi:\Xi\to\mathscr{D}$ by

$$\Phi(\lambda_1,\lambda_2)=[\eta_0,\eta_1,\eta_2].$$

Set the Picard modular group

$$\Gamma=\{g\in \mathrm{GL}_3(\mathbb{Z}[\omega]):{}^t\overline{g}Hg=H\},$$

and set $\Gamma(\sqrt{-3})=\{g\in\Gamma:g\equiv I_3\mod\sqrt{-3}\}$. The element $g=\begin{pmatrix}p_1&q_1&r_1\\p_2&q_2&r_2\\p_3&q_3&r_3\end{pmatrix}\in\Gamma$ acts on $\mathscr{D}$ by

$$g(u,v)=\left(\frac{p_3+q_3v+r_3u}{p_1+q_1v+r_1u},\frac{p_2+q_2v+r_2u}{p_1+q_1v+r_1u}\right). \tag{2.5}$$

Let $a=(a_1,a_2,a_3),b=(b_1,b_2,b_3)$ be in $\mathbb{Q}^3$. Set the Riemann theta constant

$$\vartheta\begin{bmatrix}a\\b\end{bmatrix}(\Omega)=\sum_{n\in\mathbb{Z}^3}\exp[\pi i(n+a)\Omega^t(n+a)+2\pi i(n+a)^tb],$$

here Ω is a variable on the Siegel upper half space of degree 3. We use the following Riemann theta constants and their Fourier expansions (see [S], p.327, also [K-S] formula (1.3)):

$$\vartheta_k(u,v)=\vartheta\begin{bmatrix}0&1/6&0\\k/3&1/6&k/3\end{bmatrix}(\Omega(u,v))=\sum_{\mu\in\mathbb{Z}[\omega]}\omega^{2k\mathrm{tr}\mu}H(\mu u)q^{N(\mu)} \tag{2.6}$$

with an index $k\in\mathbb{Z}$, where $\mathrm{tr}\mu=\mu+\bar{\mu},\ \mathrm{N}(\mu)=\mu\bar{\mu}$ and

$$H(u)=\exp[\frac{\pi}{\sqrt{3}}u^2]\vartheta\begin{bmatrix}1/6\\1/6\end{bmatrix}(u,-\omega^2),\quad q=\exp[\frac{2\pi}{\sqrt{3}}v].$$

Apparently it holds $\vartheta_k(u,v)=\vartheta_{k+3}(u,v)$, so k runs over $\{0,1,2\}=\mathbb{Z}/3\mathbb{Z}$.

The following properties are already established.

Fact 2.1. (i) ([P], [D-M], [T], [S] p.349) *The period map* Φ *induces a biholomorphic isomorphism from the* ξ*-space* $\mathbb{P}^2$ *to the Satake compactification* $\overline{\mathscr{D}/\Gamma(\sqrt{-3})}$ *of* $\mathscr{D}/\Gamma(\sqrt{-3})$.

(ii) ([S] p.327) *The map* $\Lambda : \mathscr{D} \longrightarrow \mathbb{P}^2$ *defined by*

$$\Lambda([\eta_0, \eta_1, \eta_2]) = [\vartheta_0(u,v)^3, \vartheta_1(u,v)^3, \vartheta_2(u,v)^3] \tag{2.7}$$

gives the inverse of the period map Φ.

(iii) ([S] p.329) *The projective group* $\Gamma(\sqrt{-3})/\{1, \omega, \omega^2\}$ *is generated by*

$$g_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \omega \end{pmatrix}, \quad g_2 = \begin{pmatrix} 1 & 0 & 0 \\ \omega - \omega^2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad g_3 = \begin{pmatrix} 1 & 0 & 0 \\ \omega - 1 & 1 & \omega - 1 \\ 1 - \omega^2 & 0 & 1 \end{pmatrix},$$
$$g_4 = \begin{pmatrix} 1 & \omega - \omega^2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad g_5 = \begin{pmatrix} 1 & \omega - 1 & \omega - 1 \\ 0 & 1 & 0 \\ 0 & 1 - \omega & 1 \end{pmatrix}.$$

Let G *denote the group generated by* $g_1, \ldots, g_5$.

(iv) ([S] p.346) *We have the automorphic property:*

$$\vartheta_k(g(u,v))^3 = \det(g)^{-1}(p_1 + q_1 v + r_1 u)^3 \, \vartheta_k(u,v)^3 \tag{2.8}$$

for $g = \begin{pmatrix} p_1 & q_1 & r_1 \\ p_2 & q_2 & r_2 \\ p_3 & q_3 & r_3 \end{pmatrix} \in G$.

3 Main Theorem

Under the relation

$$(\lambda_1, \lambda_2) = \left(\frac{\vartheta_1(u,v)^3}{\vartheta_0(u,v)^3}, \frac{\vartheta_2(u,v)^3}{\vartheta_0(u,v)^3} \right) \tag{3.1}$$

stated in Fact 2.1 , we have the following our main theorem:

Theorem 3.1.

$$\vartheta_0(u,v) = C_0 \, F_1(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, 1; 1 - \lambda_1, 1 - \lambda_2),$$
$$C_0 = \vartheta \begin{bmatrix} \frac{1}{6} \\ \frac{1}{6} \end{bmatrix} (-\omega^2),$$

here $F_1(a, b, b', c; \lambda_1, \lambda_2)$ *indicates the Appell hypergeometric function*

$$F_1(a, b, b', c; \lambda_1, \lambda_2) = \sum_{m,n \geq 0} \frac{(a, m+n)(b, m)(b', n)}{(c, m+n) m! n!} \lambda_1^m \lambda_2^n \tag{3.2}$$

with

$$(a, n) = \begin{cases} a(a+1) \cdots (a+n-1) & n > 0 \\ 1 & n = 0. \end{cases}$$

Corollary 3.1. *We have*

$$\vartheta_i(u,v)^3 = C_0^3 \, \lambda_i \left(F_1(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, 1; 1 - \lambda_1, 1 - \lambda_2) \right)^3, \quad (i = 1, 2). \tag{3.3}$$

Remark 3.1. *According to some classical literature (also in* [M-T-Y]*), it holds*

$$C_0 = \vartheta \begin{bmatrix} \frac{1}{6} \\ \frac{1}{6} \end{bmatrix} (-\omega^2) = \frac{3^{3/8}}{2\pi} \exp(\frac{5\pi\sqrt{-1}}{72})\Gamma(\frac{1}{3})^{3/2}. \tag{3.4}$$

In [K-S], a new three terms arithmetic geometric mean $M_3(a,b,c)$ is introduced. For three positive numbers a,b,c, set a new triple (a',b',c') with $a' = \frac{1}{3}(a+b+c), b'^3 + c'^3 = \frac{1}{3}(a^2b + b^2c + c^2a + ab^2 + bc^2 + ca^2), b'^3 - c'^3 = \frac{1}{3\sqrt{-3}}(a-b)(b-c)(c-a)$. Define a AGM process by

$$(a',b',c') = \psi(a,b,c).$$

We can take a nice choice of the cubic roots for b', c' so that $\psi^2(a,b,c)$ becomes to be a triple of positive numbers again. Thus, we get a unique positive number

$$M_3(a,b,c) := \lim_{n\to\infty} \psi^n(a,b,c).$$

As a consequence of Main Theorem we obtain a new proof of the three terms AGM theorem in [K-S] (p.134 Theorem 2.2) :

Corollary 3.2.

$$\frac{1}{M_3(1,x,y)} = F_1(\frac{1}{3},\frac{1}{3},\frac{1}{3},1;1-x^3,1-y^3), \quad (|x|<1, |y|<1).$$

Observing the following isogeny formula (see [K-S] Theorem 1.1 p.132)

$$\begin{cases} \vartheta_0(\sqrt{-3}u, 3v) = \frac{1}{3}(\vartheta_0 + \vartheta_1 + \vartheta_2), \\ \vartheta_1^3(\sqrt{-3}u, 3v) + \vartheta_2^3(\sqrt{-3}u, 3v) = \frac{1}{3}(\vartheta_0^2\vartheta_1 + \vartheta_1^2\vartheta_2 + \vartheta_2^2\vartheta_0 + \vartheta_0\vartheta_1^2 + \vartheta_1\vartheta_2^2 + \vartheta_2\vartheta_0^2), \\ \vartheta_1^3(\sqrt{-3}u, 3v) - \vartheta_2^3(\sqrt{-3}u, 3v) = \frac{1}{3\sqrt{-3}}(\vartheta_0 - \vartheta_1)(\vartheta_1 - \vartheta_2)(\vartheta_2 - \vartheta_0), \end{cases}$$

we obtain the above corollary by the exactly analogous argument in the introduction.

4 Proof of the Main Theorem

Note that $\vartheta_k(u,v)$ $(k=0,1,2)$ is holomorphic on $\mathscr{D}$. The period $\eta_0 = \int_{A_1}(x(x-1)(x-\lambda_1)(x-\lambda_2))^{-1/3}dx$ is a single valued holomorphic function on $\mathscr{D}$ via the relation (3.1). And η_0 has only zeros possibly on $\Phi(\{\xi_0 = 0\})$.

For an element $g = \begin{pmatrix} p_1 & q_1 & r_1 \\ p_2 & q_2 & r_2 \\ p_3 & q_3 & r_3 \end{pmatrix} \in G$, there are actions on $(u,v) \in \mathscr{D}$ given by (2.5) and on the triple (η_0, η_1, η_2) by

$$g(\eta) = (g(\eta_0), g(\eta_1), g(\eta_2)) = (p_1\eta_0 + q_1\eta_1 + r_1\eta_2, p_2\eta_0 + q_2\eta_1 + r_2\eta_2, p_3\eta_0 + q_3\eta_1 + r_3\eta_2).$$

Note that we don't have the ambiguity of the choice of the projective group in G.

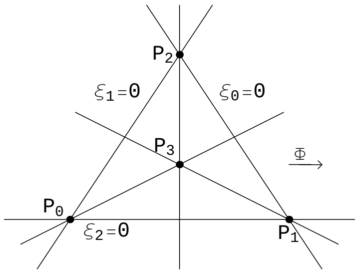
From Fact 2.1 (iii) we have

$$\frac{\vartheta_0(g(u,v))^3}{g(\eta_0)^3} = \frac{\vartheta_0^3(\mathrm{id}(u,v))}{\mathrm{id}(\eta_0)^3}.$$

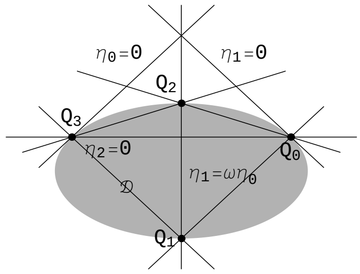
So $\vartheta_0(u,v)^3/\eta_0^3$ is invariant under the action of G, namely it is a single valued rational function on $\overline{\mathscr{D}/G}$, and that is also identified as a function on the ξ space $\mathbb{P}^2$ via the isomorphism Φ. By (2.6) we have

$$(1:\lambda_1:\lambda_2) = (\xi_0:\xi_1:\xi_2) = \left(\frac{\vartheta_0(u,v)^3}{\eta_0^3} : \frac{\vartheta_1(u,v)^3}{\eta_0^3} : \frac{\vartheta_2(u,v)^3}{\eta_0^3}\right).$$

The compactification $\overline{\mathscr{D}/G}$ is obtained by attaching 4 points corresponding to $P_0 = [\xi_0, \xi_1, \xi_2] = [1,0,0], P_1 = [0,1,0], P_2 = [0,0,1], P_3 = [1,1,1]$ to $\mathscr{D}/G$. Put $Q_i = \Phi(P_i), \quad (i = 0,1,2,3)$.



$\xi-$ space



$\eta-$ space

$\vartheta_k^3(u,v)$ is zero on the divisor $\Phi(\{\xi_k = 0\})$, and does not vanish at the point corresponding to a nondegenerate Picard curve. This is due to the argument in [S] p.316. If $\vartheta_0^3(u,v)$ has a zero outside $\Phi(\{\xi_0 = 0\})$, it should be a common zero of all $\vartheta_k^3(u,v)$'s. Q_3 is given by

$$[\eta_0, \eta_1, \eta_2] = [0, 1, 0] = \lim_{u\to 0, v\to -\infty}(u, v).$$

According to the Fourier expansion in (2.6) we have

$$\lim_{(u,v)\to(0,-\infty)} \vartheta_k(u,v) = H(0) = \vartheta \begin{bmatrix} \frac{1}{6} \\ \frac{1}{6} \end{bmatrix} (0, -\omega^2) \neq 0 \quad \text{for } k = 0, 1, 2. \tag{4.1}$$

So, there is no common divisor of $\vartheta_0^3(u,v), \vartheta_1^3(u,v), \vartheta_2^3(u,v)$ on $\Phi(\{\xi_i = \xi_j\})$ $(i \neq j)$.

Set $G' \subset \Gamma$ be the group generated by G and $\delta = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \in \Gamma$. δ induces the permutation of Q_0 and Q_3, and it is coming from the symplectic transformation

$$M_\delta = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

that acts on the basis $(B_1, B_2, B_3, A_1, A_2, A_3)$ from left. Here we note that $\vartheta_0^3(u,v)$ is a modular form with respect to the modular group in the sense that we have

$$\vartheta_0^3(g(u,v)) = \det(g)^{-1}(p_1 + q_1 v + r_1 u)^3 \vartheta_0^3(u,v) \text{ for } (u,v) \in \mathscr{D}, g = \begin{pmatrix} p_1 & q_1 & r_1 \\ p_2 & q_2 & r_2 \\ p_3 & q_3 & r_3 \end{pmatrix} \in G',$$

and that $\vartheta_1^3(u,v)$ and $\vartheta_2^3(u,v)$ are not the case. This assertion is derived by calculating the action of M_δ on the characteristics of ϑ_k. $Q_0 = \Phi(P_0)$ is given by $[\eta_0, \eta_1, \eta_2] = [1, 0, 0]$, namely $(u,v) = (0,0)$. Then we have $\delta(Q_3) = Q_0$. Thus, we have $\lim\limits_{(u,v)\to(0,0)} v^3\vartheta_0^3(u,v) = \vartheta_0^3(Q_3) = H(0)^3 \neq 0$. Hence, $\Phi(\{\xi_1 = 0\})$ and $\Phi(\{\xi_2 = 0\})$ are not contained in the divisor of $\vartheta_0^3(u,v)$. Namely, the support of the divisor of $\vartheta_0^3(u,v)$ is exactly $\Phi(\{\xi_0 = 0\})$. This is contained in the support of η_0. As we already mentioned, ϑ_0^3/η_0^3 is a rational function on the $\xi-$ space $\mathbb{P}^2$. Thus, we have

$$\frac{\vartheta_0(u,v)^3}{\eta_0^3} = \alpha$$

for some constant α. Now let us determine the value α. Suppose $0 < \lambda_1 < \lambda_2 < 1$. We have

$$\eta_0 = \int_{A_1} (z(z-1)(z-\lambda_1)(z-\lambda_2))^{-1/3} dz = c_1 \int_0^{-\infty} (z(z-1)(z-\lambda_1)(z-\lambda_2))^{-1/3} dz$$

with some constant c_1. Now recall the integral representation

$$F_1(a,b,b',c;x,y) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)} \int_1^{\infty} z^{b+b'-c}(z-1)^{c-a-1}(z-x)^{-b}(z-y)^{-b'}dz.$$

By changing the variable $z = 1 - z'$ we have

$$\eta_0 = c_2 F_1(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, 1; 1-\lambda_1, 1-\lambda_2)$$

with some constant c_2. Hence we have

$$\vartheta_0(u,v) = \beta F_1(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, 1; 1-\lambda_1, 1-\lambda_2)$$

with some constant β. If we put $\xi = P_3 = [1,1,1]$, it corresponds to $(\xi_0 : \xi_1 : \xi_2) = (1:1:1)$. So the right hand side is equal to $\beta F_1(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, 1; 0, 0) = \beta$. Recall again that $\Phi(P_3) = \lim_{u\to 0, v\to -\infty}(u,v)$. By using (4.1) we have $\beta = \vartheta \begin{bmatrix} \frac{1}{6} \\ \frac{1}{6} \end{bmatrix} (0, -\omega^2)$. Thus, we obtained the required equality.

q.e.d.

5 Degeneration to the case of Borweins' case

As a degenerate case $\lambda = \lambda_1 = \lambda_2$, we obtain the Jacobi type formula for the Borweins curves (see [B-B], [K-S] p.141)

$$w^3 = z(z-1)(z-\lambda)^2. \tag{5.1}$$

Set

$$\begin{cases} \theta_0(\tau) = \sum_{\mu\in\mathbb{Z}[\omega]} q^{N(\mu)} = \sum_{m,n\in\mathbb{Z}} \left(e^{2\pi i\tau/3}\right)^{m^2-mn+n^2}, \\ \theta_1(\tau) = \sum_{\mu\in\mathbb{Z}[\omega]} e^{2\pi i Tr(\mu)/3} q^{N(\mu)} = \sum_{m,n\in\mathbb{Z}} e^{2\pi i(m+n)/3} \left(e^{2\pi i\tau/3}\right)^{m^2-mn+n^2}, \\ q = \exp[2\pi i\tau/3], N(\mu) = \mu\overline{\mu}, Tr(\mu) = \mu + \overline{\mu}. \end{cases}$$

Putting $u = 0, \tau = -i\sqrt{3}v$ in (3.1) we have the expression of $\lambda = \lambda_1 = \lambda_2$:

$$\lambda = \frac{\theta_1(\tau)^3}{\theta_0(\tau)^3}. \tag{5.2}$$

Theorem 5.1. (Borweins [B-B] p.695) *Under the relation (5.2), we have*

$$\theta_0(\tau) = F\left(\frac{1}{3}, \frac{2}{3}, 1; 1-\lambda\right).$$

This equality is obtained just as the case $\lambda = \lambda_1 = \lambda_2$, namely the case $u = 0$, in the main theorem.

Remark 5.1. *Borweins have shown this theorem by using their AGM theorem. But we proved it directly with modular arguments. So our theorem induces their AGM theorem also. This is the context already discussed in the proof of Cor 3.2.*

References

[App] P. Appell, *Sur les Fonctions hypergéométriques de plusieurs variables les polynomes d'Hermite et autres fonctions sphériques dans l'hyperspace,* Gauthier-Villars, Paris, 1925.

[B-B] J.M. Borwein and P.B. Borwein, *A cubic counterpart of Jacobi's identity and the AGM,* Trans. Amer. Math. Soc., 323 (1991), no. 2, 691–701.

[D-M] P. Deligne and G.D. Mostow, *Monodromy of hypergeometric functions and non-lattice integral monodromy,* I.H.E.S. Pub. Math.,63(1986)

[J] C.G.J. Jacobi, *Gesammelte Werke I,* Chelsea, New York, 1969.

[K-S] K. Koike and H. Shiga, *Isogeny formulas for the Picard modular form and a three terms arithmetic geometric mean,* J. Number Theory, 124(2007), 123-141.

[M-T-Y] K. Matsumoto, T. Terasoma and S. Yamazaki, *On periods of cubic curves of the Hesse canonical form,* Preprint, August 25, 2008.

[P] E. Picard, *Sur les fonctions de deux variables indépendantes analogues aux fonctions modulaires,* Acta math., 2(1883), 114-135.

[S] H. Shiga, *On the representation of the Picard modular function by θ constants I-II,* Pub. R.I.M.S. Kyoto Univ. 24(1988), 311-360.

[T] T.Terada, *Fonctions hypergéométriques F_1 et fonctions automorphes. I,* J. Math. Soc. Japan, 1983