AN UPPER BOUND ON THE REDUCTION NUMBER OF AN IDEAL

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Abstract. Let A be a commutative ring and I an ideal of A with a reduction Q. In this paper we give an upper bound on the reduction number of I with respect to Q, when a suitable family of ideals in A is given. As a corollary it follows that if some ideal J containing I satisfies $J^2 = QJ$, then $I^{v+2} = QI^{v+1}$, where v denotes the number of generators of J/I as an A-module.

1. Introduction

Let Q, I and J be ideals of a commutative ring A such that $Q \subseteq I \subseteq J$. As is noted in [1, 2.6], if J/I is cyclic as an A-module and $J^2 = QJ$, then we have $I^3 = QI^2$. The purpose of this paper is to generalize this fact. We will show that if J/I is generated by v elements as an A-module and $J^2 = QJ$, then $I^{v+2} = QI^{v+1}$. We get this result as a corollary of the following theorem, which generalizes Rossi's assertion stated in the proof of [7, 1.3].

Theorem 1.1. *Let A be a commutative ring and $\{F_n\}_{n \geq 0}$ a family of ideals in A such that $F_0 = A$ and $IF_n \subseteq F_{n+1}$ for any $n \geq 0$. Let $k \geq 0$ be a non-negative integer and $\mathfrak{a}$ an ideal in A such that $I^{k+1} \subseteq QF_k + \mathfrak{a}F_{k+1}$. Suppose that $F_n/(QF_{n-1} + I^n)$ is generated by v_n elements for any $n \geq 0$ and $v_n = 0$ for $n \gg 0$. We put $v = \sum_{n \geq 0} v_n$. Then we have*

$$I^{v+k+1} = QI^{v+k} + \mathfrak{a}I^{v+k+1}\,.$$

If a family $\{F_n\}_{n \geq 0}$ of ideals in A satisfies all of the conditions required in 1.1 in the case where $\mathfrak{a} = (0)$, we have $F_n = QF_{n-1}$ for $n \gg 0$. As a typical example of such $\{F_n\}_{n \geq 0}$, we find $\{\widetilde{I^n}\}_{n \geq 0}$ when I contains a non-zerodivisor, where $\widetilde{I^n}$ denotes the Ratliff-Rush closure of I^n (cf. [9]). If A is an analytically unramified local ring, then $\{\overline{I^n}\}_{n \geq 0}$ is also an important example, where $\overline{I^n}$ denotes the integral closure of I^n. It is obvious that $\{J^n\}_{n \geq 0}$ always satisfies the required condition on $\{F_n\}_{n \geq 0}$ for any ideal J with $I \subseteq J \subseteq \overline{I}$.

We prove 1.1 following Rossi's argument in the proof of [7, 1.3]. However we do not assume that A/I has finite length. And furthermore we can deduce the following corollary which gives an upper bound on the reduction number $\mathrm{r}_Q(I)$ of I with respect to Q using numbers of gerators of certain A-modules.

Corollary 1.2. *Let $(A, \mathfrak{m})$ be a Noetherian local ring and $\{F_n\}_{n \geq 0}$ a family of ideals in A such that $F_0 = A$ and $IF_n \subseteq F_{n+1}$ for any $n \geq 0$. Then, if $I^{k+1} \subseteq QF_k + \mathfrak{m}F_{k+1}$ for*

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some $k \geq 0$, we have

$$\mathrm{r}_Q(I) \leq k + \sum_{n \geq 1} \mu_A(F_n/(QF_{n-1} + I^n)),$$

and furthermore it follows that

$$\mathrm{r}_Q(I) \leq 1 + \mu_A(F_1/I) + \sum_{n \geq 2} \mu_A(F_n/QF_{n-1}).$$

Throughout this paper A denotes a commutative ring. We do not assume that A is Noetherian unless otherwise specified. Furthermore I and Q denote ideals of A such that $Q \subseteq I$. We set $\mathrm{r}_Q(I) = \inf\{n \geq 0 \mid I^{n+1} = QI^n\}$. Of course, $\mathrm{r}_Q(I) = \infty$ if Q is not a reduction of I. For a finitely generated A-module M, we denote by $\mu_A(M)$ the minimal number of generators of M. If $(A, \mathfrak{m})$ is a Noetherian local ring and M is annihilated by some power of $\mathfrak{m}$, the length of M is denoted by $\ell_A(M)$.

2. Proof of Theorem 1.1

In order to prove 1.1 we prepare the following lemma, which generalizes [4, 2.3].

Lemma 2.1. *Let $I_1, I_2, \ldots, I_N$ be finite number of ideals of A. For any $1 \leq n \leq N$, we assume that I_n is generated by v_n elements and*

$$I \cdot I_n \subseteq I^{n+1} + \sum_{\ell=1}^{N} Q^{n+1-\ell} I_\ell .$$

Let $v := v_1 + v_2 + \cdots + v_N > 0$. Then, for any v elements $a_1, a_2, \ldots, a_v$ in I, there exists $\sigma \in QI^{v-1}$ such that

$$a_1 a_2 \cdots a_v - \sigma \in \bigcap_{n=1}^{N} [I^{n+v} : I_n].$$

Proof. We put $w_0 = 0$ and $w_n = v_1 + \cdots + v_n$ for $1 \leq n \leq N$. Then $0 = w_0 \leq w_1 \leq w_2 \leq \cdots \leq w_N = v$. Hence, if $1 \leq i \leq v$, we have $w_{n-1} < i \leq w_n$ for some $1 \leq n \leq N$, and we denote this number n by n_i. Now we choose elements $x_1, x_2, \ldots, x_v$ of A so that I_n is generated by $\{x_i \mid w_{n-1} < i \leq w_n\}$ for any $1 \leq n \leq N$ with $v_n \neq 0$. Then $x_i \in I_{n_i}$ and

$$a_i x_i \in I \cdot I_{n_i} \subseteq I^{n_i+1} + \sum_{\ell=1}^{N} Q^{n_i+1-\ell} I_\ell$$

for any $1 \leq i \leq v$. Hence there exists a family $\{c_{ij}\}_{1 \leq i,j \leq v}$ of elements in A such that

$$a_i x_i \equiv \sum_{j=1}^{v} c_{ij} x_j \mod I^{n_i+1} \quad \text{and} \quad c_{ij} \in Q^{n_i+1-n_j}$$

for any $1 \leq i, j \leq v$. Let $R = A[It, t^{-1}]$ and $T = A[t, t^{-1}]$, where t is an indeterminate. We regard T/R as a graded R-module, and for any $f \in T$ we denote by $\overline{f}$ the class of f

in T/R. Then we have

$$a_i t \cdot \overline{x_i t^{n_i}} = \sum_{j=1}^{v} c_{ij} t^{n_i - n_j + 1} \cdot \overline{x_j t^{n_j}}$$

for any $1 \le i, j \le v$. Here we put

$$\begin{aligned} b_{ij} &= \begin{cases} a_i - c_{ii} & \text{if } i = j \\ -c_{ij} & \text{if } i \neq j \end{cases}, \\ m_{ij} &= b_{ij} t^{n_i - n_j + 1} \in R, \text{ and} \\ e_i &= \overline{x_i t^{n_i}} \in T/R \end{aligned}$$

for any $1 \le i, j \le v$. Let us consider the $v \times v$ matrix $M = (\, m_{ij} \,)$ with entries in R. Because we have

$$M \begin{pmatrix} e_1 \\ e_2 \\ \vdots \\ e_v \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix},$$

it follows that $\Delta e_i = 0$ for any $1 \le i \le v$, where $\Delta = \det M$. Then we get

$$(*) \qquad \Delta \cdot x_i t^{n_i} \in R$$

for any $1 \le i \le v$. On the other hand, by the definition of determinant, we have

$$\Delta = \sum_{(p_1, p_2, \ldots, p_v) \in S_v} \operatorname{sgn}(p_1, p_2, \ldots, p_v) m_{1p_1} m_{2p_2} \cdots m_{vp_v},$$

where S_v denotes the set of permutations of $1, 2, \ldots, v$ and $\operatorname{sgn}(p_1, p_2, \ldots, p_v)$ denotes the signature of $(p_1, p_2, \ldots, p_v) \in S_v$. Because

$$\deg(\prod_{i=1}^{v} m_{ip_i}) = \sum_{i=1}^{v} (n_i - n_{p_i} + 1) = \sum_{i=1}^{v} n_i - \sum_{i=1}^{v} n_{p_i} + v = v,$$

we have $\prod_{i=1}^{v} m_{ip_i} = (\prod_{i=1}^{v} b_{ip_i}) t^v$. Therefore $\Delta = \delta t^v$, where δ denotes the determinant of the $v \times v$ matrix $(\, b_{ij} \,)$ with entries in A. Hence, by $(*)$ we have $\delta x_i \in I^{v+n_i}$ for any $1 \le i \le v$. This means $\delta I_n \subseteq I^{v+n}$ for any $1 \le n \le N$, and so $\delta \in \bigcap_{n=1}^{N} [I^{v+n} : I_n]$. If $(p_1, p_2, \ldots, p_v) \neq (1, 2, \ldots, v)$, then $j > p_j$ for some $1 \le j \le v$, which means $n_j \ge n_{p_j}$, and so $b_{jp_j} = -c_{jp_j} \in Q^{n_j - n_{p_j} + 1} \subseteq QI^{n_j - n_{p_j}}$. As a consequence, if $(1, 2, \ldots, v) \neq (p_1, p_2, \ldots, p_v) \in S_v$, we get

$$\prod_{i=1}^{v} b_{ip_i} = b_{jp_j} \cdot \prod_{i \neq j} b_{ip_i} \in QI^{n_j - n_{p_j}} \cdot \prod_{i \neq j} I^{n_i - n_{p_i} + 1} \subseteq Q \cdot I^{n_j - n_{p_j} + \sum_{i \neq j} (n_i - n_{p_i} + 1)} = QI^{v-1}.$$

Furthermore, as $a_i \in I$ and $c_{ii} \in Q$ for any $1 \le i \le v$, we have

$$\prod_{i=1}^{v} b_{ii} = \prod_{i=1}^{v} (a_i - c_{ii}) = a_1 a_2 \cdots a_v - d$$

for some $d \in QI^{v-1}$. Therefore, there exists $\sigma \in QI^{v-1}$ such that $\delta = a_1 a_2 \cdots a_v - \sigma$, and the proof is complete.

Proof of Theorem 1.1. If $v = 0$, then we have $F_n = I^n$ for any $n \geq 0$, and so $I^{k+1} \subseteq QF_k + \mathfrak{a}F_{k+1} = QI^k + \mathfrak{a}I^{k+1} \subseteq I^{k+1}$, which means $I^{k+1} = QI^k + \mathfrak{a}I^{k+1}$. Hence we may assume $v > 0$. For any $n \geq 0$, let us take an ideal I_n generated by v_n elements so that $F_n = QF_{n-1} + I^n + I_n$. We can easily show that

$$F_n = I^n + \sum_{\ell=0}^{n} Q^{n-\ell} I_\ell \tag{\#}$$

for any $n \geq 0$ by induction on n. Now we choose an integer N so that $N > k$ and $I_n = 0$ for any $n > N$. Then by (#) it follows that

$$I \cdot I_n \subseteq F_{n+1} = I^{n+1} + \sum_{\ell=0}^{N} Q^{n+1-\ell} I_\ell$$

for any $0 \leq n \leq N$. Let $a_1, a_2, \ldots, a_v$ be any elements of I. Then, by 2.1 there exists $\sigma \in QI^{v-1}$ such that

$$a_1 a_2 \cdots a_v - \sigma \in \bigcap_{n=0}^{N} [I^{n+v} : I_n] .$$

We put $\xi = a_1 a_2 \cdots a_v - \sigma$. Then by (#) we get

$$\xi F_n = \xi I^n + \sum_{\ell=0}^{n} Q^{n-\ell} \cdot \xi I_\ell \subseteq I^v \cdot I^n + \sum_{\ell=0}^{n} Q^{n-\ell} \cdot I^{\ell+v} \subseteq I^{v+n}$$

for any $0 \leq n \leq N$. Now the assumption that $I^{k+1} \subseteq QF_k + \mathfrak{a}F_{k+1}$ implies

$$\xi I^{k+1} \subseteq Q \cdot \xi F_k + \mathfrak{a} \cdot \xi F_{k+1} \subseteq Q \cdot I^{v+k} + \mathfrak{a} \cdot I^{v+k+1} .$$

Therefore we get

$$a_1 a_2 \cdots a_v \cdot I^{k+1} = (\xi + \sigma) I^{k+1} \subseteq QI^{v+k} + \mathfrak{a}I^{v+k+1} .$$

Then, as the elements $a_1, a_2, \ldots, a_v$ are chosen arbitrarily from I, it follows that $I^v \cdot I^{k+1} \subseteq QI^{v+k} + \mathfrak{a}I^{v+k+1} \subseteq I^{v+k+1}$. Thus we get $I^{v+k+1} = QI^{v+k} + \mathfrak{a}I^{v+k+1}$.

Proof of Corollary 1.2. We put $v = \sum_{n \geq 1} \mu_A(F_n/(QF_{n-1} + I^n))$. We may assume $v < \infty$. Then, setting $\mathfrak{a} = \mathfrak{m}$ in 1.1, it follows that $I^{v+k+1} = QI^{v+k} + \mathfrak{m}I^{v+k+1}$. Hence we get $I^{v+k+1} = QI^{v+k}$ by Nakayama's lemma, and so $\mathrm{r}_Q(I) \leq v + k$. In order to prove the second inequality, we choose k as small as possible. If $k \leq 1$, we have

$$\mathrm{r}_Q(I) \leq k + v \leq 1 + \mu_A(F_1/I) + \sum_{n \geq 2} \mu_A(F_n/QF_{n-1}) .$$

So, we assume $k \geq 2$ in the rest of this proof. In this case we have

$$\mathrm{r}_Q(I) \leq k + \mu_A(F_1/I) + \sum_{n=2}^{k} \mu_A(F_n/(QF_{n-1} + I^n)) + \sum_{n \geq k+1} \mu_A(F_n/QF_{n-1}) . \tag{$\natural$}$$

If $2 \leq n \leq k$, then $I^n \not\subseteq QF_{n-1} + \mathfrak{m}F_n$, and so the canonical surjection

$$F_n/(QF_{n-1} + \mathfrak{m}F_n) \longrightarrow F_n/(QF_{n-1} + I^n + \mathfrak{m}F_n)$$

is not injective, which means

$$\mu_A(F_n/QF_{n-1}+I^n) \leq \mu_A(F_n/QF_{n-1}) - 1\,.$$

Thus we get

$$\sum_{n=2}^{k} \mu_A(F_n/QF_{n-1}+I^n) \leq \{\sum_{n=2}^{k} \mu_A(F_n/QF_{n-1})\} - (k-1)\,.$$

Therefore the required inequality follows from ($\natural$).

3. Corollaries

In this section we collect some results deduced from 1.1 and 1.2.

Corollary 3.1. *Let J be an ideal of A such that $J \supseteq I$ and $J^2 = QJ$. If J/I is finitely generated as an A-module, then $\mathrm{r}_Q(I) \leq \mu_A(J/I) + 1$.*

Proof. We apply 1.1 setting $F_n = J^n$ for any $n \geq 0$ and $\mathfrak{a} = (0)$. Because $I^2 \subseteq J^2 = QJ$, we may put $k = 1$, and hence we get $I^{v+2} = QI^{v+1}$, where $v = \mu_A(J/I)$. Then $\mathrm{r}_Q(I) \leq v + 1$.

Corollary 3.2. *Let $(A, \mathfrak{m})$ be a two-dimensional regular local ring (or, more generally, a two-dimensional pseudo-rational local ring) such that $A/\mathfrak{m}$ is infinite. If I is an $\mathfrak{m}$-primary ideal with a minimal reduction Q, then $\mathrm{r}_Q(I) \leq \mu_A(\overline{I}/I) + 1$.*

Proof. This follows from 3.1 since $(\,\overline{I}\,)^2 = Q\overline{I}$ by [5, 5.1] (or [6, 5.4]).

Corollary 3.3. *Let $\mathfrak{p}$ be a prime ideal of A with $\mathrm{ht}\,\mathfrak{p} = g \geq 2$. Let $Q = (a_1, a_2, \ldots, a_g)$ be an ideal generated by a regular sequence contained in the k-th symbolic power $\mathfrak{p}^{(k)}$ of $\mathfrak{p}$ for some $k \geq 2$. Then we have $\mathrm{r}_Q(I) \leq \mu_A((Q:\mathfrak{p}^{(k)})/Q) + 1$ for any ideal I with $Q \subseteq I \subseteq Q:\mathfrak{p}^{(k)}$, if one of the following three conditions holds* ; (i) *$A_\mathfrak{p}$ is not a regular local ring,* (ii) *$A_\mathfrak{p}$ is a regular local ring and $g \geq 3$,* (iii) *$A_\mathfrak{p}$ is a regular local ring, $g = 2$, and $a_i \in \mathfrak{p}^{(k+1)}$ for any $1 \leq i \leq g$.*

Proof. This follows from 3.1 since $(Q:\mathfrak{p}^{(k)})^2 = Q(Q:\mathfrak{p}^{(k)})$ by [10, 3.1].

Corollary 3.4. *Let $(A, \mathfrak{m})$ be a Buchsbaum local ring. Assume that the multiplicity of A with respect to $\mathfrak{m}$ is 2 and* $\mathrm{depth}\,A > 0$. *Then, for any parameter ideal Q in A and an ideal I with $Q \subseteq I \subseteq Q:\mathfrak{m}$, we have $\mathrm{r}_Q(I) \leq \mu_A((Q:\mathfrak{m})/Q) + 1$.*

Proof. This follows from 3.1 since $(Q:\mathfrak{m})^2 = Q(Q:\mathfrak{m})$ by [3, 1.1].

In order to state the last corollary, let us recall the definition of Hilbert coefficients. Let $(A, \mathfrak{m})$ be a d-dimensional Noetherian local ring and I an $\mathfrak{m}$-primary ideal. Then there exists a family $\{\,\mathrm{e}_i(I)\,\}_{0 \leq i \leq d}$ of integers such that

$$\ell_A(\,A/I^{n+1}\,) = \sum_{i=0}^{d} (-1)^i\,\mathrm{e}_i(I)\binom{n+d-i}{d-i}$$

for $n \gg 0$. We call $\mathrm{e}_i(I)$ the i-th Hilbert coefficient of I. On the other hand, if A is an analytically unramified local ring, then $\{\,\overline{I^n}\,\}_{n\geq 0}$ is a Hilbert filtration (cf. [2]), and so there exists a family $\{\,\overline{\mathrm{e}}_i(I)\,\}_{0\leq i\leq d}$ of integers such that

$$\ell_A(\,A/\overline{I^{n+1}}\,) = \sum_{i=0}^{d} (-1)^i\,\overline{\mathrm{e}}_i(I)\binom{n+d-i}{d-i}$$

for $n \gg 0$. As is proved in [7, 1.5], if A is a two-dimensional Cohen-Macaulay local ring, then we have

$$\mathrm{r}_Q(I) \leq \mathrm{e}_1(I) - \mathrm{e}_0(I) + \ell_A(\,A/I\,) + 1$$

for any minimal reduction Q of I. We can generalize this result as follows.

Corollary 3.5. *Let $(A, \mathfrak{m})$ be a two-dimensional Cohen-Macaulay local ring with infinite residue field and I an $\mathfrak{m}$-primary ideal with a minimal reduction Q. Then we have the following inequalities.*

(1) $\mathrm{r}_Q(I) \leq \mathrm{e}_1(J) - \mathrm{e}_0(J) + \ell_A(\,A/I\,) + 1$ *for any ideal J such that $I \subseteq J \subseteq \overline{I}$.*
(2) $\mathrm{r}_Q(I) \leq \overline{\mathrm{e}}_1(I) - \overline{\mathrm{e}}_0(I) + \ell_A(\,A/I\,) + 1$, *if A is analytically unramified.*

Proof. (1) Setting $F_n = \widetilde{J^n}$ for any $n \geq 0$ in 1.2, we get

$$\begin{aligned} \mathrm{r}_Q(I) &\leq 1 + \mu_A(\widetilde{J}/I) + \sum_{n\geq 2} \mu_A(\widetilde{J^n}/Q\widetilde{J^{n-1}}) \\ &\leq 1 + \ell_A(\,\widetilde{J}/I\,) + \sum_{n\geq 2} \ell_A(\,\widetilde{J^n}/Q\widetilde{J^{n-1}}\,) \\ &= \sum_{n\geq 1} \ell_A(\,\widetilde{J^n}/Q\widetilde{J^{n-1}}\,) - \ell_A(\,I/Q\,) + 1\,. \end{aligned}$$

Because $\mathrm{e}_1(J) = \sum_{n\geq 1} \ell_A(\,\widetilde{J^n}/Q\widetilde{J^{n-1}}\,)$ by [2, 1.10] and

$$\ell_A(\,I/Q\,) = \ell_A(\,A/Q\,) - \ell_A(\,A/I\,) = \mathrm{e}_0(J) - \ell_A(\,A/I\,)\,,$$

the required inequality follows.

(2) Similarly as the proof of (1), setting $F_n = \overline{I^n}$ for any $n \geq 0$ in 1.2, we get

$$\mathrm{r}_Q(I) \leq \sum_{n\geq 1} \ell_A(\,\overline{I^n}/Q\overline{I^{n-1}}\,) - \ell_A(\,I/Q\,) + 1\,.$$

Because the depth of the associated graded ring of the filtration $\{\,\overline{I^n}\,\}_{n\geq 0}$ is positive, we have $\overline{\mathrm{e}}_1(I) = \sum_{n\geq 1} \ell_A(\,\overline{I^n}/Q\overline{I^{n-1}}\,)$ by [2, 1.9]. Hence we get the required inequality as $\ell_A(\,I/Q\,) = \overline{\mathrm{e}}_0(I) - \ell_A(\,A/I\,)$.

4. Example

In this section we give an example which shows that the maximum value stated in 3.1 can be reached. It provides an example in the case where $\dim A/I > 0$.

Example 4.1. *Let $n \geq 3$ be an integer and $S = k[X_0, X_1, \dots, X_n]$ be the polynomial ring with $n+1$ variables over a field k. Let $A = S/\mathfrak{a}$, where $\mathfrak{a}$ is the ideal of S generated by the maximal minors of the matrix*

$$\begin{pmatrix} X_0 & X_1 & \cdots & X_{n-1} \\ X_1 & X_2 & \cdots & X_n \end{pmatrix}.$$

We denote the image of X_i in A by x_i for $0 \leq i \leq n$. It is well known that A is a two-dimensional Cohen-Macaulay graded ring with the graded maximal ideal $\mathfrak{m} = (x_0, x_1, \dots, x_n)$.

(1) *Let $I = (x_0, x_1, x_n)$ and $Q = (x_0, x_n)$. Then we have $\mathfrak{m}^2 = Q\mathfrak{m}$, $\mu_A(\mathfrak{m}/I) = n-2$, and $\mathrm{r}_Q(I) = n-1$.*

(2) *Let $I = (x_0, x_1, x_{n-1})$, $J = (x_0, x_1, \dots, x_{n-1})$, and $Q = (x_0, x_{n-1})$. Then we have $\dim A/I = 1$, $J^2 = QJ$, $\mu_A(J/I) = n-3$, and $\mathrm{r}_Q(I) = n-2$.*

Proof. (1) Let $0 \leq i \leq j \leq n$. If $i = 0$ or $j = n$, then $x_i x_j \in Q\mathfrak{m}$. On the other hand, if $i > 0$ and $j < n$, then the determinant of the matrix

$$\begin{pmatrix} X_{i-1} & X_j \\ X_i & X_{j+1} \end{pmatrix}$$

is contained in $\mathfrak{a}$, and so $x_i x_j = x_{i-1} x_{j+1}$. Hence we can show that $x_i x_j \in Q\mathfrak{m}$ for any $0 \leq i \leq j \leq n$ by descending induction on $j - i$. Thus we get $\mathfrak{m}^2 = Q\mathfrak{m}$. It is obvious that $\mu_A(\mathfrak{m}/I) = n-2$. Therefore $I^n = QI^{n-1}$ by 3.1 (In fact, we have ${x_1}^n = {x_1}^{n-2} \cdot {x_1}^2 = {x_1}^{n-2} \cdot x_0 x_2 = x_0 {x_1}^{n-3} \cdot x_1 x_2 = x_0 {x_1}^{n-3} \cdot x_0 x_3 = {x_0}^2 {x_1}^{n-4} \cdot x_1 x_3 = \cdots = {x_0}^{n-2} \cdot x_1 x_{n-1} = {x_0}^{n-2} \cdot x_0 x_n = {x_0}^{n-1} x_n \in Q^n \subseteq QI^{n-1}$). In order to prove $\mathrm{r}_Q(I) = n-1$, we show ${x_1}^{n-1} \notin QI^{n-2}$. For that purpose we use the isomorphism

$$\varphi : A \longrightarrow k[\,\{\, s^{n-i} t^i \,\}_{0 \leq i \leq n}\,]$$

of k-algebras such that $\varphi(x_i) = s^{n-i} t^i$ for $0 \leq i \leq n$, where s and t are indeterminates. We have to show $\varphi(x_1)^{n-1} \notin \varphi(Q)\varphi(I)^{n-2}$. Because $\varphi(I) = (s^n, s^{n-1}t, t^n)$, we get

$$\varphi(I)^{\ell} \subseteq (\,\{\, s^{\alpha n - \beta} t^{(\ell - \alpha)n + \beta} \mid 0 \leq \alpha \leq \ell\,,\, 0 \leq \beta \leq \alpha \,\}\,)$$

for any $\ell \geq 1$ by induction on ℓ, and so

$$\varphi(Q)\varphi(I)^{n-2} \subseteq (\,\{\, s^{(\alpha+1)n - \beta} t^{(n-2-\alpha)n + \beta}\,,\, s^{\alpha n - \beta} t^{(n-1-\alpha)n + \beta} \mid 0 \leq \alpha \leq n-2\,,\, 0 \leq \beta \leq \alpha \,\}\,).$$

Therefore, if $\varphi(x_1)^{n-1} = (s^{n-1}t)^{n-1} = s^{(n-1)^2} t^{n-1} \in \varphi(Q)\varphi(I)^{n-2}$, one of the following two cases

(i) $(\alpha+1)n - \beta \leq (n-1)^2$ and $(n-2-\alpha)n + \beta \leq n-1$, or
(ii) $\alpha n - \beta \leq (n-1)^2$ and $(n-1-\alpha)n + \beta \leq n-1$

must occur for some α and β with $0 \leq \alpha \leq n-2$ and $0 \leq \beta \leq \alpha$. Suppose that the case (i) occured. Then we have

$$(\alpha+1)n - \beta \leq (n-1)n - (n-1) \text{ and } (n-2-\alpha)n \leq n-1-\beta\,.$$

As the first inequality implies

$$n - 1 - \beta \leq (n-1)n - (\alpha+1)n = (n-2-\alpha)n\,,$$

it follows that

$$n-1-\beta=(n-1)n-(\alpha+1)n\,,$$

and so

$$\alpha n-\beta=n^2-3n+1\,.$$

Then, as $\alpha n>n^2-3n=(n-3)n$, we have $n-3<\alpha\leq n-2$, which implies $\alpha=n-2$. Thus we get

$$(n-2)n-\beta=n^2-3n+1\,,$$

and so $\beta=n-1$, which contradicts to $\beta\leq\alpha$. Therefore the case (ii) must occur. Then we have

$$\alpha n-\beta\leq(n-1)n-(n-1)\ \text{ and }\ (n-1-\alpha)n\leq n-1-\beta\,.$$

As the first inequality implies

$$n-1-\beta\leq(n-1)n-\alpha n=(n-1-\alpha)n\,,$$

it follows that

$$n-1-\beta=(n-1)n-\alpha n\,,$$

and so

$$\alpha n-\beta=n^2-2n+1\,.$$

Then, as $\alpha n>n^2-2n=(n-2)n$, we get $\alpha>n-2$, which contradicts to $\alpha\leq n-2$. Thus we have seen that ${x_1}^{n-1}\notin QI^{n-2}$.

(2) Let $\mathfrak{b}=(X_0,X_1,\ldots,X_{n-1})S$. Then $\mathfrak{a}\subseteq\mathfrak{b}$, and so $\mathfrak{b}$ is the kernel of the canonical surjection $S\longrightarrow A/J$. Hence $A/J\cong k[\,X_n\,]$, which implies $\dim A/J=1$. Let $0\leq i\leq j\leq n-1$. If $i=0$ or $j=n-1$, then $x_ix_j\in QJ$. On the other hand, if $i>0$ and $j<n$, then $x_ix_j=x_{i-1}x_{j+1}$. Hence we can show that $x_ix_j\in QJ$ for any $0\leq i\leq j\leq n-1$ by descending induction on $j-i$. Thus we get $J^2=QJ$. It is obvious that $\mu_A(J/I)=n-3$. Therefore $I^{n-1}=QI^{n-2}$ by 3.1. This means $\dim A/I=\dim A/Q=\dim A/J=1$. In order to prove $\mathrm{r}_Q(I)=n-2$, we show ${x_1}^{n-2}\notin QI^{n-3}$. For that purpose we use again the isomorphism φ stated in the proof of (1). Although we have to prove $\varphi(x_1)^{n-2}\notin\varphi(Q)\varphi(I)^{n-3}$, it is enough to show

$$(s^{n-1}t)^{n-2}\notin(s^n,st^{n-1})(s^n,s^{n-1}t,st^{n-1})^{n-3}B\,,$$

where $B=k[s,t]$. Because

$$(s^{n-1}t)^{n-2}=s^{n-2}\cdot(s^{n-2}t)^{n-2}$$

in B and

$$(s^n,st^{n-1})(s^n,s^{n-1}t,st^{n-1})^{n-3}B=s^{n-2}\cdot(s^{n-1},t^{n-1})(s^{n-1},s^{n-2}t,t^{n-1})^{n-3}B\,,$$

we would like to show

$$(s^{n-2}t)^{n-2}\notin(s^{n-1},t^{n-1})(s^{n-1},s^{n-2}t,t^{n-1})^{n-3}B\,.$$

However, it can be done by the same argument as the proof of

$$(s^{n-1}t)^{n-1} \notin (s^n, t^n)(s^n, s^{n-1}t, t^n)^{n-2},$$

and hence we have proved (2).

References

[1] S. Goto, K. Nishida and K. Ozeki, *The structure of Sally modules of rank one*, Preprint (2007), arXiv: math.AC/0710.1178.

[2] A. Guerrieri and M. E. Rossi, *Hilbert coefficients of Hilbert filtrations*, J. Algebra, **199** (1998), 40–61.

[3] S. Goto and H. Sakurai, *The reduction exponent of socle ideals associated to parameter ideals in a Buchsbaum local ring of multiplicity two*, J. Math. Soc. Japan, **56** (2004), 1157–1168.

[4] S. Huckaba, *On associated graded rings having almost maximal depth*, Comm. Algebra, **26** (1998), 967–976.

[5] C. Huneke, *Complete ideals in two-dimensional regular local rings*, Commutative algebra, Math. Sci. Res. Inst. Publ., **15**, Springer, New York, 1989, 325–338.

[6] J. Lipman and B. Teissier, *Pseudo-rational local rings and a theorem of Briançon-Skoda about integral closure of ideals*, Michigan Math. J., **28** (1981), 97–116.

[7] M. E. Rossi, *A bound on the reduction number of a primary ideal*, Proc. Amer. Math. Soc., **128** (1999), 1325–1332.

[8] J. D. Sally, *On the associated graded ring of a local Cohen-Macaulay ring*, J. Math. Kyoto Univ., **17** (1977), 19–21.

[9] R. J. Ratliff and D. E. Rush, *Two notes on reductions of ideals*, Indiana Univ. Math. J., **27** (1978), 929–934.

[10] H. Wang, *Links of symbolic powers of prime ideals*, Math. Z., **256** (2007), 749–756.

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