

# NONLINEAR EVOLUTION EQUATIONS GENERATED BY SUBDIFFERENTIALS WITH NONLOCAL CONSTRAINTS

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**Abstract.** We consider an abstract formulation for a class of parabolic quasi-variational inequalities or quasi-linear PDEs, which are generated by subdifferentials of convex functions with various nonlocal constraints depending on the unknown functions. In this paper we specify a class of convex functions  $\{\varphi^t(v; \cdot)\}$  on a real Hilbert space  $H$ , with parameters  $0 \leq t \leq T$  and  $v$  in a set of functions from  $[-\delta_0, T]$ ,  $0 < \delta_0 < \infty$ , into  $H$ , in order to formulate an evolution equation of the form

$$u'(t) + \partial\varphi^t(u; u(t)) \ni f(t), \quad 0 < t < T, \quad \text{in } H.$$

Our objective is to discuss the existence question for the Cauchy problem.

## 1. Introduction

For positive numbers  $\delta_0$ ,  $T$ , we are given sets  $V(-\delta_0, t)$ ,  $0 \leq t \leq T$ , of functions from  $(-\delta_0, t)$  into a real Hilbert space  $H$  and a family  $\{\varphi^s(v; \cdot)\}_{0 \leq s \leq t}$  of proper, lower semicontinuous convex functions  $\varphi^s(v; \cdot)$  with parameters  $s \in [0, t]$  and  $v \in V(-\delta_0, t)$ ; here  $\varphi^s(v; \cdot)$  continuously depends upon  $v \in V(-\delta_0, t)$  in a certain nonlocal way (see section 2 for the detail definition). We consider a nonlinear evolution equation of the form:

$$u'(t) + \partial\varphi^t(u; u(t)) \ni f(t), \quad 0 < t < T, \quad \text{in } H, \quad (1.1)$$

subject to the initial condition

$$u(t) = u_0(t), \quad -\delta_0 \leq t \leq 0, \quad \text{in } H, \quad (1.2)$$

where  $\partial\varphi^t(u; \cdot)$  is the subdifferential of convex function  $\varphi^t(u; \cdot)$  on  $H$ ,  $u' = \frac{du}{dt}$  and  $u_0 : [-\delta_0, 0] \rightarrow H$  and  $f : (0, T) \rightarrow H$  are prescribed as the initial and forcing functions, respectively. This is a sort of functional differential equations generated by subdifferentials of  $\varphi^t(v; \cdot)$  with a nonlocal dependence upon  $v$ . The objective of this paper is to specify a class of convex functions  $\{\varphi^s(v; \cdot)\}_{0 \leq s \leq t}$  as well as its nonlocal dependence upon  $v \in$

$V(-\delta_0, t)$  in order that Cauchy problem  $\{(1.1), (1.2)\}$  admits at least one local or global in time solution  $u$ .

Variational problems are often called “quasi-variational problems”, when the constraints depend upon the unknowns. The stationary cases have been dealt with in many papers, for instance, [2,5,10,13,14], but there are not so many papers dealing with the time-evolution problems, because it is not expected for solutions to have much regularity in time. We recall some papers [11,15,16] for time evolution quasi-variational inequalities. In papers [11,16], the so-called monotonicity property of the mapping  $v \rightarrow \varphi^s(v; \cdot)$  is used as one of key tools in their treatment. However, the monotonicity property is too restrictive in many important applications, as examples of section 5 suggest. They evolved the theory of quasi-variational evolution equations with a concept of weak solutions. Our main theorems (Theorems 3.1, 4.1 and 4.2) of this paper ensure the existence of strong solutions without assuming the monotonicity property of the mapping  $v \rightarrow \varphi^s(v; \cdot)$ . A similar attempt was made in the paper [15] for an evolution problem arising in the theory of semiconductor.

The solvability of evolution equations of the form (1.1) seems delicate, as a simple example shows below. Let us consider a scalar evolution equation

$$u'(t) + \partial I_{[2u(t), \infty)}(u(t)) \ni 1, \quad 0 < t < T, \quad (1.3)$$

where  $I_{[2v, \infty)}$  is the indicator function of the real interval  $[2v, \infty)$  and  $\partial I_{[2v, \infty)}$  is its subdifferential, namely

$$I_{[2v, \infty)}(z) := \begin{cases} 0, & \text{if } z \geq 2v, \\ \infty, & \text{if } z < 2v, \end{cases} \quad \partial I_{[2v, \infty)}(z) := \begin{cases} 0, & \text{if } z > 2v, \\ (-\infty, 0], & \text{if } z = 2v, \\ \emptyset, & \text{if } z < 2v. \end{cases} \quad (1.4)$$

Any solution  $u$  of (1.3) satisfies that  $u(t) \geq 2u(t)$ , hence  $u(t) \leq 0$ . Also, because of (1.4),  $u'(t) \geq 1$ , if (1.3) holds at time  $t$ . Therefore, when the initial condition is  $u(0) = 0$ , (1.3) has no solution, since  $u(t) > 0$  for any  $t > 0$ . This is an example which shows that the Cauchy problem has no solution, even if the mapping  $v \rightarrow \varphi^t(v; \cdot)$  is regular enough.

In this paper we shall specify a class of  $\{\varphi^s(v; \cdot)\}$  of convex functions on  $H$  and give a nonlocal dependence of  $\{\varphi^s(v; \cdot)\}$  upon  $v$  in order that the Cauchy problem (1.1)-(1.2) has a local in time solution or more restrictedly a global in time solution.

The solvability of problem (1.1)-(1.2) is based on that of evolution equations generated by the subdifferentials of time dependent convex functions  $\psi^t(\cdot)$  of the form:

$$u'(t) + \partial \psi^t(u(t)) \ni f(t), \quad 0 < t < T, \text{ in } H \quad (1.5)$$

subject to the initial condition  $u(0) = u_0$ . Therefore, prior to (1.1)-(1.2) we shall recall the important class of  $\psi^t(\cdot)$  which guarantees the well-posedness of Cauchy problems for equation (1.5). The main part of this theory was evolved in [3,7,8,17].

As a typical example of equation (1.1), we apply our theorems to the following system

of inequalities:

$$\begin{aligned}
u_t - \Delta u &\geq f(x, t) \quad \text{on } Q := \Omega \times (0, T), \\
u &\geq k_c(u; \cdot, \cdot) \quad \text{on } Q, \\
(u_t - \Delta u - f(x, t))(u - k_c(u; \cdot, \cdot)) &= 0 \quad \text{on } Q, \\
\frac{\partial u}{\partial n} &\geq 0, \quad \frac{\partial u}{\partial n}(u - k_c(u; \cdot, \cdot)) = 0 \quad \text{on } \Sigma := \partial\Omega \times (0, T);
\end{aligned} \tag{1.6}$$

here  $\Omega$  is a bounded smooth domain in  $\mathbf{R}^N$ ,  $f$  is a given function on  $Q$ ,  $k_c$  is a integral mapping of the form:

$$k_c(v; x, t) = \int_{-\delta_0}^t \int_{\Omega} \rho(x - y, t - s, v(y, s)) dy ds, \tag{1.7}$$

where  $\rho$  is a smooth function with respect to all the variables on  $\mathbf{R}^N \times \mathbf{R} \times \mathbf{R}$ . The above system (1.6) is reformulated as a parabolic variational inequality of the form:

$$\begin{aligned}
u &\in W^{1,2}(-\delta_0, T; L^2(\Omega)) \cap L^\infty(-\delta_0, T; H^1(\Omega)) \text{ with } u \geq k_c(u; \cdot, \cdot) \text{ a.e. on } Q; \\
\int_Q \{u_t(u - w) + \nabla u \cdot \nabla(u - w)\} dx dt &\leq \int_Q f(x, t)(u - w) dx dt, \\
\forall w &\in L^2(0, T; H^1(\Omega)) \text{ with } w \geq k_c(u; \cdot, \cdot) \text{ a.e. on } Q.
\end{aligned} \tag{1.8}$$

In the system (1.8) the constraint  $k_c = k_c(u; \cdot, \cdot)$  depends upon the unknown  $u$ . Moreover, it is easy to check that (1.8) is written in the form (1.1) by using the following convex function  $\varphi^s(v; \cdot)$  on  $H := L^2(\Omega)$  given by

$$\varphi^s(v; z) := \begin{cases} \frac{1}{2} \int_{\Omega} |\nabla z|^2 dx, & \text{if } z \in H^1(\Omega), z \geq k_c(v; \cdot, s) \text{ a.e. on } \Omega, \\ +\infty, & \text{otherwise .} \end{cases}$$

Thus equation (1.1) is an abstract formulation which includes a class of parabolic quasi-variational inequalities.

*(Notation and fundamental concepts)*

In general, for a given real Banach space  $X$  we denote by  $|\cdot|_X$  the norm in  $X$ .

Throughtout this paper, let  $H$  be a real Hilbert space with inner product  $(\cdot, \cdot)_H$  and norm  $|\cdot|_H$ . Given a proper, lower semi-continuous (l.s.c.) and convex function  $\psi(\cdot)$  on  $H$  we use the usual notation:

- $D(\psi) := \{z \in H; \psi(z) < \infty\}$  (effective domain).
- $\partial\psi$  is the subdifferential of  $\psi$ , which is a (multivalued) mapping in  $H$  and defined by

$$z^* \in \partial\psi(z) \iff (z^*, v - z)_H \leq \psi(v) - \psi(z), \quad \forall v \in H$$

with domain  $D(\partial\psi) := \{z \in H; \partial\psi(z) \neq \emptyset\} (\subset D(\psi))$ .

There is an important concept of convergence for convex functions, which was introduced by Mosco [12] in order to characterize the convergence of solutions to variational inequalities. Let  $\{\psi_n\}$  be a sequence of proper l.s.c. and convex functions on  $H$ . Then it is said that  $\psi_n$  converges to a proper, l.s.c. and convex function  $\psi$  on  $H$  in the sense of Mosco, if the following two conditions (M1) and (M2) are fulfilled:

$$(M1) \liminf_{n \rightarrow \infty} \psi_n(z) \geq \psi(z) \text{ for every } z \in H.$$

$$(M2) \text{ For each } z \in D(\psi) \text{ there is a sequence } \{z_n\} \text{ in } H \text{ such that } z_n \rightarrow z \text{ in } H \text{ and } \psi_n(z_n) \rightarrow \psi(z) \text{ as } n \rightarrow \infty.$$

We refer various basic properties about convex functions to monographs [1,4,9].

## 2. A class of time-dependent convex functions

Given a family  $\{\psi^t\} := \{\psi^t\}_{0 \leq t \leq T}$  of time-dependent proper, l.s.c. and convex functions  $\psi^t$  on  $H$  for a positive finite number  $T$ , let us consider an evolution equation generated by the subdifferential  $\partial\psi^t$  in the following form:

$$(CP) \quad \begin{cases} u'(t) + \partial\psi^t(u(t)) \ni f(t), & 0 < t < T, \text{ in } H, \\ u(0) = u_0 & \text{in } H, \end{cases} \quad (2.1)$$

where  $f$  and  $u_0$  are respectively prescribed in  $L^2(0, T; H)$  and  $H$ . We say that  $u$  is a solution of (CP) on  $[0, T]$ , if  $u \in C([0, T]; H) \cap W_{loc}^{1,2}((0, T]; H)$ ,  $\psi^{(\cdot)}(u) \in L^1(0, T)$ ,  $u(0) = u_0$  and  $f(t) - u'(t) \in \partial\psi^t(u(t))$  holds for a.e.  $t \in (0, T)$ . When the data  $\{\psi^t\}$ ,  $u_0$ ,  $f$  are explicitly indicated, (CP) is denoted by  $(CP; \{\psi^t\}, u_0, f)$ .

Now, we specify a class of families  $\{\psi^t\}$  of time-dependent convex functions on  $H$  so that problem (2.1) admits a solution. Let  $\{a_r\} := \{a_r; 0 \leq r < \infty\}$  and  $\{b_r\} := \{b_r; 0 \leq r < \infty\}$  be subsets consisting of non-negative functions in  $L^2(0, T)$  and  $L^1(0, T)$ , respectively. Then we define the class  $G(\{a_r\}, \{b_r\})$  of  $\{\psi^t\}$  as follows.

**Definition 2.1.** We denote by  $G(\{a_r\}, \{b_r\})$  the set of all families  $\{\psi^t\} := \{\psi^t\}_{0 \leq t \leq T}$  of proper (i.e. not identically  $\infty$ ), l.s.c., non-negative and convex function  $\psi^t(\cdot)$  on  $H$  satisfying that  $\forall r > 0$ ,  $0 \leq \forall s \leq \forall t \leq T$ ,  $\forall z \in D(\psi^s)$  with  $|z|_H \leq r$ ,  $\exists \tilde{z} \in D(\psi^t)$  such that

$$|\tilde{z} - z|_H \leq \int_s^t a_r(\tau) d\tau (1 + \psi^s(z)^{\frac{1}{2}}), \quad \psi^t(\tilde{z}) - \psi^s(z) \leq \int_s^t b_r(\tau) d\tau (1 + \psi^s(z)).$$

We may assume without loss of generality that  $a_r$ ,  $b_r$  are non-decreasing with respect to  $r > 0$ , namely  $a_{r_1} \geq a_{r_2}$ ,  $b_{r_1} \geq b_{r_2}$  a.e. on  $[0, T]$ , if  $r_1 > r_2$ . If the time interval  $[0, T]$  is required to indicate explicitly, we denote  $G(\{a_r\}, \{b_r\})$  by  $G_{[0, T]}(\{a_r\}, \{b_r\})$ .

Furthermore, let  $\{M_r\}_{0 \leq r < \infty}$  be a family of non-negative numbers. We then put

$$\mathcal{G}(\{M_r\}) = \bigcup_{|a_r|_{L^2(0, T)} \leq M_r, |b_r|_{L^1(0, T)} \leq M_r, 0 \leq \forall r < \infty} G(\{a_r\}, \{b_r\}); \quad (2.2)$$

this is denoted by  $\mathcal{G}_{[0, T]}(\{M_r\})$ , when the interval  $[0, T]$  is indicated explicitly.

We recall an existence-uniqueness result on (CP).

**Theorem 2.1.** (cf. [3,7,8,17]) *Assume that  $\{\psi^t\} \in G(\{a_r\}, \{b_r\})$ . Let  $f \in L^2(0, T; H)$  and  $u_0 \in \overline{D(\psi^0)}$ . Then, (CP) has one and only one solution  $u$  on  $[0, T]$  such that*

$$\sqrt{t}u' \in L^2(0, T; H), \quad \sup_{0 < t \leq T} t\psi^t(u(t)) < \infty.$$

Moreover, if  $u_0 \in D(\psi^0)$ , then

$$u' \in L^2(0, T; H), \quad \sup_{0 \leq t \leq T} \psi^t(u(t)) < \infty.$$

We recall briefly the proof of the above theorem, since the key ideas for the solvability of our quasi-variational evolution problem (1.1) are found there.

The construction of a solution of (2.1) is made by showing the convergence of the solutions  $u_\lambda$  of the following approximate problems, with real parameters  $\lambda \in (0, 1]$ , as  $\lambda \downarrow 0$ :

$$u'_\lambda(t) + \partial\psi_\lambda^t(u_\lambda(t)) = f(t) \quad \text{in } H \text{ for a.e. } t \in [0, T], \quad (2.3)$$

with initial condition  $u_\lambda(0) = u_0$ , where  $\psi_\lambda^t$  is the Moreau-Yosida approximation, i.e.

$$\psi_\lambda^t(v) := \inf_{z \in H} \left\{ \frac{1}{2\lambda} |v - z|_H^2 + \psi^t(z) \right\}, \quad \forall v \in H.$$

In order to get the uniform estimates for approximate solutions  $u_\lambda$  with respect to  $\lambda \in (0, 1]$  we derive the following key inequality from the time-dependence condition on  $\psi^t(\cdot)$  mentioned in Definition 2.1:

$$\frac{d}{dt} \psi_\lambda^t(u_\lambda(t)) - (\partial\psi_\lambda^t(u_\lambda(t)), u'_\lambda(t))_H \quad (2.4)$$

$$\leq a_r(t) |\partial\psi_\lambda^t(u_\lambda(t))| (\psi_\lambda^t(u_\lambda(t))^{\frac{1}{2}} + 1) + b_r(t) (\psi_\lambda^t(u_\lambda(t)) + 1), \quad \text{a.e. } t \in [0, T],$$

for any  $r > |u_\lambda|_{L^\infty(0, T; H)}$ .

First of all, taking the inner product of the both sides of (2.3) and  $u_\lambda(t) - h(t)$  for any function  $h \in W^{1,2}(0, T; H)$  with  $\psi^{(\cdot)}(h) \in L^1(0, T)$ , we have

$$\frac{d}{dt} \left\{ \frac{1}{2} |u_\lambda(t)|_H^2 - (u_\lambda(t), h(t))_H \right\} + (u_\lambda(t), h'(t) - f(t))_H + \psi_\lambda^t(u_\lambda(t)) \quad (2.5)$$

$$\leq \psi_\lambda^t(h(t)) - (f(t), h(t))_H, \quad \text{a.e. } t \in [0, T];$$

note here that the existence of such a function  $h$  is also shown from our condition on the time-dependence of  $\psi^t(\cdot)$ , i.e.,  $\{\psi^t\} \in G(\{a_r\}, \{b_r\})$ , mentioned in Definition 2.1. Applying the Gronwall's lemma to (2.5) yields an inequality of the form

$$|u_\lambda|_{L^\infty(0, T; H)}^2 + \int_0^T \psi_\lambda^t(u_\lambda(t)) dt \leq R_1 (|u_0|_H + |f|_{L^2(0, T; H)}), \quad \forall \lambda \in (0, 1], \quad (2.6)$$

where  $R_1(\cdot)$  is a non-negative and non-decreasing function from  $[0, \infty)$  into  $[0, \infty)$  which depends only on the class  $\mathcal{G}(\{M_r\})$ .

Next, taking the inner product of the both sides of (2.3) and  $u'_\lambda$ , we obtain that

$$|u'_\lambda(\tau)|_H^2 + (\partial\psi_\lambda^t(u_\lambda(\tau)), u'_\lambda(\tau))_H = (f(\tau), u'_\lambda(\tau))_H, \quad \text{a.e. } \tau \in (0, T).$$

Using inequality (2.4) in the above relation, we see for any  $r > R_1 (|u_0|_H + |f|_{L^2(0,T;H)})$  that

$$\frac{d}{d\tau}\psi_\lambda^\tau(u_\lambda(\tau)) + \frac{1}{2}|u'_\lambda(\tau)|_H^2 \leq k_r(\tau)(\psi_\lambda^\tau(u_\lambda(\tau)) + 1), \quad \text{a.e. } \tau \in (0, T), \quad \forall \lambda \in (0, 1], \quad (2.7)$$

where

$$k_r(\tau) := 6 (|a_r(\tau)|^2 + |b_r(\tau)| + |f(\tau)|_H^2), \quad \text{a.e. } \tau \in [0, T]. \quad (2.8)$$

Further, multiplying the both sides of (2.7) by  $\tau$ , we get

$$\frac{d}{d\tau}\{\tau\psi_\lambda^\tau(u_\lambda(\tau))\} + \frac{\tau}{2}|u'_\lambda(\tau)|_H^2 \leq (\tau k_r(\tau) + 1)(\psi_\lambda^\tau(u_\lambda(\tau)) + 1) \quad (2.9)$$

$$\text{a.e. } \tau \in (0, T), \quad \forall \lambda \in (0, 1],$$

From these inequalities (2.6), (2.7) and (2.9) we see that  $\{u_\lambda\}$  is bounded in  $W^{1,2}(0, T; H)$  and  $\{\psi_\lambda^t(u_\lambda)\}$  is bounded in  $L^\infty(0, T)$ , and moreover by the usual monotonicity argument that  $u_\lambda$  converges to the solution  $u$  of (CP) as  $\lambda \downarrow 0$  in the sense that

$$u_\lambda \rightarrow u \quad \text{in } C([0, T]; H), \quad \text{weakly in } W^{1,2}(0, T; H),$$

$$\partial\psi_\lambda(u_\lambda) \rightarrow \xi \quad \text{in } L^2(0, T; H), \quad \xi(t) := f(t) - u'(t) \in \partial\psi^t(u(t)) \quad \text{a.e. } t \in (0, T).$$

Accordingly, integrating energy inequalities (2.7) and (2.9) in time and letting  $\lambda \downarrow 0$ , we obtain the following estimates for the solution  $u$  of (CP):

$$|u|_{L^\infty(0,T;H)}^2 + \int_0^T \psi^t(u(t))dt \leq R_1 (|u_0|_H + |f|_{L^2(0,T;H)}); \quad (2.10)$$

$$\psi^t(u(t)) - \psi^s(u(s)) + \frac{1}{2} \int_s^t |u'(\tau)|_H^2 d\tau \leq \int_s^t k_r(\tau)(\psi^\tau(u(\tau)) + 1)d\tau, \quad (2.11)$$

$$t\psi^t(u(t)) - s\psi^s(u(s)) + \frac{1}{2} \int_s^t \tau |u'(\tau)|_H^2 d\tau \leq \int_s^t (\tau k_r(\tau) + 1)(\psi^\tau(u(\tau)) + 1)d\tau, \quad (2.12)$$

for all  $s, t$  with  $0 \leq s \leq t \leq T$  and all  $r > R_1 (|u_0|_H + |f|_{L^2(0,T;H)})$ .

The results mentioned above are summarized in the following theorem.

**Theorem 2.2.** *Let  $\mathcal{G}(\{M_r\})$  be as given by (2.2). Then we have:*

(i) *For each positive number  $p_1$  there is a positive constant  $P_1$ , depending on  $\mathcal{G}(\{M_r\})$ , such that*

$$|u|_{L^\infty(0,T;H)}^2 + |\sqrt{t}u'|_{L^2(0,T;H)}^2 + \sup_{0 \leq t \leq T} t\psi^t(u(t)) + \int_0^T \psi^t(u(t))dt \leq P_1 \quad (2.13)$$

for every solution  $u$  of (CP), whenever  $|u_0|_H + |f|_{L^2(0,T;H)} \leq p_1$ .

(ii) For each positive number  $p_2$  there is a positive constant  $P_2$ , depending on  $\mathcal{G}(\{M_r\})$ , such that

$$|u'|_{L^2(0,T;H)}^2 + \sup_{0 \leq t \leq T} \psi^t(u(t)) \leq P_2 \quad (2.14)$$

for every solution  $u$  of (CP), whenever  $|u_0|_H + \psi^0(u_0) + |f|_{L^2(0,T;H)} \leq p_2$ .

In construction of solutions of our quasi-variational evolution equations the next convergence theorem plays an important role together with the above Theorem 2.2. Our result is based on the concept of convergence of convex functions due to Mosco.

**Theorem 2.3.** *Let  $\{\psi_n^t\}$  be a sequence in  $\mathcal{G}(\{M_r\})$  and  $\{u_{0n}\}$  and  $\{f_n\}$  be sequences in  $H$  and  $L^2(0, T; H)$ , respectively, such that*

(a)  $\psi_n^t$  converges to  $\psi^t$  on  $H$  in the sense of Mosco as  $n \rightarrow \infty$  for every  $t \in [0, T]$ , where  $\{\psi^t\}$  is a family in  $\mathcal{G}(\{M_r\})$ .

(b)  $u_{0n} \in \overline{D(\psi_n^0)}$ ,  $u_0 \in \overline{D(\psi^0)}$ ,  $u_{0n} \rightarrow u_0$  in  $H$ , and  $f_n \rightarrow f$  in  $L^2(0, T; H)$  as  $n \rightarrow \infty$ .

Then the solution  $u_n$  of (CP;  $\{\psi_n^t\}, u_{0n}, f_n$ ) tends to the solution  $u$  of (CP;  $\{\psi^t\}, u_0, f$ ) in the sense that

$$u_n \rightarrow u \text{ in } C([0, T]; H), \quad \sqrt{t}u'_n \rightarrow \sqrt{t}u' \text{ weakly in } L^2([0, T]; H), \quad (2.15)$$

$$\int_0^T \psi_n^t(u_n) dt \rightarrow \int_0^T \psi^t(u) dt \text{ as } n \rightarrow \infty. \quad (2.16)$$

Moreover, if  $\{\psi_n^0(u_{0n})\}$  is bounded, then

$$u'_n \rightarrow u' \text{ weakly in } L^2([0, T]; H), \quad \psi_n^t(u_n(t)) \rightarrow \psi^t(u(t)) \text{ for a.e } t \in [0, T]. \quad (2.17)$$

For the detail proofs of Theorems 2.1, 2.2 and 2.3, see [8; Chapter 1].

### 3. Local existence result

In order to formulate functions  $\varphi^t(v; \cdot)$  precisely we introduce a time-independent, non-negative, proper, l.s.c. and convex function  $\varphi_0(\cdot)$  on  $H$  such that

( $\varphi_0$ ) the set  $\{z \in H; |z|_H \leq r, \varphi_0(z) \leq r\}$  is compact in  $H$  for each  $r \geq 0$ .

Let  $\delta_0$  be a fixed positive number and  $T > 0$  be a finite time. For each  $t \in [0, T]$  we define a closed convex subset  $\mathcal{V}(-\delta_0, t)$  of  $W^{1,2}(-\delta_0, t; H)$  by

$$\mathcal{V}(-\delta_0, t) := \{v; V_{[-\delta_0, t]}(v) < \infty\} \quad (3.1)$$

with

$$V_{[-\delta_0, t]}(v) := \sup_{-\delta_0 \leq s \leq t} \varphi_0(v(s)) + |v(0)|_H^2 + |v'|_{L^2(-\delta_0, t; H)}^2 \quad (3.2)$$

where  $v'(t) = \frac{dv(t)}{dt}$ .

Now, to each  $v \in \mathcal{V}(-\delta_0, t)$  a family  $\{\varphi^s(v; \cdot)\}_{0 \leq s \leq t}$  of functions  $\varphi^s(v; \cdot)$  on  $H$  is assigned such that

(Φ1)  $\varphi^s(v; z)$  is proper, l.s.c., non-negative and convex in  $z \in H$ , and it is determined by  $s \in [0, t]$  and  $v$  on  $[-\delta_0, s]$ ; namely, for  $v_1, v_2 \in \mathcal{V}(-\delta_0, t)$ , we have  $\varphi^s(v_1, \cdot) \equiv \varphi^s(v_2, \cdot)$  on  $H$  whenever  $v_1 \equiv v_2$  on  $[-\delta_0, s]$ ;

(Φ2)  $\varphi^s(v; z) \geq \varphi_0(z)$ ,  $\forall v \in \mathcal{V}(-\delta_0, t)$ ,  $0 \leq \forall s \leq \forall t \leq T$ ;

(Φ3) If  $0 \leq s_n \leq t \leq T$ ,  $v_n \in \mathcal{V}(-\delta_0, t)$ ,  $\sup_{n \in \mathbf{N}} V_{[-\delta_0, t]}(v_n) < \infty$ ,  $s_n \rightarrow s$  and  $v_n \rightarrow v$  in  $C([-\delta_0, t]; H)$ , then  $\varphi^{s_n}(v_n; \cdot) \rightarrow \varphi^s(v; \cdot)$  on  $H$  in the sense of Mosco.

We give the definition of solutions for evolution equation (1.1).

**Definition 3.1.** Let  $u_0 \in C([-\delta_0, 0]; H)$  and  $f \in L^2(0, T; H)$ . Then we say that  $u$  is a solution of the Cauchy problem

$$CP(u_0, f) \quad \begin{cases} u'(t) + \partial\varphi^t(u; u(t)) \ni f(t), & 0 < t < T, \\ u = u_0 & \text{on } [-\delta_0, 0] \end{cases}$$

on  $[0, T]$ , if  $u$  satisfies that  $u \in C([-\delta_0, T]; H)$ ,  $u = u_0$  on  $[-\delta_0, 0]$ ,  $u \in W^{1,2}(\delta, T; H)$  for every (small)  $\delta > 0$ ,  $\varphi^{(\cdot)}(u; u(\cdot)) \in L^1(0, T)$  and  $f(t) - u'(t) \in \partial\varphi^t(u; u(t))$  for a.e.  $t \in (0, T)$ .

Next, in order to formulate our local existence result for  $CP(u_0, f)$  we introduce the following function spaces: given any function  $u_0$  in  $\mathcal{V}(-\delta_0, 0)$ ,  $0 < R < \infty$  and  $t \in [0, T]$ , we put

$$\mathcal{V}(u_0; -\delta_0, t) := \{v \in \mathcal{V}(-\delta_0, t); v = u_0 \text{ on } [-\delta_0, 0]\}, \quad (3.3)$$

and

$$\mathcal{V}_R(u_0; -\delta_0, t) := \left\{ v \in \mathcal{V}(u_0; -\delta_0, t); \sup_{0 \leq s \leq t} \left\{ \varphi_0(v(s)) + |v'|_{L^2(0, s; H)}^2 \right\} \leq R \right\}. \quad (3.4)$$

We are in a position to state a local existence result for problem  $CP(u_0, f)$ .

**Theorem 3.1.** Let  $0 < T < \infty$  and  $u_0 \in \mathcal{V}(-\delta_0, 0)$  with  $\varphi^0(u_0; u_0(0)) < \infty$ . Assume that there are positive numbers  $T_0 \leq T$  and  $R > \varphi^0(u_0; u_0(0))$ , a family  $\{M_r\}_{0 \leq r < \infty}$  of positive numbers  $M_r$  and a set  $\{\{\varphi^t(v; \cdot)\}; v \in \mathcal{V}_R(u_0; -\delta_0, T_0)\}$  of families  $\{\varphi^t(v; \cdot)\}_{0 \leq t \leq T_0}$  of convex functions satisfying the following condition (\*):

(\*) There are two families  $\{a_r^v; v \in \mathcal{V}_R(u_0; -\delta_0, T_0), 0 \leq r < \infty\}$  of non-negative functions in  $L^2(0, T_0)$  and  $\{b_r^v; v \in \mathcal{V}_R(u_0; -\delta_0, T_0), 0 \leq r < \infty\}$  of non-negative functions in  $L^1(0, T_0)$  such that

(H1)  $|a_r^v|_{L^2(0, T_0)} \leq M_r$  and  $|b_r^v|_{L^1(0, T_0)} \leq M_r$  for all  $r > 0$  and all  $v \in \mathcal{V}_R(u_0; -\delta_0, T_0)$ , and  $\{\varphi^t(v; \cdot)\} \in G(\{a_r^v\}, \{b_r^v\})$  for all  $v \in \mathcal{V}_R(u_0; -\delta_0, T_0)$ ;

(H2) for each finite  $r > 0$  and  $\varepsilon > 0$  there is a positive number  $\delta_{r\varepsilon} > 0$  such that

$$\int_0^{\delta_{r\varepsilon}} (a_r^v(\tau)^2 + b_r^v(\tau)) d\tau < \varepsilon, \quad \forall v \in \mathcal{V}_R(u_0; -\delta_0, T_0).$$

Then, for each  $f \in L^2(0, T_0; H)$ , problem  $CP(u_0, f)$  has at least one solution  $u$  on an interval  $[0, T']$  with  $0 < T' \leq T_0$  such that  $u \in \mathcal{V}(-\delta_0, T'; H)$  and  $\sup_{0 \leq t \leq T'} \varphi^t(u; u(t)) < \infty$ .

In the rest of this section we give a proof of Theorem 3.1. Let  $v$  be any element in  $\mathcal{V}_R(u_0; -\delta_0, T_0)$ . Then, by (H1),  $\{\varphi^t(v; \cdot)\} \in G(\{a_r^v\}, \{b_r^v\})$  with  $|a_r^v|_{L^2(0, T_0)} \leq M_r$  and  $|b_r^v|_{L^1(0, T_0)} \leq M_r$  for all  $r \geq 0$ . Now, consider the problem

$$\begin{cases} u'(t) + \partial\varphi^t(v; u(t)) \ni f(t), & \text{a.e. } t \in [0, T_0], \\ u(0) = u_0(0). \end{cases} \quad (3.5)$$

By virtue of Theorem 2.1, this problem has a unique solution  $u \in W^{1,2}(0, T_0; H)$  such that  $\sup_{0 \leq t \leq T_0} \varphi^t(v; u(t)) < \infty$ . On account of (ii) of Theorem 2.2 we have the following uniform estimates with respect to  $v \in \mathcal{V}_R(u_0; -\delta_0, T_0)$ :

$$|u'|_{L^2(0, T_0; H)}^2 + \sup_{0 \leq t \leq T_0} \varphi^t(v; u(t)) \leq P_2, \text{ hence } |u|_{L^\infty(0, T_0; H)} \leq |u_0|_H + \sqrt{T_0 P_2} =: r_0, \quad (3.6)$$

where  $P_2$  is a positive constant independent of  $v \in \mathcal{V}_R(u_0; -\delta_0, T_0)$

Moreover, we have:

**Lemma 3.1.** *Let  $\tau_0$  be a positive number such that  $R > \varphi^0(u_0; u_0(0)) + \tau_0$ . Then there exists a positive number  $T_1$  such that*

$$\varphi^t(v; u(t)) + |u'|_{L^2(0, T_1; H)}^2 \leq \varphi^0(u_0; u_0(0)) + \tau_0, \quad \forall t \in [0, T_1], \quad (3.7)$$

for any  $v \in \mathcal{V}_R(u_0; -\delta_0, T_0)$  and the solution  $u$  of (3.5).

**Proof.** By taking the inner product between the both sides of the equation in (3.5) and  $u'(t)$ , we have

$$|u'(t)|_H^2 + (\xi(t), u'(t))_H = (f(t), u'(t))_H, \quad \text{a.e. } t \in [0, T_0], \quad (3.8)$$

where  $\xi(t) := f(t) - u'(t) \in \partial\varphi^t(v; u(t))$  for a.e.  $t \in [0, T_0]$ . By (3.6), we have  $|u|_{L^\infty(0, T_0; H)} < r := r_0 + 1$  and hence (cf. (2.4)) it holds that

$$\begin{aligned} & \frac{d}{dt} \varphi^t(v; u(t)) - (\xi(t), u'(t))_H \\ & \leq a_r^v(t) |u'(t) - f(t)|_H (\varphi^t(v; u(t))^{\frac{1}{2}} + 1) + b_r^v(t) (\varphi^t(v; u(t)) + 1), \quad \text{a.e. } t \in [0, T_0]. \end{aligned} \quad (3.9)$$

Using this inequality, we see from (3.8) that

$$\begin{aligned} & |u'(t)|_H^2 + \frac{d}{dt} \varphi^t(v; u(t)) \\ & \leq |f(t)|_H |u'(t)|_H + (P_2^{\frac{1}{2}} + 1) a_r^v(t) (|u'(t)|_H + |f(t)|_H) + (P_2 + 1) b_r^v(t) \end{aligned}$$

for a.e.  $t \in [0, T_0]$ . Hence, for any  $t \in [0, T_0]$ ,

$$\begin{aligned} & \int_0^t |u'(\tau)|_H^2 d\tau + \varphi^t(v; u(t)) \\ & \leq \varphi^0(u_0; u_0(0)) + P_2^{\frac{1}{2}} \left( \int_0^t |f(\tau)|_H^2 d\tau \right)^{\frac{1}{2}} \\ & \quad + \left( \int_0^t a_r^v(\tau)^2 d\tau \right)^{\frac{1}{2}} \cdot 2(P_2 + 1) (P_2^{\frac{1}{2}} + |f|_{L^2(0, T_0; H)}) + (P_2 + 1) \int_0^t b_r^v(\tau) d\tau. \end{aligned}$$

Therefore, by condition (H2) there exists a small positive number  $T_1$ , independent of  $v \in \mathcal{V}_R(u_0; -\delta_0, T_0)$ , such that

$$P_2^{\frac{1}{2}} \left( \int_0^{T_1} |f(\tau)|_H^2 d\tau \right)^{\frac{1}{2}} + \left( \int_0^{T_1} a_r^v(\tau)^2 d\tau \right)^{\frac{1}{2}} \cdot 2(P_2 + 1)(P_2^{\frac{1}{2}} + |f|_{L^2(0, T_0; H)}) \\ + (P_2 + 1) \int_0^{T_1} b_r^v(\tau) d\tau \leq \tau_0.$$

Thus we have (3.7).  $\diamond$

**Proof of Theorem 3.1:** By (3.4) and assumption  $(\varphi_0)$  about the level set compactness of  $\varphi_0$ ,  $\mathcal{V}_R(u_0; -\delta_0, T_0)$  is non-empty, compact and convex in  $C([-\delta_0, T_0]; H)$ . Now, consider a mapping  $S : \mathcal{V}_R(u_0; -\delta_0, T_0) \rightarrow \mathcal{V}(-\delta_0, T_0; H)$  which is defined as follows:

$$[Sv](t) = \begin{cases} u_0(t), & \text{for } t \in [-\delta_0, 0], \\ u(t), & \text{for } t \in (0, T_1], \\ u(T_1), & \text{for } t \in (T_1, T_0], \end{cases} \quad (3.10)$$

where  $u$  is the solution of (3.5) associated with  $v \in \mathcal{V}_R(u_0; -\delta_0, T_0)$  and  $T_1$  is the same number as in Lemma 3.1. Then, for every  $v \in \mathcal{V}_R(u_0; -\delta_0, T_0)$ , it follows from the definition of  $S$  and Lemma 3.1 that  $Sv \in \mathcal{V}(u_0; -\delta_0, T_0)$  and

$$\sup_{0 \leq s \leq T_0} \left\{ \varphi_0([Sv](s)) + |[Sv]'|_{L^2(0, s; H)}^2 \right\} = \sup_{0 \leq s \leq T_1} \left\{ \varphi_0(u(s)) + |u'|_{L^2(0, s; H)}^2 \right\} \\ \leq \sup_{0 \leq s \leq T_1} \left\{ \varphi^s(v; u(s)) + |u'|_{L^2(0, s; H)}^2 \right\} \\ \leq \varphi^0(u_0; u_0(0)) + \tau_0 \\ \leq R,$$

where  $u$  is the solution of (3.5). Thus  $S$  maps  $\mathcal{V}_R(u_0; -\delta_0, T_0)$  into itself.

Next, we show the continuity of  $S$  in  $\mathcal{V}_R(u_0; -\delta_0, T_0)$  with respect to the topology of  $C([-\delta_0, T_0]; H)$ . Let  $\{v_n\}$  be any sequence in  $\mathcal{V}_R(u_0; -\delta_0, T_0)$  and suppose that  $v_n \rightarrow v$  in  $C([-\delta_0, T_0]; H)$  (as  $n \rightarrow \infty$ ). It is clear that  $v \in \mathcal{V}_R(u_0; -\delta_0, T_0)$  and  $\{V_{[-\delta_0, T_0]}(v_n)\}$  is bounded. By assumption  $(\Phi 3)$ , we see that  $\varphi^t(v_n; \cdot) \rightarrow \varphi^t(v; \cdot)$  on  $H$  in the sense of Mosco for every  $t \in [0, T_0]$ . Therefore, according to Theorem 2.3, the solution  $u_n$  of (3.5) corresponding to  $v = v_n$  converges to the solution  $u$  of (3.5) in the sense that

$$u_n \rightarrow u \text{ in } C([0, T_0]; H), \quad u'_n \rightarrow u' \text{ weakly in } L^2(0, T_0; H),$$

This means that  $Sv_n \rightarrow Sv$  in  $C([-\delta_0, T_0]; H)$ .

We are in a position to apply the fixed-point theorem for continuous mappings in compact and convex sets. Applying it to the mapping  $S$  we see that  $S$  has at least one fixed-point  $u_*$  in  $\mathcal{V}_R(u_0; -\delta_0, T_0)$ , i.e.  $Su_* = u_*$ . Denoting by  $u$  the restriction of  $u_*$  on  $[-\delta_0, T_1]$ , we easily check from the definition (3.10) that  $u$  is a solution of  $CP(u_0, f)$  on the time interval  $[0, T_1]$ .  $\diamond$

#### 4. Global existence result

Let  $\varphi_0$  be the same as in the previous section as well as  $\delta_0 > 0$  and  $T > 0$ . In this section, we consider a closed convex subset  $\tilde{\mathcal{V}}(-\delta_0, t)$  of  $L^2(-\delta_0, t; H)$  for each  $t \in [0, T]$ , as is defined below, in place of  $\mathcal{V}(-\delta_0, t)$ .

For each  $t \in [0, T]$  we define

$$\tilde{\mathcal{V}}(-\delta_0, t) := \{v; \tilde{V}_{[-\delta_0, t]}(v) < \infty\}, \quad (4.1)$$

where

$$\tilde{V}_{[-\delta_0, t]}(v) := |v|_{L^\infty(-\delta_0, t; H)}^2 + \int_{-\delta_0}^t \varphi_0(v(s)) ds. \quad (4.2)$$

Now, we suppose that to each  $v \in \tilde{\mathcal{V}}(-\delta_0, t)$  a family  $\{\varphi^s(v; \cdot)\}_{0 \leq s \leq t}$  of functions  $\varphi^s(v; \cdot)$  on  $H$  is assigned such that

- ( $\tilde{\Phi}1$ )  $\varphi^s(v; z)$  is proper, l.s.c., non-negative and convex in  $z \in H$ , and it is determined by  $s \in [0, t]$  and  $v$  on  $[-\delta_0, s]$ ; namely, for  $v_1, v_2 \in \tilde{\mathcal{V}}(-\delta_0, t)$ , we have  $\varphi^s(v_1, \cdot) \equiv \varphi^s(v_2, \cdot)$  on  $H$  whenever  $v_1 = v_2$  a.e. on  $(-\delta_0, s)$ ;
- ( $\tilde{\Phi}2$ )  $\varphi^s(v; z) \geq \varphi_0(z)$ ,  $\forall v \in \tilde{\mathcal{V}}(-\delta_0, t)$ ,  $0 \leq \forall s \leq \forall t \leq T$ ;
- ( $\tilde{\Phi}3$ ) If  $0 \leq s_n \leq t \leq T$ ,  $v_n \in \tilde{\mathcal{V}}(-\delta_0, t)$ ,  $\sup_{n \in \mathbf{N}} \tilde{V}_{[-\delta_0, t]}(v_n) < \infty$ ,  $s_n \rightarrow s$  and  $v_n \rightarrow v$  in  $L^2(-\delta_0, t; H)$ , then  $\varphi^{s_n}(v_n; \cdot) \rightarrow \varphi^s(v; \cdot)$  on  $H$  in the sense of Mosco.

Next, we define a function space  $\tilde{\mathcal{V}}_M(-\delta_0, t)$  for each  $M > 0$  and  $t \in [0, T]$  by

$$\tilde{\mathcal{V}}_M(-\delta_0, t) := \{v \in \tilde{\mathcal{V}}(-\delta_0, t); \tilde{V}_{[-\delta_0, t]}(v) \leq M\}.$$

In order to show the existence of a solution of  $CP(u_0, f)$  on the whole interval  $[0, T]$  we relax assumptions (H1) and (H2) as follows: For each  $M > 0$  there is a family  $\{M_r\}_{0 \leq r < \infty}$  of positive numbers  $M_r$  and a set  $\{\{\varphi^t(v; \cdot)\}; v \in \tilde{\mathcal{V}}_M(-\delta_0, T)\}$  of families  $\{\varphi^t(v; \cdot)\}_{0 \leq t \leq T}$  of convex functions satisfying the following condition (\*\*):

(\*\*) There are two families  $\{a_r^v; v \in \tilde{\mathcal{V}}_M(-\delta_0, T), 0 \leq r < \infty\}$  of non-negative functions in  $L^2(0, T)$  and  $\{b_r^v; v \in \tilde{\mathcal{V}}_M(-\delta_0, T), 0 \leq r < \infty\}$  of non-negative functions in  $L^1(0, T)$  such that

- ( $\tilde{H}1$ )  $|a_r^v|_{L^2(0, T)} \leq M_r$  and  $|b_r^v|_{L^1(0, T)} \leq M_r$  for all  $r > 0$  and all  $v \in \tilde{\mathcal{V}}_M(-\delta_0, T)$ , and  $\{\varphi^t(v; \cdot)\} \in G(\{a_r^v\}, \{b_r^v\})$  for all  $v \in \tilde{\mathcal{V}}_M(-\delta_0, T)$ ;
- ( $\tilde{H}2$ ) for each finite  $r > 0$  and  $\varepsilon > 0$  there is a positive number  $\delta_{r\varepsilon} > 0$  such that

$$\int_t^{t+\delta_{r\varepsilon}} (a_r^v(\tau)^2 + b_r^v(\tau)) d\tau < \varepsilon, \quad \forall t \in [0, T - \delta_{r\varepsilon}], \quad \forall v \in \tilde{\mathcal{V}}_M(-\delta_0, T).$$

It should be noted that these conditions are independent of initial data. Moreover we require the following assumption ( $\tilde{H}3$ ):

( $\tilde{H}3$ ) there are a positive number  $R_0$  and a family  $\{h_v\} := \{h_v; v \in \tilde{\mathcal{V}}(-\delta_0, T)\}$  of functions in  $W^{1,2}(0, T; H)$  such that

$$|h_v|_{W^{1,2}(0, T; H)} \leq R_0, \quad \int_0^T \varphi^t(v; h_v(t)) dt \leq R_0, \quad \forall v \in \tilde{\mathcal{V}}(-\delta_0, T).$$

We first show the existence of a solution  $CP(u_0, f)$  on the whole interval  $[0, T]$  for good initial values  $u_0$ .

**Theorem 4.1.** *Suppose that ( $\tilde{H}1$ ) and ( $\tilde{H}2$ ) hold for every  $M > 0$  as well as ( $\tilde{H}3$ ). Let  $u_0 \in \mathcal{V}(-\delta_0, 0)$  with  $\varphi^0(u_0; u_0(0)) < \infty$  and  $f$  be any function in  $L^2(0, T; H)$ . Then  $CP(u_0, f)$  has at least one solution  $u$  on  $[0, T]$  such that*

$$u \in W^{1,2}(0, T; H), \quad \sup_{0 \leq t \leq T} \varphi^t(u; u(t)) < \infty.$$

**Proof.** It is clear that ( $\Phi1$ ) – ( $\Phi3$ ) automatically satisfied, if ( $\tilde{\Phi}1$ ) – ( $\tilde{\Phi}3$ ) hold, and that ( $H1$ ) and ( $H2$ ) follow immediately from ( $\tilde{H}1$ ) and ( $\tilde{H}2$ ). Therefore, according to Theorem 3.1,  $CP(u_0, f)$  has a solution  $u$  on a certain time interval  $[0, \tau] (\subset [0, T])$  such that

$$u \in W^{1,2}(0, \tau; H), \quad \sup_{0 \leq t \leq \tau} \varphi^t(u; u(t)) < \infty.$$

Consider an ordered set  $Z$  given by

$$Z := \{(u, \tau); 0 < \tau \leq T, u \text{ is a solution of } CP(u_0, f) \text{ on } [0, \tau]\}$$

with an order  $\prec$  defined by

$$(u_1, \tau_1) \prec (u_2, \tau_2) \iff \tau_1 \leq \tau_2, \quad u_1 = u_2 \text{ on } [-\delta_0, \tau_1].$$

Then, by the local existence result mentioned above,  $Z$  is non-empty. Now, let  $Y$  be any totally ordered set in  $Z$  with respect to the above order  $\prec$ . Then, putting  $\hat{u}(t) = u(t)$  if  $(u, \tau) \in Y$  and  $0 \leq t \leq \tau$ , we see that  $\hat{u}$  is well defined on the interval  $[0, \hat{\tau}]$  with  $\hat{\tau} := \sup_{(u, \tau) \in Y} \tau$ . Moreover, we obtain that  $\hat{u}_0 := \lim_{t \uparrow \hat{\tau}} \hat{u}(t)$  exists in  $H$ . In fact, since  $\hat{u}$  is a solution of  $CP(u_0, f)$  on any compact interval  $[0, \tau]$  with  $0 < \tau < \hat{\tau}$ , it follows (cf. (2.5)) that

$$\frac{d}{dt} \left\{ \frac{1}{2} |\hat{u}(t)|_H^2 - (\hat{u}(t), h(t))_H \right\} + (\hat{u}(t), h'(t) - f(t))_H + \varphi^t(\hat{u}; \hat{u}(t)) \quad (4.3)$$

$$\leq \varphi^t(\hat{u}; h(t)) - (f(t), h(t))_H, \quad \text{for a.e. } t \in [0, \tau],$$

if  $0 < \tau < \hat{\tau}$ ,  $h \in W^{1,2}(0, \tau; H)$  and  $\varphi^{(\cdot)}(\hat{u}; h(\cdot))$  is integrable on  $[0, \tau]$ . Here, we use assumption ( $\tilde{H}3$ ) as follows. Take an increasing sequence  $\{\tau_n\}$  with  $\tau_n \uparrow \hat{\tau}$  and define a sequence  $\{u_n\}$  of functions by

$$u_n(t) = \begin{cases} \hat{u}(t) & \text{for } t \in [-\delta_0, \tau_n] \\ \hat{u}(\tau_n) & \text{for } t \in [\tau_n, T]. \end{cases}$$

Since  $\{u_n\} \subset \tilde{\mathcal{V}}(-\delta_0, T)$ , we see from  $(\tilde{H}3)$  that there are functions  $h_n$  for all  $n$  such that

$$|h_n|_{W^{1,2}(0,T;H)} \leq R_0, \quad \sup_{n \in \mathbf{N}} \int_0^T \varphi^t(u_n; h_n(t)) dt \leq R_0.$$

Noting that  $\varphi^t(u_n; h_n(t)) = \varphi^t(\hat{u}; h_n(t))$  a.e. on  $[0, \tau_n]$ , we infer from (4.3) with  $h = h_n$  that

$$\sup_{n \in \mathbf{N}} |\hat{u}|_{L^\infty(0, \tau_n; H)} < \infty, \quad \sup_{n \in \mathbf{N}} \int_0^{\tau_n} \varphi^t(\hat{u}; \hat{u}(t)) dt < \infty. \quad (4.4)$$

Hence,  $\hat{u} \in L^\infty(0, \hat{\tau}; H)$  and  $\varphi^{(\cdot)}(\hat{u}; \hat{u}(\cdot))$  is integrable on  $[0, \hat{\tau}]$ , namely  $\hat{u} \in \tilde{\mathcal{V}}(-\delta_0, \hat{\tau})$ . This shows by  $(\tilde{H}1)$  that  $\{\varphi^t(\hat{u}; \cdot)\} \in \mathcal{G}_{[0, T]}(\{M_r\})$  for some family  $\{M_r\} := \{M_r\}_{0 \leq r < \infty}$  of positive numbers. By Theorem 2.1, the Cauchy problem

$$w'(t) + \partial \varphi^t(\hat{u}; w(t)) \ni f(t), \quad 0 < t < \hat{\tau}, \quad w(0) = u_0(0).$$

has a unique solution  $w$  on the interval  $[0, \hat{\tau}]$  such that

$$w \in W^{1,2}(0, \hat{\tau}; H), \quad \sup_{0 \leq t \leq \hat{\tau}} \varphi^t(\hat{u}; w(t)) < \infty.$$

Since  $w = \hat{u}$  on  $[0, \hat{\tau}]$ , it follows that  $\hat{u}(\tau_n) = w(\tau_n) \rightarrow w(\hat{\tau})$ . If  $w$  is denoted by  $\hat{u}$ , the element  $(\hat{u}, \hat{\tau})$  is an upper bound of  $Y$ . Therefore, by virtue of Zorn's lemma, we conclude that  $Z$  has at least one maximal element  $(u^*, \tau^*)$ .

If  $\tau^* = T$  is shown, then  $u^*$  is a solution of  $CP(u_0, f)$  on  $[0, T]$ , namely it is enough to prove  $\tau^* = T$  for the completeness of the proof. Assume that  $\tau^* < T$ . Then, since  $\varphi^{\tau^*}(u^*; u^*(\tau^*)) < \infty$ , it follows that  $u^*$  is extended beyond time  $\tau^*$  as a solution of  $CP(u_0, f)$ . In fact, we consider the problem

$$\begin{cases} \tilde{u}'(t) + \partial \tilde{\varphi}^t(\tilde{u}; \tilde{u}(t)) \ni \tilde{f}(t), & 0 < t < \tilde{T}, \\ \tilde{u} = \tilde{u}_0^* & \text{on } [-\tilde{\delta}_0, 0], \end{cases} \quad (4.5)$$

where  $\tilde{T} := T - \tau^*$ ,  $\tilde{\delta}_0 := \delta_0 + \tau^*$ ,  $\tilde{u}_0^*(t) = u^*(t + \tau^*)$  for  $t \in [-\tilde{\delta}_0, 0]$ ,  $\tilde{f}(t) := f(t + \tau^*)$  for  $t \in (0, \tilde{T})$  and

$$\tilde{\varphi}^t(v; \cdot) := \varphi^{t+\tau^*}(v(\cdot + \tau^*); \cdot), \quad \forall v \in \tilde{\mathcal{V}}(-\tilde{\delta}_0, t), \quad 0 < t \leq \tilde{T}.$$

It is easy to see from  $(\tilde{H}1)$  and  $(\tilde{H}2)$  that assumptions  $(H1)$  and  $(H2)$  of Theorem 3.1 are satisfied for the family  $\{\tilde{\varphi}^t(v; \cdot)\}_{0 \leq t \leq \tilde{T}}$ , initial datum  $\tilde{u}_0^*$  and any  $R > \tilde{\varphi}^0(\tilde{u}_0^*; \tilde{u}_0^*(0))$ . Therefore, problem (4.5) has a solution  $\tilde{u}$  on a certain interval  $[0, \tilde{T}']$  such that

$$\tilde{u} \in W^{1,2}(0, \tilde{T}'; H), \quad \sup_{0 \leq t \leq \tilde{T}'} \tilde{\varphi}^t(\tilde{u}; \tilde{u}(t)) < \infty.$$

Putting

$$u(t) := \begin{cases} u^*(t) & \text{for } t \in [-\delta_0, \tau^*) \\ \tilde{u}(t - \tau^*) & \text{for } t \in [\tau^*, \tau^* + \tilde{T}'], \end{cases}$$

we observe that  $u \in W^{1,2}(0, \tau^* + \tilde{T}'; H)$ ,  $\sup_{0 \leq t \leq \tau^* + \tilde{T}'} \varphi^t(u; u(t)) < \infty$  and  $u$  is a solution of  $CP(u_0, f)$  on  $[0, \tau^* + \tilde{T}']$ . This contradicts the fact that  $(u^*, \tau^*)$  is maximal in  $Z$ . Consequently,  $\tau^* = T$  must be true.  $\diamond$

Finally we show the existence of a solution of  $CP(u_0, f)$  for a little bit more general class of initial data.

**Theorem 4.2.** *Suppose that  $(\tilde{\Phi}1)$ ,  $(\tilde{\Phi}2)$  and  $(\tilde{\Phi}3)$  hold and that  $(\tilde{H}1)$  and  $(\tilde{H}2)$  hold for every  $M > 0$  as well as  $(\tilde{H}3)$ . Let  $u_0 \in \tilde{\mathcal{V}}(-\delta_0, 0) \cap C([-\delta_0, 0]; H)$  such that there is a sequence  $\{u_{0n}\}$  in  $\mathcal{V}(-\delta, 0)$  with  $\varphi^0(u_{0n}; u_{0n}(0)) < \infty$  satisfying that*

$$\sup_{n \in \mathbf{N}} \tilde{V}_{[-\delta_0, 0]}(u_{0n}) < \infty, \quad u_{0n} \rightarrow u_0 \quad \text{in } C([-\delta_0, 0]; H). \quad (4.6)$$

Then  $CP(u_0, f)$  has at least one solution  $u$  on  $[0, T]$  such that

$$u \in C([0, T]; H), \quad \sqrt{t}u' \in L^2(0, T; H), \quad \sup_{0 < t \leq T} t\varphi^t(u; u(t)) < \infty. \quad (4.7)$$

**Proof.** Since  $u_{0n} \in \mathcal{V}(-\delta_0, 0)$  and  $\varphi^0(u_{0n}; u_{0n}(0)) < \infty$ , by virtue of Theorem 4.1 problem  $CP(u_{0n}, f)$  has at least one solution  $u_n$  on  $[0, T]$ , i.e.

$$u_n'(t) + \xi_n(t) = f(t), \quad \xi_n(t) \in \partial\varphi^t(u_n; u_n(t)), \quad \text{a.e. } t \in (0, T), \quad (4.8)$$

and

$$u_n = u_{0n} \quad \text{on } [-\delta_0, 0],$$

such that  $u_n \in W^{1,2}(0, T; H)$  and  $\sup_{0 \leq t \leq T} \varphi^t(u_n; u_n(t)) < \infty$ . Also, note from  $(\tilde{H}3)$  that there is a sequence  $\{h_n\}$  such that

$$\sup_{n \in \mathbf{N}} \left\{ |h_n|_{W^{1,2}(0, T; H)}^2 + \int_0^T \varphi^t(u_n; h_n(t)) dt \right\} < \infty. \quad (4.9)$$

Taking the inner product of the both sides of (4.8) and  $u_n(t) - h_n(t)$ , we get (cf. (2.5))

$$\begin{aligned} & \frac{d}{dt} \left\{ \frac{1}{2} |u_n(t)|_H^2 - (u_n(t), h_n(t))_H \right\} + (u_n(t), h_n'(t) - f(t))_H + \varphi^t(u_n; u_n(t)) \\ & \leq \varphi^t(u_n; h_n(t)) - (f(t), h_n(t))_H \end{aligned}$$

for a.e.  $t \in (0, T)$ . Integrating this inequality in time, we obtain with the help of the Gronwall's lemma and (4.9) that

$$\sup_{n \in \mathbf{N}} \left\{ |u_n|_{L^\infty(0, T; H)}^2 + \int_0^T \varphi^t(u_n; u_n(t)) dt \right\} < \infty.$$

Therefore we have for some constant  $M > 0$

$$\tilde{V}_{[-\delta_0, T]}(u_n) \leq M, \quad \forall n = 1, 2, \dots$$

Hence, by condition  $(\tilde{H}1)$ , there is a family  $\{M_r\}$  of positive numbers  $M_r$  such that

$$\{\varphi^t(u_n; \cdot)\} \in \mathcal{G}(\{M_r\}), \quad \forall n = 1, 2, \dots .$$

Furthermore, from (i) of Theorem 2.2 and our assumption (4.6) it follows that there is a positive constant  $P_1$  satisfying

$$|u_n|_{L^\infty(0,T;H)}^2 + |\sqrt{t}u'_n|_{L^2(0,T;H)}^2 + \sup_{0 < t \leq T} t\varphi^t(u_n; u_n(t)) + \int_0^T \varphi^t(u_n; u_n(t))dt \leq P_1, \quad (4.10)$$

for all  $n = 1, 2, \dots$ , and by  $(\tilde{\Phi}2)$ ,

$$\varphi_0(u_n(t)) \leq \frac{P_1}{t}, \quad \forall t \in (0, t], \quad \forall n = 1, 2, \dots . \quad (4.11)$$

Since the level set of  $\varphi_0$  is compact in  $H$ , by (4.10) and (4.11) it is easy to extract a subsequence  $\{u_{n_k}\}$  from  $\{u_n\}$  such that  $u_{n_k} \rightarrow u$  in  $C_{loc}((0, T]; H)$  and hence in  $L^2(0, T; H)$  (as  $k \rightarrow \infty$ ) for a certain  $u \in \tilde{\mathcal{V}}(u_0; -\delta_0; T)$ . This shows by our assumption  $(\tilde{\Phi}3)$  that  $\varphi^t(u_{n_k}; \cdot) \rightarrow \varphi^t(u; \cdot)$  on  $H$  in the sense of Mosco for every  $t \in [0, T]$ . Here, apply Theorem 2.3 to the sequence of problems

$$u'_{n_k}(t) + \partial\varphi^t(u_{n_k}; u_{n_k}(t)) \ni f(t), \quad 0 < t < T, \quad u_{n_k}(0) = u_{n_k 0}(0),$$

to see that  $u_{n_k}$  converges in  $C([0, T]; H)$  to the solution  $w$  of

$$w'(t) + \partial\varphi^t(u; w(t)) \ni f(t), \quad 0 < t < T, \quad w(0) = u_0(0).$$

Then, clearly,  $w = u$  on  $(0, T]$  and thus  $u$  must be a solution of  $CP(u_0, f)$  on  $[0, T]$  and satisfies (4.7).  $\diamond$

## 5. Obstacle problems

We begin this section with some artificial examples in order to explore our assumptions  $(H1) - (H2)$  for local existence in time or  $(\tilde{H}1) - (\tilde{H}3)$  for global existence in time.

**Example 5.1.** Let  $H := \mathbf{R}$ ,  $\delta_0$  and  $T$  be fixed positive numbers. We consider a scalar quasi-variational inequality, choosing  $\varphi_0 \equiv 0$  on  $\mathbf{R}$  and

$$\varphi^s(v; z) := \begin{cases} 0, & \text{if } z \in [k_c(v; s), \infty), \\ \infty, & \text{otherwise,} \end{cases} \quad \forall v \in W^{1,2}(-\delta_0, t), \quad 0 \leq \forall s \leq \forall t \leq T, \quad (5.1)$$

where

$$k_c(v; s) := 2v(0) + 2 \int_0^s |v'(\tau)|^p d\tau, \quad \forall v \in W^{1,2}(-\delta_0, t), \quad 0 \leq \forall s \leq \forall t \leq T, \quad (5.2)$$

for a fixed number  $p$  with  $0 < p \leq 1$ . It is easy to check conditions  $(\varphi_0)$  and  $(\Phi1) - (\Phi3)$  for  $\varphi_0$  and  $\varphi^t(v, \cdot)$ , respectively.

Now, we consider

$$u'(t) + \partial\varphi^t(u; u(t)) \ni f(t), \quad 0 < t < T, \quad (5.3)$$

with initial condition

$$u(t) = u_0(t) \quad \text{for all } t \in [-\delta_0, 0], \quad (5.4)$$

where  $f$  is given in  $L^2(0, T)$  and  $u_0$  in  $W^{1,2}(-\delta_0, 0)$  with  $u_0(0) \leq 0$  (hence  $\varphi^0(u_0; u_0(0)) < \infty$ ). For such an initial datum  $u_0$  and any number  $M > 0$ , we choose

$$a_r^v(\tau) := 2|v'(\tau)|^p, \quad b_r^v(\tau) := 0 \quad \text{for a.e. } \tau \in (0, T),$$

$$\forall v \in W^{1,2}(-\delta_0, T) \text{ with } v = u_0 \text{ on } [-\delta_0, 0] \text{ and } |v'|_{L^2(0,T)}^2 \leq M, \quad \forall r \geq 0.$$

Then, condition  $(*) = \{(H1) - (H2)\}$  with  $T_0 := T$  is satisfied; in fact, given  $z \geq k_c(v; s)$ ,  $\tilde{z} = z - k_c(v; s) + k_c(v; t)$  satisfies that

$$|\tilde{z} - z| \leq 2 \int_s^t |v'(\tau)|^p d\tau, \quad \varphi^t(v; \tilde{z}) - \varphi^s(v; z) = 0.$$

Since  $a_r^v := 2|v'|^p \in L^{\frac{2}{p}}(0, T)$ ,  $(H1)$  holds. Also, if  $0 < p < 1$ , then  $(H2)$  holds, too. When  $p = 1$ , as is easily checked,  $(H2)$  does not hold. If  $0 < p < 1$  and  $u_0(0) \leq 0$ , our Theorem 3.1 says that the scalar problem (5.3)-(5.4) has a local in time solution. However, in case  $p = 1$  and  $u_0(0) = 0$ , the problem has no solution. By the way, when  $p = \frac{1}{2}$  and  $u_0 \equiv 0$  on  $[-\delta_0, 0]$ , it is easy to see that the function  $u(t) = 4t$  is a solution of (5.3)-(5.4) on  $[0, T]$ .

**Example 5.2.** Let  $\rho_0$  be a smooth function on  $\mathbf{R}$  and define  $k_c(v; \cdot)$  by

$$k_c(v; s) := \int_{-\delta_0}^s \rho_0(s - \tau)v(\tau)d\tau, \quad \forall v \in L^2(-\delta_0, t), \quad 0 \leq \forall s \leq \forall t \leq T.$$

We define  $\varphi^s(v; \cdot)$  by (5.1) for this obstacle function  $k_c(v; \cdot)$ . It is easy to see that conditions  $(\tilde{\Phi}1) - (\tilde{\Phi}3)$  are fulfilled. Also, for any  $M > 0$ , conditions  $(\tilde{H}1)$  and  $(\tilde{H}2)$  hold for  $a_r^v \equiv M \max|\rho_0|$  and  $b_r^v \equiv 0$ . Therefore, by Theorem 3.1, problem (5.3)-(5.4) has a local in time solution, if  $f \in L^2(0, T)$  and the initial datum  $u_0$  is given so that  $u_0(0) \geq k_c(u_0; 0)$ . Furthermore, if the obstacle function is replaced by

$$k_c(v; s) := \int_{-\delta_0}^s \rho_0(s - \tau) \min\{v(\tau), m_0\}d\tau$$

with a positive constant  $m_0$ , then  $k_c$  is bounded by  $m_1 := m_0(\delta_0 + T) \max|\rho_0|$  from above and hence condition  $(\tilde{H}3)$  is satisfied; in fact, we can choose as  $h_v$  in  $(\tilde{H}3)$  the constant function  $m_1$ . Accordingly, by Theorem 4.1, problem (5.3) and (5.4) has a solution on the whole interval  $[0, T]$ .

Next, we give two typical applications to parabolic partial differential inequalities with the unknown dependent obstacles; one of them is the one mentioned in the introduction.

**Example 5.3.** Let  $\Omega$  be a bounded domain in  $\mathbf{R}^N$  with smooth boundary  $\Gamma := \partial\Omega$ . We put  $Q := \Omega \times (0, T)$ ,  $0 < T < \infty$ , and  $H := L^2(\Omega)$ . As the proper, l.s.c. convex function  $\varphi_0$  on  $L^2(\Omega)$  we take

$$\varphi_0(z) := \begin{cases} \frac{1}{2} \int_{\Omega} |\nabla z|^2 dx, & \forall z \in H^1(\Omega), \\ \infty, & \text{otherwise.} \end{cases} \quad (5.5)$$

Also, let  $\rho(\cdot, \cdot, \cdot)$  be a smooth function on  $\mathbf{R}^N \times \mathbf{R} \times \mathbf{R}$ . Assume that  $\rho$  and its partial derivatives  $\rho_i(x, t, v) := \frac{\partial}{\partial x_i} \rho(x, t, v)$ ,  $i = 1, 2, \dots, N$ , satisfy

$$|\rho(x, s, v_1) - \rho(x, t, v_2)| + \sum_{i=1}^N |\rho_i(x, s, v_1) - \rho_i(x, t, v_2)| \leq c_0(|t - s| + |v_1 - v_2|), \quad (5.6)$$

$$\forall (x, s, v_1), \forall (x, t, v_2) \in \mathbf{R}^N \times \mathbf{R} \times \mathbf{R},$$

where  $c_0$  is a positive constant. We define

$$\varphi^s(v; z) := \begin{cases} \frac{1}{2} \int_{\Omega} |\nabla z|^2 dx, & \text{if } z \in H^1(\Omega) \text{ and } z \geq k_c(v; \cdot, s) \text{ a.e. on } \Omega, \\ \infty, & \text{otherwise,} \end{cases} \quad (5.7)$$

$$\forall v \in L^2(-\delta_0, t; H^1(\Omega)), \quad 0 \leq \forall s \leq \forall t \leq T,$$

where  $\delta_0$  is a fixed positive number and

$$k_c(v; x, s) = \int_{-\delta_0}^s \int_{\Omega} \rho(x - y, s - \tau, v(y, \tau)) dy d\tau, \quad \forall (x, s) \in \Omega \times [-\delta_0, t].$$

It is easy to see that conditions  $(\varphi_0)$  and  $(\tilde{\Phi}1) - (\tilde{\Phi}3)$  are fulfilled by  $\varphi_0(\cdot)$  and  $\varphi^s(\cdot; \cdot)$  given by (5.5) and (5.7), respectively. Next, we check condition  $(*)$  or  $(**)$ . Let  $M > 0$  be any number and  $v$  be any function in

$$\tilde{\mathcal{V}}_M(-\delta_0, T) := \{v; |v|_{L^\infty(-\delta_0, T; L^2(\Omega))}^2 + \frac{1}{2} |\nabla v|_{L^2(-\delta_0, T; L^2(\Omega))}^2 \leq M\}. \quad (5.8)$$

Then, for each function  $z \in H^1(\Omega)$  with  $z \geq k_c(v; \cdot, s)$  a.e. on  $\Omega$  and  $0 \leq s \leq t \leq T$ , the function  $\tilde{z} := z - k_c(v; \cdot, s) + k_c(v; \cdot, t)$  satisfies that  $\tilde{z} \in H^1(\Omega)$  and  $\tilde{z} \geq k_c(v; \cdot, t)$  a.e. on  $\Omega$ . From (5.6) it follows that

$$\begin{aligned} & |\tilde{z}(x) - z(x)| \\ & \leq |k_c(v; x, t) - k_c(v; x, s)| \\ & \leq c_0(\delta_0 + s)|\Omega||s - t| + \int_s^t \int_{\Omega} (|\rho(x - y, t - \tau, 0)| + c_0|v(y, \tau)|) dy d\tau \\ & \leq c_0(\delta_0 + s)|\Omega||s - t| + \{|\Omega||\rho(\cdot, \cdot, 0)|_{L^\infty(Q)} + c_0|\Omega|^{\frac{1}{2}}|v|_{L^\infty(-\delta_0, T; L^2(\Omega))}\}|t - s| \\ & \leq c_1(M)|t - s| \end{aligned}$$

for all  $v \in \tilde{\mathcal{V}}_M(-\delta_0, T)$ , where  $|\Omega|$  denotes the volume of  $\Omega$  and

$$c_1(M) := c_0(\delta_0 + T)|\Omega| + |\Omega| |\rho(\cdot, \cdot, 0)|_{L^\infty(Q)} + c_0 |\Omega|^{\frac{1}{2}} \sqrt{M}.$$

Similarly,

$$\begin{aligned} & \left| \frac{\partial}{\partial x_i} (k_c(v; x, s) - k_c(v; x, t)) \right| \\ & \leq c_0(\delta_0 + s) |\Omega| |s - t| + \int_s^t \int_\Omega (|\rho_i(x - y, t - \tau, 0)| + c_0 |v(y, \tau)|) dy d\tau, \\ & \leq c_1(M) |t - s| \end{aligned}$$

for  $i = 1, 2, \dots, N$  and all  $v \in \tilde{\mathcal{V}}_M(-\delta_0, T)$ . Therefore,

$$|\tilde{z} - z|_{L^2(\Omega)} \leq c_1(M) |\Omega|^{\frac{1}{2}} |t - s|$$

and

$$\begin{aligned} \frac{1}{2} |\nabla \tilde{z}|_{L^2(\Omega)}^2 - \frac{1}{2} |\nabla z|_{L^2(\Omega)}^2 & \leq |\nabla z|_{L^2(\Omega)} \cdot c_1(M) \sqrt{N|\Omega|} |t - s| + c_1(M)^2 NT |\Omega| |t - s|^2 \\ & \leq c_2(M) |t - s| \left( \frac{1}{\sqrt{2}} |\nabla z|_{L^2(\Omega)} + 1 \right), \end{aligned}$$

where  $c_2(M) := \sqrt{2} c_1(M) \sqrt{N|\Omega|} + c_1(M)^2 NT |\Omega|$ ; namely,

$$\varphi^t(v; \tilde{z}) - \varphi^s(v; z) \leq c_2(M) |t - s| (\varphi^s(v; z)^{\frac{1}{2}} + 1).$$

Thus, putting  $a_r^v := c_1(M) |\Omega|^{\frac{1}{2}}$  and  $b_r^v := c_2(M)$  for all  $v \in \tilde{\mathcal{V}}(-\delta_0, T)$  and all  $r \geq 0$ , we see that  $(\tilde{H}1)$  and  $(\tilde{H}2)$  hold. Therefore, by virtue of Theorem 3.1, for given  $f \in L^2(Q)$  and  $u_0 \in W^{1,2}(-\delta_0, 0; L^2(\Omega)) \cap L^\infty(-\delta_0, 0; H^1(\Omega))$  with  $u_0(\cdot, 0) \geq k_c(u_0; \cdot, 0)$  a.e. on  $\Omega$ , the quasi-variational problem, denoted by  $(QV1)$ , formulated on  $Q' := \Omega \times (0, T')$ ,  $0 < T' \leq T$ ,

$$\begin{aligned} & u \in W^{1,2}(-\delta_0, T'; L^2(\Omega)) \cap L^\infty(-\delta_0, T'; H^1(\Omega)) \text{ with } u \geq k_c(u; \cdot, \cdot) \text{ a.e. on } Q'; \\ & \int_Q \{u_t(u - w) + \nabla u \cdot \nabla(u - w)\} dx dt \leq \int_Q f(x, t)(u - w) dx dt, \\ & \quad \forall w \in L^2(0, T'; H^1(\Omega)) \text{ with } w \geq k_c(u; \cdot, \cdot) \text{ a.e. on } Q', \\ & u = u_0 \text{ a.e. on } \Omega \times [-\delta_0, 0], \end{aligned}$$

has at least one solution  $u$  on a certain interval  $[0, T'] \subset [0, T]$ . Further suppose that  $\rho$  is bounded from above on  $\mathbf{R}^N \times \mathbf{R} \times \mathbf{R}$ . Then, so is  $k_c$  on  $\tilde{\mathcal{V}}(-\delta_0, T) \times \mathbf{R}^N \times \mathbf{R}$ , that is,  $k_c \leq k^*$  for a certain positive constant  $k^*$ . In this case,  $\varphi^t(v; k^*) = 0$  for all  $v \in \tilde{\mathcal{V}}(-\delta_0, T)$  and  $t \in [0, T]$ , which shows that  $(\tilde{H}3)$  holds. In such a case, our Theorem 4.1 says that problem  $(QV1)$  has a solution  $u$  on the whole interval  $[0, T]$ .

**Example 5.4.** Let  $\Omega$ ,  $\Gamma$ ,  $Q$  and  $\Sigma$  be as in Example 5.3, as well as  $0 < \delta_0 < \infty$  and  $0 < T < \infty$ . Also, we take as  $\varphi_0$  the same function as in Example 5.3, too. Let

$\Omega_i$ ,  $i = 1, 2, \dots, n$ , be a finite number of smooth subdomains of  $\Omega$  such that  $\overline{\Omega}_i \subset \Omega$ ,  $i = 1, 2, \dots, n$ , and  $\overline{\Omega}_k \cap \overline{\Omega}_j = \emptyset$  if  $k \neq j$ . We define a mapping  $\Lambda_i : L^2(-\delta_0, t; H_0^1(\Omega)) \rightarrow C([0, t])$ ,  $0 \leq s \leq t \leq T$ , by

$$[\Lambda_i v](s) := \int_{-\delta_0}^t \int_{\Omega_i} \gamma_i(s - \tau, v(x, \tau)) dx ds, \quad \forall v \in L^2(-\delta_0, t; H_0^1(\Omega)), \quad \forall s \in [0, t],$$

for each  $i = 1, 2, \dots, n$ , where  $\gamma_i(\cdot, \cdot)$  are Lipschitz continuous functions on  $\mathbf{R} \times \mathbf{R}$ ,  $i = 1, 2, \dots, n$ , i.e.

$$|\gamma_i(\tau_1, v_1) - \gamma_i(\tau_2, v_2)| \leq c_0(|\tau_1 - \tau_2| + |v_1 - v_2|), \quad \forall (\tau_1, v_1), \forall (\tau_2, v_2) \in \mathbf{R} \times \mathbf{R}, \quad (5.9)$$

for a positive constant  $c_0$ . Now, for each  $v \in L^2(-\delta_0, t; H_0^1(\Omega))$  and  $s \in [0, T]$ , consider

$$K(v; s) := \{z \in H_0^1(\Omega); |\nabla z| \leq k_c([\Lambda_i v](s)) \text{ a.e. on } \Omega_i, i = 1, 2, \dots, n\},$$

where  $k_c(\cdot)$  is a smooth and strictly positive function on  $\mathbf{R}$ . Clearly  $K(v; s)$  is a closed convex subset of  $H_0^1(\Omega)$  and non-empty, since  $0 \in K(v; s)$ .

We consider a quasi-variational problem, denoted by (QV2), to find a function  $u$  such that

$$\begin{aligned} u &\in L^2(-\delta_0, T; H_0^1(\Omega)) \cap W^{1,2}(0, T; L^2(\Omega)) \text{ with } u(t) \in K(u; t) \text{ for all } t \in [0, T], \\ \int_0^T \int_{\Omega} u_t(u - w) dx dt + \int_0^T \int_{\Omega} \nabla u \cdot \nabla(u - w) dx dt &\leq \int_0^T \int_{\Omega} f(u - w) dx dt, \\ \forall w &\in L^2(-\tau_0, T; H_0^1(\Omega)) \text{ with } w(t) \in K(u; t) \text{ for a.e. } t \in (0, T), \\ u(t) &= u_0(t) \text{ for all } t \in [-\delta_0, 0], \end{aligned}$$

where  $f$  is given in  $L^2(0, T; L^2(\Omega))$  and  $u_0$  in  $W^{1,2}(-\delta_0, 0; L^2(\Omega)) \cap L^\infty(-\delta_0, 0; H^1(\Omega))$  with  $u_0(0) \in K(u_0; 0)$ . Defining a proper, l.s.c. and convex function  $\varphi^s(v; \cdot)$  on  $L^2(\Omega)$  for each  $v \in \tilde{\mathcal{V}}(-\delta_0, t; L^2(\Omega))$  (cf.(5.9)),  $0 \leq s \leq t \leq T$ , by

$$\varphi^s(v; z) := \begin{cases} \frac{1}{2} \int_{\Omega} |\nabla z|^2 dx, & \text{if } z \in K(v; s) \\ \infty, & \text{otherwise} \end{cases}$$

we see easily that problem (QV2) can be described as a Cauchy problem of the form  $CP(u_0, f)$  on  $[0, T]$ . It is easy to check that our convex functions  $\varphi_0$  and  $\varphi^s(v; \cdot)$  satisfy conditions  $(\varphi_0)$  and  $(\tilde{\Phi}1) - (\tilde{\Phi}3)$ .

Let us verify assumptions in Theorems 4.1 (and hence Theorem 3.1). Choose a collection  $\{\eta_k\}_{0 \leq k \leq n}$  of smooth non-negative functions on  $\mathbf{R}^N$  corresponding to the family  $\{\Omega_i\}$  of subdomains of  $\Omega$  such that

$$\eta_k = 1 \text{ on } \Omega_k, \quad 1 \leq \forall k \leq n, \quad \text{and} \quad \sum_{k=0}^n \eta_k = 1 \text{ on } \Omega. \quad (5.10)$$

Now, given  $M > 0$  and any function  $v$  with  $|v|_{L^\infty(-\delta_0, T; L^2(\Omega))} + \frac{1}{2} |\nabla v|_{L^2(-\delta_0, T; L^2(\Omega))} \leq M$ . For any  $0 \leq s \leq t \leq T$  and  $z \in K(v; s)$ , we put

$$\tilde{z} = \eta_0 z + \sum_{i=1}^n \eta_i \cdot \frac{k_i(v; s)}{k_i(v; t)} z, \quad k_i(v; \cdot) := k_c([\Lambda_i v](\cdot)), \quad i = 1, 2, \dots, n.$$

Then we see from (5.10) that

$$\nabla \tilde{z} = \frac{k_i(v; t)}{k_i(v; s)} \nabla z \quad \text{on } \Omega_i, \quad i = 1, 2, \dots, n, \quad \text{namely, } \tilde{z} \in K(v; t).$$

Besides, we have

$$|\tilde{z}(x) - z(x)| \leq \sum_{i=1}^n \eta_i(x) \frac{|k_i(v; t) - k_i(v; s)|}{k_i(v; s)} \cdot |z(x)|$$

and

$$\begin{aligned} \nabla \tilde{z}(x) &= \left\{ \eta_0(x) + \sum_{i=1}^n \frac{k_i(v; t)}{k_i(v; s)} \eta_i(x) \right\} \nabla z(x) \\ &\quad + \left\{ \nabla \eta_0(x) + \sum_{i=1}^n \frac{k_i(v; t)}{k_i(v; s)} \nabla \eta_i(x) \right\} z(x) \\ &= \nabla z(x) + \left\{ \sum_{i=1}^n \frac{k_i(v; t) - k_i(v; s)}{k_i(v; s)} \eta_i(x) \right\} \nabla z(x) \\ &\quad + \left\{ \sum_{i=1}^n \frac{k_i(v; t) - k_i(v; s)}{k_i(v; s)} \nabla \eta_i(x) \right\} z(x). \end{aligned}$$

Since  $k_i(v; \tau)$  is Lipschotz continuous in  $\tau$ , i.e.  $|k_i(v; t) - k_i(v; s)| \leq c_i(M)|t - s|$ , where

$$c_i(M) := c_0(T + \delta_0)|\Omega_i| + |\Omega_i| \max |\gamma_i(\cdot, 0)| + c_0\{|\Omega_i|M\}^{\frac{1}{2}}, \quad i = 1, 2, \dots, n,$$

we obtain from the above relations that

$$|\tilde{z} - z|_{L^2(\Omega)} \leq c_*(M)|t - s|, \quad \varphi^t(v; t) - \varphi^s(v; z) \leq c_*(M)|t - s|(\varphi^s(v; s)^{\frac{1}{2}} + 1),$$

where  $c_*(M)$  is a positive constant depending only on  $M$ . Thus conditions ( $\tilde{H}1$ ) and ( $\tilde{H}2$ ) are satisfied by the families of functions  $a_r^v \equiv b_r^v \equiv c_*(M)$ , and ( $\tilde{H}3$ ) is trivially satisfied by  $h_v \equiv 0$ . Accordingly, by Theorem 4.1, our quasi-variational problem (QV2) has at least one solution  $u$  on the whole interval  $[0, T]$ .

Some other applications are found in a recent paper [5] treating one dimensional gradient obstacle problems of parabolic type.

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