Extended Gauss AGM and corresponding Picard modular forms

Kenji Koike and Hironori Shiga

August 11,2006

0 Introduction

For two positive numbers a, b, set $\psi_g(a,b) = (\frac{a+b}{2}, \sqrt{ab})$ and inductively set $\psi_g^n(a,b) = \psi_g(\psi_g^{n-1}(a,b)) = (a_n, b_n)$. Then we have a common limit

$$M_g(a,b) = \lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n.$$

This is called the Gauss Arithmetic Geometric Mean (we use the abbreviation Gauss AGM). Concerning this Gauss AGM we have two classical identities both we can find in the literature of Gauss himself.

Theorem 0.1 (Gauss [GaT] 1799) *For $x \in (0,1)$ we have*

$$\frac{1}{AGM(1,x)} = \frac{1}{\pi}\int_1^\infty \frac{dz}{\sqrt{z(z-1)(z-\lambda)}} = F(\frac{1}{2}, \frac{1}{2}, 1; \lambda),\ (\lambda = 1 - x^2). \tag{0.1}$$

The right hand side F means the Gauss hypergeometric function.

Setting the Jacobi theta function and theta constant for $p, q \in \{0,1\}^2$

$$\vartheta\begin{bmatrix} p \\ q \end{bmatrix}(z,\tau) = \sum_{n\in\boldsymbol{Z}} \exp[\pi i(n+\frac{p}{2})^2\tau + 2\pi i(n+\frac{p}{2})(z+\frac{q}{2})],$$
$$\vartheta\begin{bmatrix} p \\ q \end{bmatrix}(\tau) = \sum_{n\in\boldsymbol{Z}} \exp[\pi i(n+\frac{p}{2})^2\tau + \pi i(n+\frac{p}{2})q],$$

we have

Theorem 0.2 (Gauss [GaH] 1818)

$$\begin{cases} \vartheta\begin{bmatrix} 0 \\ 0 \end{bmatrix}^2(2\tau) = \frac{1}{2}\left(\vartheta\begin{bmatrix} 0 \\ 0 \end{bmatrix}^2(\tau) + \vartheta\begin{bmatrix} 0 \\ 1 \end{bmatrix}^2(\tau)\right) \\ \vartheta\begin{bmatrix} 0 \\ 1 \end{bmatrix}^2(2\tau) = \vartheta\begin{bmatrix} 0 \\ 0 \end{bmatrix}(\tau)\vartheta\begin{bmatrix} 0 \\ 1 \end{bmatrix}(\tau). \end{cases} \tag{0.2}$$

The latter theorem shows the relation of the (coefficients of the realized) elliptic curves corresponding to two isogenous torus $\boldsymbol{C}/\boldsymbol{Z} + \tau\boldsymbol{Z}$ and $\boldsymbol{C}/\boldsymbol{Z} + 2\tau\boldsymbol{Z}$. So in general this theorem is referred as the isogeny formula for the Jacobi theta constants.

Any way these two theorems are telling us a very interesting story concerned with AGM, periods of algebraic varieties, hypergeometric functions and modular forms at a same time. But still now there is no sufficiently nice generalization for it.

In this article we show an extended story of Gauss AGM to two variables case by using Picard modular forms for $\boldsymbol{Q}(\sqrt{-1})$ studied by Matsumoto [Mat1] which is coming from one of the Terada and Deligne-Mostow list.

1 Extended Gauss AGM and the Appell F_1 (First Main Theorem)

1.1 Extended AGM system

At first let us consider a trivially extended Gauss AGM system

$$(a, b, \sqrt{ab}) \mapsto \left(\frac{a+b}{2}, \sqrt{ab}, \sqrt{\frac{a+b}{2}\sqrt{ab}} \right).$$

We generalize it to the system of three terms. Let a, b and c be positive real numbers. Set

$$\psi(a, b, c) = \left(\frac{a+b}{2}, c, \sqrt[4]{\frac{1}{4}(c^2 + ab)(c^2 + \frac{a^2+b^2}{2})} \right), \tag{1.1}$$

and set

$$(a_n, b_n, c_n) = \psi(a_{n-1}, b_{n-1}, c_{n-1}).$$

Theorem 1.1 *Suppose $0 < a, b, c$, then the sequences $\{a_n\}, \{b_n\}$ and $\{c_n\}$ have a common limit.*

Definition 1.1 *Let us denote the above limit by $M(a, b, c)$, and call it Extended Gauss AGM.*

Lemma 1.1 *We have*

$$\min\{a, b, c\} \le \min\{a_1, b_1, c_1\} \le \mathrm{Max}\{a_1, b_1, c_1\} \le \mathrm{Max}\{a, b, c\}.$$

[proof].

Set $\min = \min\{a, b, c\}, \mathrm{Max} = \mathrm{Max}\{a, b, c\}$.

Obviously we have $a_1, b_1 \in [\min, \mathrm{Max}]$.

Set

$$c_1^\flat = \sqrt{\frac{c^2 + ab}{2}},\ c_1^\sharp = \sqrt{\frac{1}{2}\left(c^2 + \frac{a^2+b^2}{2}\right)}.$$

We see easily $c_1^\flat, c_1^\sharp \in [\min, \mathrm{Max}]$. Because $c^2 + ab \le c^2 + \frac{a^2+b^2}{2}$, we have

$$c_1^\flat \le c_1 \le c_1^\sharp.$$

So we have $c_1 \in [\min, \mathrm{Max}]$

q.e.d.

Remark 1.1 *It happens that*

$$\min\{a, b, c\} = \min\{a_1, b_1, c_1\} < \mathrm{Max}\{a_1, b_1, c_1\} = \mathrm{Max}\{a, b, c\}.$$

For example $(a, b, c) = (1, 1, 2)$.

The following Proposition assures Theorem 1.1.

Proposition 1.1 *Suppose $0 < \mu \le a, b, c \le \nu$ for fixed positive numbers μ, ν. Then*

$$\mathrm{Max}\{a_2, b_2, c_2\} - \min\{a_2, b_2, c_2\} \le \rho(\mathrm{Max} - \min)$$

for some positive number $\rho(\mu, \nu) < 1$.

[proof].

Assume $a \le b \le c$. Then

$$a_2 = \frac{a_1 + b_1}{2} = \frac{(a+b)/2 + c}{2} \ge \frac{a+c}{2},$$

and

$$c_2 \ge c_2^\flat = \sqrt{\frac{c_1^2 + a_1 b_1}{2}} \ge \sqrt{\frac{(c_1^\flat)^2 + a_1 b_1}{2}} = \frac{1}{2}\sqrt{(c+a)(c+b)} \ge \frac{a+c}{2}.$$

We have

$$b_2 = c_1 \ge c_1^\flat = \sqrt{\frac{c^2 + ab}{2}} \ge \sqrt{\frac{c^2 + a^2}{2}} \ge \sqrt{ac}.$$

By cosidering obvious inequality $a_2, b_2, c_2 \le c$, we have

$$\frac{\mathrm{Max}\{a_2, b_2, c_2\} - \min\{a_2, b_2, c_2\}}{c-a} \le \frac{c - \sqrt{ac}}{c-a} = \frac{\sqrt{c}}{\sqrt{c} + \sqrt{a}} \le \frac{\sqrt{\nu}}{\sqrt{\mu} + \sqrt{\nu}} < 1.$$

Assume $a \le c \le b$, then we have

$$a_2 = \frac{a_1 + b_1}{2} = \frac{(a+b)/2 + c}{2} \ge \sqrt{\frac{a+b}{2}a},$$

and

$$c_2 \ge c_2^\flat = \sqrt{\frac{c_1^2 + a_1 b_1}{2}} \ge \sqrt{\frac{(c_1^\flat)^2 + a_1 b_1}{2}} = \frac{1}{2}\sqrt{(c+a)(c+b)} \ge \sqrt{\frac{a+b}{2}a}.$$

We have

$$b_2 = c_1 \ge c_1^\flat = \sqrt{\frac{c^2 + ab}{2}} \ge \sqrt{\frac{a+b}{2}a}.$$

So we have

$$\frac{\mathrm{Max}\{a_2, b_2, c_2\} - \min\{a_2, b_2, c_2\}}{b-a} \le \frac{b - \sqrt{\frac{a+b}{2}a}}{b-a}.$$

By putting $b/a = t$, we have

$$\frac{\mathrm{Max}\{a_2, b_2, c_2\} - \min\{a_2, b_2, c_2\}}{b-a} \le \frac{t - \sqrt{(1+t)/2}}{t-1}.$$

We have

$$\frac{d}{dt}\left(\frac{t - \sqrt{(1+t)/2}}{t-1}\right) = \frac{2(t-1)^2}{3\sqrt{2} + \sqrt{2}\,t + 4\sqrt{1+t}} > 0$$

for $t > 1$. So

$$\mathrm{Max}_{t \in (1, \nu/\mu]} \frac{t - \sqrt{(1+t)/2}}{t-1} = \frac{\nu - \sqrt{\mu(\mu+\nu)/2}}{\nu - \mu} < 1.$$

This argument works for the case $c \le a \le b$ also.

So we obtain the required inequality.

q.e.d.

1.2 Functional equations

Set

$$x^\sharp y^\sharp = \frac{2(x+y)}{(1+\sqrt{xy})^2} \qquad \frac{x^\sharp + y^\sharp}{2} = \frac{(\sqrt{x} + \sqrt{y})\sqrt{(1+x)(1+y)}}{(1+\sqrt{xy})^2}$$

for real numbers $0 < x, y < 1$, namely

$$x^\sharp = \frac{(\sqrt{x}+\sqrt{y})\sqrt{(1+x)(1+y)} + i(\sqrt{x}-\sqrt{y})\sqrt{(1-x)(1-y)}}{(1+\sqrt{xy})^2}$$
$$y^\sharp = \frac{(\sqrt{x}+\sqrt{y})\sqrt{(1+x)(1+y)} - i(\sqrt{x}-\sqrt{y})\sqrt{(1-x)(1-y)}}{(1+\sqrt{xy})^2},$$

here we choose real positive square roots.
We have

Proposition 1.2

$$M\left(1,\ \sqrt{xy},\ \sqrt{\frac{x+y}{2}}\right) = \frac{1+\sqrt{xy}}{2} M\left(1,\ \sqrt{x^\sharp y^\sharp},\ \sqrt{\frac{x^\sharp + y^\sharp}{2}}\right).$$

[proof]. It holds

$$M\left(1,\ \sqrt{xy},\ \sqrt{\frac{x+y}{2}}\right) = M\left(\frac{1+\sqrt{xy}}{2},\ \sqrt{\frac{x+y}{2}},\ \sqrt[4]{\frac{1}{4}(\frac{x+y}{2}+\sqrt{xy})(\frac{x+y}{2}+\frac{1+xy}{2})}\right)$$
$$= M\left(\frac{1+\sqrt{xy}}{2},\ \sqrt{\frac{x+y}{2}},\ \frac{1}{2}\sqrt[4]{(\sqrt{x}+\sqrt{y})^2(1+x)(1+y)}\right)$$
$$= \frac{1+\sqrt{xy}}{2} M\left(1,\ \sqrt{\frac{2(x+y)}{(1+\sqrt{xy})^2}},\ \sqrt[4]{\frac{(\sqrt{x}+\sqrt{y})^2(1+x)(1+y)}{(1+\sqrt{xy})^4}}\right)$$
$$= \frac{1+\sqrt{xy}}{2} M\left(1,\ \sqrt{\frac{2(x+y)}{(1+\sqrt{xy})^2}},\ \sqrt{\frac{(\sqrt{x}+\sqrt{y})\sqrt{(1+x)(1+y)}}{(1+\sqrt{xy})^2}}\right)$$
$$= \frac{1+\sqrt{xy}}{2} M\left(1,\ \sqrt{x^\sharp y^\sharp},\ \sqrt{\frac{x^\sharp + y^\sharp}{2}}\right).$$

q.e.d.

Let $F_1(a, b, b', c; x, y)$ be the Appell hypergeometric function

$$F_1(a, b, b', c; x, y) = \sum_{m,n \geq 0} \frac{(a, m+n)(b, m)(b', n)}{(c, m+n)m!n!} x^m y^n, \quad (|x|, |y| < 1)$$

of complex variables x, y with the Pochmammer notation

$$(a, k) = \begin{cases} a(a+1)\cdots(a+k-1), & (k \geq 1) \\ 0, & (k = 0). \end{cases}$$

This is a solution of the differential equation

$$E_1(a, b, b', c) : \begin{cases} x(1-x)f_{xx} + (1-x)yf_{xy} + (c-(a+b+1)x)f_x - byf_y - abf = 0, \\ y(1-y)f_{yy} + (1-y)xf_{xy} + (c-(a+b'+1)y)f_y - b'xf_x - ab'f = 0, \\ (x-y)f_{xy} - b'f_x + bf_y = 0, \end{cases}$$

and it is characterized as the solution $f(x, y)$ which is holomorphic at $(x, y) = (0, 0)$ and $f(0, 0) = 1$.

Theorem 1.2 *We have*

$$F_1(\frac{1}{2}, \frac{1}{4}, \frac{1}{4}, 1; 1-x^2, 1-y^2) = \frac{2}{1+\sqrt{xy}} F_1(\frac{1}{2}, \frac{1}{4}, \frac{1}{4}, 1; 1-(x^\sharp)^2, 1-(y^\sharp)^2)$$

for real variables $0 < x, y < 1$.

[proof]. First let us note the system $E_1\left(\frac{1}{2},\frac{1}{4},\frac{1}{4},1\right)$:

$$\begin{cases} P_1 = P_1(x,y,\partial_x,\partial_y) = x(1-x)\partial_x^2 + y(1-x)\partial_x\partial_y + (1-\frac{7}{4}x)\partial_x - \frac{1}{4}y\partial_y - \frac{1}{8} \\ P_2 = P_2(x,y,\partial_x,\partial_y) = y(1-y)\partial_y^2 + x(1-y)\partial_x\partial_y + (1-\frac{7}{4}y)\partial_y - \frac{1}{4}x\partial_x - \frac{1}{8} \\ P_3 = (1-x)\partial_x P_2 - (1-y)\partial_y P_1 + \frac{1}{4}(P_1-P_2) = \frac{1}{2}(x-y)\partial_x\partial_y - \frac{1}{8}(\partial_x-\partial_y) \end{cases} \tag{1.2}$$

gives the differential equation for $F_1(\frac{1}{2},\frac{1}{4},\frac{1}{4},1;x,y)$.

From the definition of $x^\sharp$ and $y^\sharp$ we have

$$1-(x^\sharp)^2 = \frac{(\sqrt{(1-x^2)(1-y^2)} - i(x-y))^2}{(1+\sqrt{xy})^4}$$
$$1-(y^\sharp)^2 = \frac{(\sqrt{(1-x^2)(1-y^2)} + i(x-y))^2}{(1+\sqrt{xy})^4}.$$

Putting $x=X^2$, $y=Y^2$ we are requested to show

$$\frac{1+XY}{2}F_1(1-X^4,\ 1-Y^4) = F_1(\Phi_1(X,Y),\ \Phi_2(X,Y)) \tag{1.3}$$

with

$$\Phi_1(X,Y) = \left(\frac{\sqrt{(1-X^4)(1-Y^4)} - i(X^2-Y^2)}{(1+XY)^2}\right)^2, \quad \Phi_2(X,Y) = \left(\frac{\sqrt{(1-X^4)(1-Y^4)} + i(X^2-Y^2)}{(1+XY)^2}\right)^2.$$

Now we are going to describe the differential operators which annihilate $F_1(1-X^4,\ 1-Y^4)$ and $F_1(\Phi_1(X,Y),\ \Phi_2(X,Y))$, respectively.

Lemma 1.2 *The system*

$$\begin{cases} Q_1 = \frac{1-X^4}{16X^2}\partial_X^2 + \frac{X(1-Y^4)}{16Y^3}\partial_X\partial_Y - \frac{1}{4}X\partial_X + \frac{1-Y^4}{8Y^3}\partial_Y - \frac{1}{8} \\ Q_2 = \frac{1-Y^4}{16Y^2}\partial_Y^2 + \frac{Y(1-X^4)}{16X^3}\partial_X\partial_Y - \frac{1}{4}Y\partial_Y + \frac{1-X^4}{16X^3}\partial_X - \frac{1}{8} \\ Q_3 = -\frac{X^4-Y^4}{32X^3Y^3}\partial_X\partial_Y + \frac{1}{32X^3}\partial_X - \frac{1}{32Y^3}\partial_Y \end{cases} \tag{1.4}$$

annihilates $F_1(1-X^4,\ 1-Y^4)$.

[proof]. We get the above system by just putting

$$x = 1-X^4, \quad y = 1-Y^4, \quad \partial_x = -\frac{1}{4X^3}\partial_X, \quad \partial_y = -\frac{1}{4Y^3}\partial_Y$$

in the system (1.2).

Lemma 1.3 *Putting* $R_i = (1+XY)\cdot Q_i \cdot (1+XY)^{-1}$ *the system*

$$\begin{cases} R_1 = \frac{1-X^4}{16X^2}\partial_X^2 + \frac{X(1-Y^4)}{16Y^3}\partial_X\partial_Y - \frac{X^4+2Y^4+4X^3Y^3+X^4Y^4}{16X^2Y^3(1+XY)}\partial_X + \frac{(1-Y^4)}{16Y^3(1+XY)}\partial_Y - \frac{Y^3(X^2-Y^2)-X^3(1-Y^4)}{8X^2Y^3(1+XY)^2} \\ R_2 = \frac{1-Y^4}{16Y^2}\partial_Y^2 + \frac{Y(1-X^4)}{16X^3}\partial_X\partial_Y - \frac{Y^4+2X^4+4Y^3X^3+Y^4X^4}{16Y^2X^3(1+YX)}\partial_Y + \frac{(1-X^4)}{16X^3(1+YX)}\partial_X - \frac{X^3(Y^2-X^2)-Y^3(1-X^4)}{8Y^2X^3(1+YX)^2} \\ R_3 = -\frac{1}{32X^3Y^3}\left((X^4-Y^4)\partial_X\partial_Y - \frac{X^5+Y^3}{1+XY}\partial_X + \frac{X^2+Y^5}{1+XY}\partial_Y - \frac{2(X^4-Y^4)}{(1+XY)^2}\right) \end{cases}$$

annihilates $\dfrac{1+XY}{2}F_1(1-X^4,1-Y^4)$.

Let $S_1(S_2)$ be the operator obtained from $P_1(P_2)$, respectively, by substituting

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \Phi_1(X,Y) \\ \Phi_2(X,Y) \end{bmatrix}, \quad \begin{bmatrix} \partial_x \\ \partial_y \end{bmatrix} = \begin{bmatrix} \partial\Phi_1/\partial X & \partial\Phi_2/\partial X \\ \partial\Phi_1/\partial Y & \partial\Phi_2/\partial Y \end{bmatrix}^{-1} \begin{bmatrix} \partial_X \\ \partial_Y \end{bmatrix}.$$

They take some complicated shapes, but

$$T_1 = 8(X^2+Y^2)(1-XY)^3(S_1+S_2), \qquad T_2 = \frac{8(X^2-Y^2)(1-XY)^3}{i\sqrt{(1-X^4)(1-Y^4)}}(S_1-S_2)$$

can be represented in an explicit way:

Lemma 1.4

$$\begin{aligned} T_1 = \quad & Y(1-X^4)^2(X-Y^3)\partial_X^2 + (1-X^4)(1-Y^4)(X^2+Y^2)(1-XY)\partial_X\partial_Y \\ & +X(1-Y^4)^2(Y-X^3)\partial_Y^2 - (1-X^4)(X^3-Y+2X^4Y+4XY^2-4X^2Y^3-X^3Y^4-Y^5)\partial_X \\ & -(1-Y^4)(Y^3-X+2Y^4X+4YX^2-4Y^2X^3-Y^3X^4-X^5)\partial_Y - 2(X^2+Y^2)(1-XY)^3 \\ T_2 = \quad & Y(1-X^4)(Y-X^3)\partial_X^2 - (X^4-Y^4)(1-XY)\partial_X\partial_Y - X(1-Y^4)(X-Y^3)\partial_Y^2 \\ & +(X^5-X^2Y+2X^6Y-X^3Y^2-Y^3)\partial_X - (Y^5-Y^2X+2Y^6X-Y^3X^2-X^3)\partial_Y. \end{aligned}$$

By comparing the results in Lemma 1.3 and Lemma 1.4 we obtain:

Lemma 1.5

$$\begin{aligned} \frac{1}{16XY}T_1 = \quad & X(1-X^4)(X-Y^3)R_1 \\ & +Y(1-Y^4)(Y-X^3)R_2 + 2(1-X^4)(1-Y^4)(X^2-Y^2)R_3 \\ \frac{1}{16XY}T_2 = \quad & X(Y-X^3)R_1 - Y(X-Y^3)R_2 \\ & +2(XY+X^2Y^2-X^3Y^3+X^4Y^4-X^4-Y^4)R_3. \end{aligned}$$

So T_1 and T_2 annihilate $\frac{1+XY}{2}F_1(1-X^4,\ 1-Y^4)$. Hence S_1 and S_2 annihilate $\frac{1+XY}{2}F_1(1-X^4,\ 1-Y^4)$ also. Namely $\frac{1+XY}{2}F_1(1-X^4,\ 1-Y^4)$ satisfies the same differential equation S_1+S_2 with three dimensional solution space as $F_1(\Phi_1(X,Y),\ \Phi_2(X,Y))$.

This system has unique holomorphic solution at $(X,Y)=(1,1)$ up to a constant factor. So we have the required equality.

q.e.d.

1.3 First Main theorem

Theorem 1.3 *Assume* $0<x,y<1$ *and set* $\lambda = 1-x^2$, $\mu = 1-y^2$. *Then we have the following trinity theorem for Extended Gauss AGM:*

$$\frac{1}{M(1,\sqrt{xy},\sqrt{(x+y)/2})} = \frac{1}{\pi}\int_1^\infty \frac{du}{\sqrt[4]{u^2(u-1)^2(u-\lambda)(u-\mu)}} = F_1(\frac{1}{2},\frac{1}{4},\frac{1}{4},1;\lambda,\mu). \qquad (1.5)$$

[proof]. According to Proposition 1.2 and Theorem 1.2 , we know that $\dfrac{1}{M(1,\sqrt{xy},\sqrt{(x+y)/2})}$ and $F_1(\frac{1}{2},\frac{1}{4},\frac{1}{4},1;\lambda,\mu)$ satisfy the same functional equation. Set

$$g(x,y) = F_1(\frac{1}{2},\frac{1}{4},\frac{1}{4},1;\lambda,\mu)M(1,\sqrt{xy},\sqrt{(x+y)/2}).$$

Then we have

$$g(x,y) = g(x^\sharp, y^\sharp).$$

Note that we have

$$\lim_{n\to\infty}\frac{b_n}{c_n} = 1$$

for the Extended AGM system (a_n, b_n, c_n). So, by the iteration of the procedure $(x, y) \mapsto (x^\sharp, y^\sharp)$, we have $g(x, y) = 1$. Hence the left hand side and the right hand side in (1.5) coincide. The second equality in the statement of the Theorem is nothing but the integral representation of the Appell hypergeometric function. Namely we obtained the required equalities.

q.e.d.

2 Modular interpretation and isogeny formula

2.1 Matsumoto hyperelliptic modular function for $C(x, y)$ (sum up with corrections)

Here we review the result of K. Matsumoto [Mat1] with some modifications and small corrections. Let us start from the family of algebraic curves

$$C(x,y) : w^4 = z^2(z-1)^2(z-x)(z-y) \qquad ((x,y) \in \boldsymbol{C}^2 - \{xy(x-1)(y-1)(x-y) = 0\}). \tag{2.1}$$

$C(x, y)$ has a model of compact nonsinguler hyperelliptic curve of genus 3. We denote it by the same notation. The differentials

$$dz/w, z(z-1)dz/w^3, z^2(z-1)dz/w^3$$

form a basis of holomorphic 1-forms on $C(x, y)$.

For the moment suppose x and y are real and satisfy $1 < x < y$. Let α be the automorphism $(z, w) \mapsto (z, \sqrt{-1}w)$ of $C(x, y)$.

Let us make three 1-cycles. Choose branches $w^{(0)}, w^{(1)}, w^{(2)}, w^{(3)}$ of w over $\{\mathrm{Im}\ z \geq 0\} - \{0, 1, x, y\}$ so as to be $\alpha(z, w^{(k)}) = (z, w^{(k+1)})$ with $w^{(4)} = w^{(0)}$. For real numbers z_1, z_2 let $[z_1, z_2 >$ be an oriented line segment connecting z_1 and z_2 in this order. Let $[z_1, z_2 >^{(k)}$ be an arc on $C(x, y)$ over $[z_1, z_2 >^{(k)}$ with the branch $w^{(k)}$. Set

$$A_1 = [0, -\infty >^{(0)} + [-\infty, 0 >^{(2)},\ B_1 = [1, 0 >^{(0)} + [0, 1 >^{(2)},\ A_3 = [y, x >^{(0)} + [x, y >^{(1)} .$$

They are considered as 1-cycles on $C(x, y)$. By making their analytic continuations we can define multi-valued analytic functions

$$a_1(x,y) = \int_{A_1} \frac{dz}{w},\ a_2(x,y) = \int_{B_1} \frac{dz}{w},\ a_5(x,y) = \int_{A_3} \frac{dz}{w}$$

on $\boldsymbol{C}^2 - \{xy(x-1)(y-1)(x-y) = 0\}$. These setting and notation are the same in [Mat1].

[Fact 1] $a_1(x, y),\ a_2(x, y),\ a_5(x, y)$ satisfy the differential equation $E_1(\frac{1}{2}, \frac{1}{4}, \frac{1}{4})$ of the Appell hypergeometric function $F_1(\frac{1}{2}, \frac{1}{4}, \frac{1}{4}; x, y)$ and they form a basis of the space of solutions.

We define the Schwarz map for $E_1(\frac{1}{2}, \frac{1}{4}, \frac{1}{4})$ by

$$\Phi : (x, y) \mapsto (u, v) = (a_5/a_1, a_2/a_1) \in \boldsymbol{C}^2.$$

[Fact 2] The image of Φ is contained in the hyperball

$$\boldsymbol{B}^2 = \{(u, v) \in \boldsymbol{C}^2 :\ 2\,\mathrm{Im}\ v - |u|^2 > 0\}$$

and it is open dense in $\boldsymbol{B}^2$. The map Φ has a continuation on $\boldsymbol{P}^1 \times \boldsymbol{P}^1 - \{(0, 0), (1, 1), (\infty, \infty)\}$ and the image is equal to $\boldsymbol{B}^2$.

[Fact 3] The fundamental group $\pi_1(\boldsymbol{C}^2 - \{xy(x-1)(y-1)(x-y) = 0\}, *)$ induces the monodromy group G of Φ acting on $\boldsymbol{B}^2$. The following five transformations give a generator system of G:

$$g_1 = \begin{pmatrix} 1 & 0 & 0 \\ 1+i & 1 & 1-i \\ -1-i & 0 & i \end{pmatrix},$$

$$g_2 = \begin{pmatrix} 2+i & -1-i & -1-i \\ 1+i & -i & -1-i \\ 1-i & -1+i & i \end{pmatrix},$$

$$g_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix},$$

$$g_4 = \begin{pmatrix} i & 1-i & 1-i \\ 0 & i & 0 \\ 0 & -1-i & -1 \end{pmatrix},$$

$$g_5 = \begin{pmatrix} 2+i & -1-i & 1-i \\ 1+i & -i & 1-i \\ -1-i & 1+i & i \end{pmatrix},$$

where g_i $(i = 1, 2, 3, 4, 5)$ acts on ${}^t(1, v, u)$ from left. Their orders are $4, 4, 2, 4, 4$ and eigen values are

$$\{1, 1, i\}, \{1, 1, i\}, \{1, 1, -1\}, \{-1, i, i\}, \{1, 1, i\},$$

respectively (the original article contains a printing error for g_5).

The Schawarz map Φ induces a biholomorphic correspondence

$$\boldsymbol{P}^1 \times \boldsymbol{P}^1 \cong \overline{\boldsymbol{B}^2/G},$$

and three points $(0, 0), (1, 1), (\infty, \infty)$ correspond to the boundaries $\overline{\boldsymbol{B}^2/G} - \boldsymbol{B}^2/G$.

[Fact 5] We can realize $\boldsymbol{B}^2$ in the form

$$D = \{\xi = [\xi_0, \xi_1, \xi_2] \in \boldsymbol{P}^2 : \ \xi H {}^t\overline{\xi} < 0\},$$
$$H = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

The map

$$\Omega(u, v) = \begin{pmatrix} v + iu^2/2 & -u^2/2 & -iu \\ -u^2/2 & v - iu^2/2 & u \\ -iu & u & i \end{pmatrix}$$

gives an modular embedding of $\boldsymbol{B}^2$ into the Siegel upper half space of degree 3. Set

$$G_0 = U(2, 1, \boldsymbol{Z}[\sqrt{-1}]) = \{g \in GL(3, \boldsymbol{Z}[\sqrt{-1}]) : \ gH {}^t\overline{g} = H\}.$$

G_0 becomes a restriction of $Sp(6, \boldsymbol{Z})$ on $\Omega(\boldsymbol{B}^2)$. Note that the projective group $\overline{G_0} = G_0/\langle\sqrt{-1}\rangle$ is isomorphic to

$$SU(2, 1, \boldsymbol{Z}[\sqrt{-1}]) = \{g \in SL(3, \boldsymbol{Z}[\sqrt{-1}]) : \ gH {}^t\overline{g} = H\}.$$

[Fact 6] Set

$$g_6 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & i \end{pmatrix}.$$

Then the group generated by G and g_6 together with $\sqrt{-1}$ becomes the congruent subgroup

$$G(1 + \sqrt{-1}) = \{g \in G_0 : \ g \equiv \mathrm{id} \pmod{(1 + \sqrt{-1})}\}.$$

Fact 7] For $p = (p_1, p_2, p_3), q = (q_1, q_2, q_3) \in \boldsymbol{Z}^3$, let

$$\Theta \begin{bmatrix} p_1 & p_2 & p_3 \\ q_1 & q_2 & q_3 \end{bmatrix} (u, v) = \vartheta \begin{bmatrix} p_1 & p_2 & p_3 \\ q_1 & q_2 & q_3 \end{bmatrix} (\Omega(u, v))$$
$$= \sum_{n=(n_1,n_2,n_3)\in\boldsymbol{Z}^3} \exp[\pi i(n + \frac{p}{2})\Omega(u, v){}^t(n + \frac{p}{2}) + \pi i(n + \frac{p}{2}){}^t q]$$

be the Riemann theta constant with half integral characteristics defined on $\boldsymbol{B}^2$ via the embedding Ω.

Theorem 2.1 *(Matsumoto hyperelliptic theta map theorem)*

$$(x,y)=\left(\frac{\Theta^4\begin{bmatrix}1&0&0\\0&0&1\end{bmatrix}(u,v)}{\Theta^4\begin{bmatrix}1&0&0\\0&1&1\end{bmatrix}(u,v)},\ \frac{\Theta^4\begin{bmatrix}0&1&0\\0&0&1\end{bmatrix}(u,v)}{\Theta^4\begin{bmatrix}0&1&0\\1&0&1\end{bmatrix}(u,v)}\right).$$

Remark 2.1 *The denominators and the numerators on the right hand side have common zero on some divisors of $\boldsymbol{B}^2$. But according to the argument on the period map, we have extensions $x=x(u,v)$, $y=y(u,v)$ as holomorphic maps on the whole domain $\boldsymbol{B}^2$.*

[Fact 8]

Lemma 2.1 (Fourier expansion)

$$\begin{aligned}&\Theta\begin{bmatrix}p_1&p_2&p_3\\q_1&q_1&q_3\end{bmatrix}(u,v)\\&=\sum_{n_1,n_2}\exp[-\frac{\pi}{2}\{(n_1+\frac{p_1}{2})+i(n_2+\frac{p_2}{2})\}^2u^2]\exp[\pi i\{(n_1+\frac{p_1}{2})q_1+(n_2+\frac{p_2}{2})q_2\}]\\&\times\vartheta\begin{bmatrix}p_3\\q_3\end{bmatrix}\left(\{(n_2+\frac{p_2}{2})-i(n_1+\frac{p_1}{2})\}u,i\right)\exp[\pi i\{(n_1+\frac{p_1}{2})^2+(n_2+\frac{p_2}{2})^2\}v]\end{aligned}$$

Lemma 2.2 (Matsumoto exchange formula （The multiplication factor is misplaced in the original paper)) *We have*

$$\exp[\pi i p_1q_1+\frac{\pi i}{2}p_3q_3]\cdot\Theta\begin{bmatrix}p_2&p_1&q_3\\q_2&q_1&p_3\end{bmatrix}(u,v)=\Theta\begin{bmatrix}p_1&p_2&p_3\\q_1&q_2&q_3\end{bmatrix}(u,v).$$

2.2 9 theta constants with monodromy invariant characteristics

Recall the transformation formula for theta constants:

Proposition 2.1 (Transformation formula (see for example [I])) *Let $g\in G_0$ and $N_g=\begin{pmatrix}A_g&B_g\\C_g&D_g\end{pmatrix}$ be its symplectic representation. Then we have*

$$\Theta\left[N_g\circ\begin{bmatrix}a\\b\end{bmatrix}\right](g\circ(u,v))=\varepsilon(g,a,b)\,(\det\ (C_g\Omega(u,v)+D_g))^{1/2}\,\Theta\begin{bmatrix}a\\b\end{bmatrix}(u,v).$$

Where

$$N_g\circ\begin{bmatrix}a\\b\end{bmatrix}=\begin{bmatrix}D_ga-C_gb+(1/2)\mathrm{dv}(C_g{}^tD_g)\\-B_ga+A_gb+(1/2)\mathrm{dv}(A_g{}^tB_g)\end{bmatrix},$$

("dv" *indicates the diagonal vector) and $\varepsilon(g,a,b)$ is a certain 8-th root of unity.*

By use of the transformation formula we obtain the following two propositions. Because the calculation is straight forward we omit the detail.

Proposition 2.2 (1) *There are* 36 *even theta constants $\vartheta\begin{bmatrix}a\\b\end{bmatrix}(\Omega)$ with $a^tb\equiv 0\,(\mathrm{mod}\,2)$. Among them we have* 20 *different $\Theta\begin{bmatrix}a\\b\end{bmatrix}(u,v)$:*

number	*name*	*characteristic* $\{a, b\}$
n.1	Θ_{h1}	$\{\{0,0,0\},\{0,0,0\}\}$
n.2		$\{\{1,0,0\},\{0,0,0\}\}$
n.3		$\{\{0,0,0\},\{1,0,0\}\}$
n.4		$\{\{0,0,1\},\{0,0,0\}\}$
n.5	Θ_{h2}	$\{\{1,1,0\},\{0,0,0\}\}$
n.6	Θ_{yN}	$\{\{1,0,1\},\{0,0,0\}\}$
n.7		$\{\{1,0,0\},\{0,1,0\}\}$
n.8	Θ_{xN}	$\{\{1,0,0\},\{0,0,1\}\}$
n.9	Θ_{h3}	$\{\{0,0,0\},\{1,1,0\}\}$
n.10	Θ_{xZ}	$\{\{0,0,1\},\{1,0,0\}\}$
n.11	Θ_{yZ}	$\{\{0,0,1\},\{0,1,0\}\}$
n.12		$\{\{1,1,1\},\{0,0,0\}\}$
n.13	Θ_{yD}	$\{\{1,0,1\},\{0,1,0\}\}$
n.14	Θ_{xD}	$\{\{1,0,0\},\{0,1,1\}\}$
n.15		$\{\{0,0,1\},\{1,1,0\}\}$
n.16		$\{\{1,0,1\},\{1,0,1\}\}$
n.17		$\{\{1,1,1\},\{1,1,0\}\}$
n.18		$\{\{1,1,1\},\{1,0,1\}\}$
n.19		$\{\{1,0,1\},\{1,1,1\}\}$
n.20		$\{\{1,1,0\},\{1,1,0\}\}$

Table 3.1

(2) *We have* 9 *even theta's with* $N_g \circ \begin{bmatrix} a \\ b \end{bmatrix} \equiv \begin{bmatrix} a \\ b \end{bmatrix} \pmod 2$ *for* $g \in G$*, those are indicated in Table 3.1 with proper names.*

We call the nine theta constants in Proposition 2.2(2) G-invariant thetas.

We note here that the rational representation of the transformation

$$\begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix} \in G_0$$

is given by

$$\begin{pmatrix} \mathrm{Re}\ b_2 & -\mathrm{Im}\ b_2 & -\mathrm{Im}\ b_3 & \mathrm{Re}\ b_1 & -\mathrm{Im}\ b_1 & \mathrm{Re}\ b_3 \\ \mathrm{Im}\ b_2 & \mathrm{Re}\ b_2 & \mathrm{Re}\ b_3 & \mathrm{Im}\ b_1 & \mathrm{Re}\ b_1 & \mathrm{Im}\ b_3 \\ \mathrm{Im}\ c_2 & \mathrm{Re}\ c_2 & \mathrm{Re}\ c_3 & \mathrm{Im}\ c_1 & \mathrm{Re}\ c_1 & \mathrm{Im}\ c_3 \\ \mathrm{Re}\ a_2 & -\mathrm{Im}\ a_2 & -\mathrm{Im}\ a_3 & \mathrm{Re}\ a_1 & -\mathrm{Im}\ a_1 & \mathrm{Re}\ a_3 \\ \mathrm{Im}\ a_2 & \mathrm{Re}\ a_2 & \mathrm{Re}\ a_3 & \mathrm{Im}\ a_1 & \mathrm{Re}\ a_1 & \mathrm{Im}\ a_3 \\ \mathrm{Re}\ c_2 & -\mathrm{Im}\ c_2 & -\mathrm{Im}\ c_3 & \mathrm{Re}\ c_1 & -\mathrm{Im}\ c_1 & \mathrm{Re}\ c_3 \end{pmatrix}.$$

and that we have

$$\det g \cdot \det\left(C_g\Omega(u,v) + D_g\right) = (a_1 + a_2 v + a_3 u)^2,$$
$$\frac{\partial(g(u,v))}{\partial(u,v)} = \frac{\det g}{(a_1 + a_2 v + a_3 u)^3} \quad \text{for } g = \begin{pmatrix} a_1 & a_2 & a_3 \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{pmatrix} \in G_0.$$

2.3 generator systems of the structure ring

A holomorphic function $f(u, v)$ on $\boldsymbol{B}^2$ is said to be a modular form of weight d with respect to G_0 provided

$$f(g(u,v)) = ((a_1 + a_2 v + a_3 u))^d f(u,v), \quad g = \begin{pmatrix} a_1 & a_2 & a_3 \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{pmatrix} \in G_0. \qquad (2.2)$$

We define the modular form for other discrete groups in the same way. We use the following notation:

$M_d(G)$: the vector space of modular forms of weight d with respect to G,

$M_d(G(1+i))$: the vector space of modular forms of weight d with respect to $G(1+i)$,

$M(G)(M(G(1+i)))$: the graded ring of modular forms with respect to $G(G(1+i))$, respectively.

We note that the definition (2.2) has a meaning only for d of divisible by 4.

Proposition 2.3 *Every fourth power of the G-invariant theta constant is a modular form of weight 4 with respect to G, namely it satisfies*

$$\Theta^4\begin{bmatrix} a \\ b \end{bmatrix}(g\circ(u,v)) = (a_1+a_2v+a_3u)^4\Theta^4\begin{bmatrix} a \\ b \end{bmatrix}(u,v), \quad g=\begin{pmatrix} a_1 & a_2 & a_3 \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{pmatrix}\in G.$$

[proof]. According to Proposition 2.1, for any G-invariant theta it holds

$$\Theta^4\begin{bmatrix} a \\ b \end{bmatrix}(g\circ(u,v)) = \varepsilon(g,a,b)^4(a_1+a_2v+a_3u)^4\Theta^4\begin{bmatrix} a \\ b \end{bmatrix}(u,v), \quad g=\begin{pmatrix} a_1 & a_2 & a_3 \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{pmatrix}\in G.$$

Because $\varepsilon(g,a,b)^4(a_1+a_2v+a_3u)^4\in\{1,-1\}$ we may calculate the approximate value of the ratio of $\Theta^4\begin{bmatrix} a \\ b \end{bmatrix}(g\circ(u,v))/(a_1+a_2v+a_3u)^4\Theta^4\begin{bmatrix} a \\ b \end{bmatrix}(u,v)$ just to know the signature. As a consequence we obtain the required equalities.

q.e.d.

Note that we have $G(1+i)=\langle g_1,\dots,g_5,g_6,\sqrt{-1}\rangle$ and that we have the index between two projective groups $[\overline{G(1+i)},\overline{G}]=2$. And the generator g_6 of $\overline{G(1+i)}/\overline{G}$ induces the involution $\iota:(x,y)\mapsto(y,x)$. So we obtain the diagram with the Segre embedding map $(u,v)\mapsto[X_0,X_1,X_2,X_3]\in\boldsymbol{P}^3$:

$$\begin{array}{ccc}
\boldsymbol{P}^1\times\boldsymbol{P}^1 & \xrightarrow{\quad\Phi\quad} & \overline{\boldsymbol{B}^2/G}\cong\{X_0X_3=X_1X_2\}\subset\boldsymbol{P}^3 \\
(x,y) & & (u,v) \\
\downarrow\pi & & \downarrow\pi \\
\boldsymbol{P}^2=(\boldsymbol{P}^1\times\boldsymbol{P}^1)/\iota & \xrightarrow{\quad\Psi\quad} & \overline{\boldsymbol{B}^2/G(1+i)}\cong\boldsymbol{P}^2
\end{array}$$

Diagram 1: Period diagram

Proposition 2.4 1) *We have*

$$\dim M_4(G)=4\ \ \dim M_8(G)=9.$$

2) *$M_4(G)$ is generated by Θ^4_{xN}, Θ^4_{yN}, Θ^4_{h1}, Θ^4_{h2}.*

[proof]. Note that $\boldsymbol{B}^2$ is the uniformization of the orbifold $\boldsymbol{P}^1\times\boldsymbol{P}^1$ with the arrangement of weighted divisors:

name	divisor	weight
D_{0x}	$\{(x,y):\ x=0\}$	4
D_{0y}	$\{(x,y):\ y=0\}$	4
D_{1x}	$\{(x,y):\ x=1\}$	4
D_{1y}	$\{(x,y):\ y=0\}$	4
$D_{\infty x}$	$\{(x,y):\ x=\infty\}$	4
$D_{\infty y}$	$\{(x,y):\ y=\infty\}$	4
D_{xy}	$\{(x,y):\ x=y\}$	2

.

So the divisor of $du \wedge dv$ corresponds to

$$-\frac{3}{4}D_{0x} - \frac{3}{4}D_{0y} - \frac{3}{4}D_{1x} - \frac{3}{4}D_{1y} - \frac{1}{2}D_{xy} - \frac{3}{4}D_{\infty x} - \frac{3}{4}D_{\infty y}$$

via the period map Φ. Suppose $f(u,v)$ belongs to $M_4(G)$. Then the divisor $(f(u,v)(du \wedge dv)^{4/3})$ should satisfy

$$(f(u,v)(du \wedge dv)^{4/3}) \geq -D_{0x} - D_{0y} - D_{1x} - D_{1y} - \frac{2}{3}D_{xy} - D_{\infty x} - D_{\infty y}.$$

That is equivalent to give an element of $\mathrm{H}^0(\boldsymbol{P}^1 \times \boldsymbol{P}^1, \mathscr{O}(\frac{11}{3}D_{0x} + \frac{11}{3}D_{0y} + \frac{4}{3}K))$. Where K denotes the canonical divisor. Because we have $K = -2D_{0x} - 2D_{0y}$, we know the isomorphism

$$M_4(G) \cong \mathrm{H}^0(\boldsymbol{P}^1 \times \boldsymbol{P}^1, \mathscr{O}(D_{0x} + D_{0y})).$$

The vector space $\mathrm{H}^0(\boldsymbol{P}^1 \times \boldsymbol{P}^1, \mathscr{O}(D_{0x} + D_{0y})$ is generated by the system $\{1, x, y, xy\}$, and it is four dimensional.

By the same way we have

$$M_8(G) \cong \mathrm{H}^0(\boldsymbol{P}^1 \times \boldsymbol{P}^1, \mathscr{O}(2D_{0x} + 2D_{0y}).$$

So we obtain $\dim M_8(G) = 9$.

Observing the definition of $\Omega(u,v)$ we know that

$$\Theta \begin{bmatrix} p_1 & p_2 & p_3 \\ q_1 & q_2 & q_3 \end{bmatrix} (0, v) = \vartheta \begin{bmatrix} p_1 \\ q_1 \end{bmatrix} (v) \cdot \vartheta \begin{bmatrix} p_2 \\ q_2 \end{bmatrix} (v) \cdot \vartheta \begin{bmatrix} p_3 \\ q_3 \end{bmatrix} (v).$$

So our modular forms Θ_{xN}^4, Θ_{xD}^4, Θ_{yD}^4, Θ_{h2}^4 are linearly independent, and they give a basis of $M_4(G)$.

q.e.d.

Proposition 2.5 *The space $M_4(G)$ is generated by fourth powers of G-invariant thetas, and all algebraic relations are induced from*

$$\begin{aligned}
&\Theta_{xN}^4 - \Theta_{xD}^4 = \Theta_{yN}^4 - \Theta_{yD}^4 \\
&\Theta_{h2}^4 = 2(\Theta_{xN}^4 - \Theta_{xD}^4) \\
&\Theta_{h1}^4 - \Theta_{h3}^4 = 2(\Theta_{xD}^4 + \Theta_{yN}^4) \\
&\Theta_{xZ}^4 - \Theta_{yZ}^4 = \Theta_{xN}^4 - \Theta_{yN}^4 \\
&\Theta_{h1}^4 = 2(\Theta_{xZ}^4 + \Theta_{yZ}^4)
\end{aligned}$$

$$\begin{aligned}
&\Theta_{xD}^4 \Theta_{yN}^4 = \Theta_{xZ}^4 (\Theta_{xN}^4 - (\Theta_{xD}^4) \\
&\Theta_{xN}^4 \Theta_{yD}^4 = \Theta_{yZ}^4 (\Theta_{yN}^4 - \Theta_{yD}^4).
\end{aligned}$$

[proof]. Let us consider the truncation of the Fourier expansions of the fourth powers of our G invariant Theta constants up to some fixed hight ℓ. Set V_ℓ be the vector space generated by these polynomials. There is the canonical surjective homomorphism $\overline{\omega} : M_4(G) \to V_\ell$. We can easily find that V_ℓ is four dimensional. So $\overline{\omega}$ should be an isomorphism. Then we can obtain the above relations by explicit calculations of their coefficients. The situation is the same for the quadratic relations.

q.e.d.

Remark 2.2 *By observing the last identity in the above Proposition we have a system of generators* $\Theta_{xN}^4, \Theta_{yZ}^4, \Theta_{yN}^4 - \Theta_{yD}^4, \Theta_{yD}^4$ *of the structure ring* $\boldsymbol{C}[X_0, X_1, X_2, X_3]/(X_0X_3 - X_1X_2)$ *of* $\overline{\boldsymbol{B}^2/G} \cong \boldsymbol{P}^1 \times \boldsymbol{P}^1$.

2.4 CM-isogeny on $Jac(C(x,y))$

Let us consider an isogeny map of $\boldsymbol{B}^2$:

$$\sigma(u,v) = ((1+i)u, 2v).$$

It is induced from the $\boldsymbol{Q}$-symplectic transformation:

$$N_\sigma = \begin{pmatrix} A & O \\ O & {}^tA^{-1} \end{pmatrix}, \quad A = \begin{pmatrix} 1 & 1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

and we have

$$N_\sigma \circ \tau = A\tau\, {}^tA, \quad \tau \in \boldsymbol{H}_3.$$

Set

$$\begin{aligned}
&A_0 = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}, \\
&\Lambda = \{(n_1+n_2, n_1-n_2) \in \boldsymbol{Z}^2 : n_1, n_2 \in \boldsymbol{Z}\} \\
&\vartheta^\Lambda_{a',b'}(\tau') := \sum_{m\in\Lambda} \exp[\pi i(m+\frac{a'}{2})\tau'^t(m+\frac{a'}{2}) + 2\pi i(m+\frac{a'}{2})\frac{{}^tb'}{2}], \quad \tau' \in \boldsymbol{H}_2, a', b' \in \boldsymbol{Z}^2 \\
&\vartheta^\sharp_{a,b}(\tau) := \vartheta_{a,b}(A\tau\, {}^tA), \quad \tau \in \boldsymbol{H}_3, a, b \in \boldsymbol{Z}^3 \\
&\vartheta^\sharp_{a',b'}(\tau') := \vartheta_{a',b'}(A_0\tau'\, {}^tA_0), \quad \tau' \in \boldsymbol{H}_2, a', b' \in \boldsymbol{Z}^2,
\end{aligned}$$

where $\boldsymbol{H}_g$ denotes the Siegel upper half space of degree g.

By elementary caluculations we have

Lemma 2.3

$$\vartheta^\sharp_{a',b'}(\tau') = \vartheta^\Lambda_{a'A_0, b'\,{}^tA_0^{-1}}(\tau').$$

According to Matsumoto, Minowa and Nishimura [Mat-Min-Nish] we have

Theorem 2.2 (Λ-theta formula)

$$\begin{pmatrix} \vartheta^\Lambda_{a',b'}(\tau') \\ \vartheta^\Lambda_{a'+\{2,0\},b'}(\tau') \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & \exp[-\frac{\pi i}{2}(a_1+a_2)] \\ 1 & -\exp[-\frac{\pi i}{2}(a_1+a_2)] \end{pmatrix}\begin{pmatrix} \vartheta_{a',b'}(\tau) \\ \vartheta_{a',b'+\{1,1\}}(\tau') \end{pmatrix}$$

By virtue of Theirem 2.2 together with Lemma 2.3 we obtain:

Proposition 2.6

$$\begin{aligned}
\vartheta^\sharp\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}(\tau) &= \frac{1}{2}\vartheta\begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}(\tau) - \frac{1}{2}\vartheta\begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}(\tau) \\
\vartheta^\sharp\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}(\tau) &= \frac{1}{2}\vartheta\begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}(\tau) + \frac{1}{2}\vartheta\begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}(\tau) \\
\vartheta^\sharp\begin{bmatrix} 0 & 0 & 0 \\ 1 & 1 & 0 \end{bmatrix}(\tau) &= \vartheta\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}(\tau) \\
\vartheta^\sharp\begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}(\tau) &= \frac{1+i}{2}\vartheta\begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}(\tau) \\
\vartheta^\sharp\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}(\tau) &= \frac{1}{2}\vartheta\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}(\tau) + \frac{1}{2}\vartheta\begin{bmatrix} 0 & 0 & 0 \\ 1 & 1 & 0 \end{bmatrix}(\tau) \\
\vartheta^\sharp\begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}(\tau) &= \frac{1}{2}\vartheta\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}(\tau) - \frac{1}{2}\vartheta\begin{bmatrix} 0 & 0 & 0 \\ 1 & 1 & 0 \end{bmatrix}(\tau).
\end{aligned}$$

2.5 modular interpretation of the extended Gauss AGM

Set

$$H_1(u,v) = \Theta_{h1}(u,v) = \vartheta\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}(\Omega(u,v)),$$

$$H_3(u,v) = \Theta_{h3}(u,v) = \vartheta\begin{bmatrix} 0 & 0 & 0 \\ 1 & 1 & 0 \end{bmatrix}(\Omega(u,v)),$$

$$Z(u,v) = \Theta_Z(u,v) = \vartheta\begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}(\Omega(u,v)).$$

Now we have

Theorem 2.3 (Second Main Theorem = CM-isogeny formula) *It holds*

$$H_1((1+i)u,2v)) = \frac{1}{2}\left(H_1(u,v)+H_3(u,v)\right)$$
$$H_3((1+i)u,2v)) = Z(u,v)$$
$$Z((1+i)u,2v)) = \sqrt[4]{\frac{1}{4}\left(Z^2+H_1H_3)(Z^2+\frac{H_1^2+H_3^2}{2}\right)}.$$

[proof]. The first and the second equalities are already obtained in Proposition 2.6. So we concentrate to show the last equality. The theta constants $\Theta\begin{bmatrix}0&0&0\\0&0&0\end{bmatrix}(u,v)$ and $\Theta\begin{bmatrix}0&0&0\\0&0&0\end{bmatrix}(u,v)$ are not G-invariant thetas. Setting $g_i = \begin{pmatrix} a_{i1} & a_{i2} & a_{i3} \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{pmatrix}$ $(i=1,\dots,5)$, according to Proposition 2.1 we have

$$\Theta\begin{bmatrix}0&0&0\\1&0&0\end{bmatrix}^2(g_i(u,v)) = (a_{i1}+a_{i2}v+a_{i3}u)^2\Theta\begin{bmatrix}0&0&0\\1&0&0\end{bmatrix}^2(g_i(u,v)) \quad (i=1,3),$$
$$\Theta\begin{bmatrix}1&1&1\\0&1&1\end{bmatrix}^2(g_i(u,v)) = (a_{i1}+a_{i2}v+a_{i3}u)^2\Theta\begin{bmatrix}1&1&1\\0&1&1\end{bmatrix}^2(g_i(u,v)) \quad (i=1,3),$$
$$\Theta\begin{bmatrix}0&0&0\\1&0&0\end{bmatrix}^2(g_i(u,v)) = (-1)^{i+1}(a_{i1}+a_{i2}v+a_{i3}u)^2\Theta\begin{bmatrix}1&1&1\\0&1&1\end{bmatrix}^2(g_i(u,v)) \quad (i=2,4,5)$$
$$\Theta\begin{bmatrix}1&1&1\\0&1&1\end{bmatrix}^2(g_i(u,v)) = (-1)^{i+1}(a_{i1}+a_{i2}v+a_{i3}u)^2\Theta\begin{bmatrix}0&0&0\\1&0&0\end{bmatrix}^2(g_i(u,v)) \quad (i=2,4,5).$$

Then $\Theta\begin{bmatrix}0&0&0\\1&0&0\end{bmatrix}^4(u,v)+\Theta\begin{bmatrix}1&1&1\\0&1&1\end{bmatrix}^4(u,v)$ and $\Theta\begin{bmatrix}0&0&0\\1&0&0\end{bmatrix}^2(u,v)\Theta\begin{bmatrix}1&1&1\\0&1&1\end{bmatrix}^2(u,v)$ belong to $M_4(G)$. By observing the truncations we obtain

$$\Theta\begin{bmatrix}0&0&0\\1&0&0\end{bmatrix}^4(u,v)+\Theta\begin{bmatrix}1&1&1\\0&1&1\end{bmatrix}^4(u,v) = \frac{1}{2}\left(\Theta_{h1}^4(u,v)-\Theta_{h2}^4(u,v)+\Theta_{h3}^4(u,v)\right)$$
$$\Theta\begin{bmatrix}0&0&0\\1&0&0\end{bmatrix}^2(u,v)\Theta\begin{bmatrix}1&1&1\\0&1&1\end{bmatrix}^2(u,v) = \frac{1}{2}\left(\Theta_{xN}^4(u,v)-\Theta_{yN}^4(u,v)\right).$$

By elliminating $\Theta\begin{bmatrix}1&1&1\\0&1&1\end{bmatrix}$ from these two equalities we obtain the description of $Z(u,v)^4$ in terms of $\Theta_{h1},\Theta_{h2},\Theta_{h3},\Theta_{xN}$ and Θ_{yN}. Proposition 2.6 gives the description of $\Theta_{h1}^4((1+i)u,2v)-\Theta_{h2}^4((1+i)u,2v)+\Theta_{h3}^4((1+i)u,2v)$ and $\Theta_{xN}^4((1+i)u,2v)-\Theta_{yN}^4((1+i)u,2v)$ in terms of $\Theta_{h1}(u,v),\Theta_{h3}(u,v)$ and $Z(u,v)$. As a consequence we obtain the required relation.

q.e.d.

References

[GaT] C,F, Gauss, Mathematisches Tagebuch 1796-1814, Ostwalds Kalassiker der exakte Wissenschaften 256,*Geest und Portig*, Leibzig,1976.

[GaH] C,F, Gauss, Hundert Theoreme über die neuen Transzendenten, 1818

[I] J. Igusa, Theta functions, *Springer-Verlag* , Heidelberg, New-York(1972).

[Ko-Shi] K. Koike and H. Shiga, Isogeny formulas for the Picard modular form and a three terms arithmetic geometric mean, J. Number Theory, to appear

[Bi-La] Ch. Birkenhake and H. Lange, Complex Abelian Varieties, *Springer-Verlag* (1992).

[Mat1] K. Matsumoto, On Modular Functions in 2 Variables Attached to a Family of Hyperelliptic Curves of Genus 3, Ann. S.N.S. Pisa, Ser IV-XVI(1989), 557-578.

[Mat-Min-Nish] K. Matsumoto, T. Minowa and R. Nishimura, Automorphic forms on the 5-dimensional complex ball with respect to the Picard modular group over $\boldsymbol{Z}[i]$, to appear in Hokkaido Math. J. .

[Mum] D. Mumford, Tata Lectures on Theta I, *Birkhäuser*, Boston-Basel-Stuttgard(1983).

[Re-Ta] Resnikoff and Tai, On the Structure of a Graded Ring of Automorphic Forms on the 2-dimensional Complex Ball, Math. Ann. 238 (1978), 97-117.