The Variants of Kappa Function

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Abstract: M. KANEKO and M.YOSHIDA introduced the kappa function with the property $J(\kappa(z)) = \lambda(z)$ in [KY], where J is the elliptic modular function and λ is the Jacobi's elliptic λ-function. In this paper the author constructs the variants of $\kappa(z)$ using other pairs of simple modular functions instead of $(J(z), \lambda(z))$, and gives explicit Fourier expansions for them.

1. Variants of the λ function

We rewrite some well-known factor about the modular function.

Put

$$\Gamma = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z}) \;\middle|\; ad - bc = \pm 1 \right\},$$

$$\Gamma(2) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z}) \;\middle|\; \begin{matrix} a \equiv d \equiv 1 \\ b \equiv c \equiv 0 \end{matrix} \pmod 2 \right\},$$

$$\Gamma_{1,2} = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z}) \;\middle|\; ab \equiv cd \equiv 0 \pmod 2 \right\}.$$

Setting

$$\gamma_i = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},\ \gamma_{\rho_2} = \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix},\ \gamma_\infty = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix},\ \gamma_0 = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix},$$

we have

$$\Gamma = \langle \gamma_i, \gamma_{\rho_2} \rangle,\ \Gamma(2) = \langle \gamma_\infty, \gamma_0 \rangle,\ \Gamma_{1,2} = \langle \gamma_i, \gamma_0 \rangle.$$

And we define

$$\Gamma^*(2) = \left\langle \begin{pmatrix} -\sqrt{2} & -\sqrt{2} \\ 1/\sqrt{2} & 0 \end{pmatrix}, \begin{pmatrix} 0 & -\sqrt{2} \\ 1/\sqrt{2} & 0 \end{pmatrix} \right\rangle.$$

Because of

$$\begin{pmatrix} 0 & -\sqrt{2} \\ 1/\sqrt{2} & 0 \end{pmatrix}^{-1} \begin{pmatrix} 0 & -\sqrt{2} \\ 1/\sqrt{2} & -\sqrt{2} \end{pmatrix} \begin{pmatrix} 0 & -\sqrt{2} \\ 1/\sqrt{2} & 0 \end{pmatrix} = \begin{pmatrix} -\sqrt{2} & -\sqrt{2} \\ 1/\sqrt{2} & 0 \end{pmatrix},$$

we have

$$\Gamma^*(2) = \langle \gamma_{1+i}, \gamma_{\sqrt{2}i} \rangle,$$

with

$$\gamma_{1+i} = \begin{pmatrix} 0 & -\sqrt{2} \\ 1/\sqrt{2} & -\sqrt{2} \end{pmatrix} \text{ and } \gamma_{\sqrt{2}i} = \begin{pmatrix} 0 & -\sqrt{2} \\ 1/\sqrt{2} & 0 \end{pmatrix}.$$

Any of these groups acts on the upper half complex plane $\mathbb{H}$.

(1) The modular function $\lambda^\sharp$:

Let us make a biholomorphic map from the fundamental domain $F(2) = \{z \in \mathbb{H}; 0 < \mathrm{Re}z < 1, |z - \frac{1}{2}$ of $\Gamma(2)$, that is indicated as a shaded part in Figure 1.1, onto the lower half complex plane $\mathbb{H}^-$. We assume the boundary points 0, 1 and ∞ of $\mathbb{H}^-$ correspond to 0, $i\infty$ and 1, respectively. By the Schwarz reflection principle we obtain a modular function $\lambda^\sharp(z)$ defined on the whole $\mathbb{H}$.

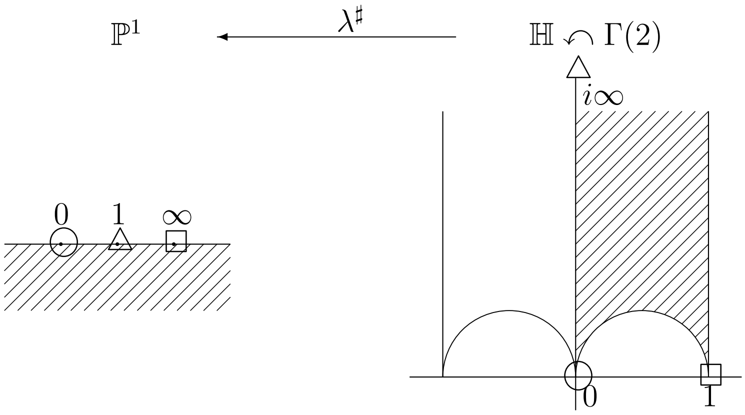


FIGURE 1.1

Here $\lambda^\sharp$ is the function λ in [KY] and we have

$$\lambda^\sharp(z) = \frac{\theta_0(z)^4}{\theta_3(z)^4}, \qquad \left(\theta_0(z) = \sum_{n\in\mathbb{Z}} (-1)^n q^{n^2}, \theta_3(z) = \sum_{n\in\mathbb{Z}} q^{n^2}\right),$$

where $q = e^{\pi i z}$.

(2) The fundamental domain of $\lambda^\flat$:

Let us make a biholomorphic map from the fundamental domain $F(2) = \{z \in \mathbb{H}; 0 < \mathrm{Re}z < 1, |z - \frac{1}{2}$ of $\Gamma(2)$, that is indicated as a shaded part in Figure 1.2, onto the lower half complex plane $\mathbb{H}$. We assume the boundary points 0, 1 and ∞ of $\mathbb{H}$ correspond to 1, $i\infty$ and 0, respectively. By the Schwarz reflection principle we obtain a modular function $\lambda^\flat(z)$ defined on the whole $\mathbb{H}$.

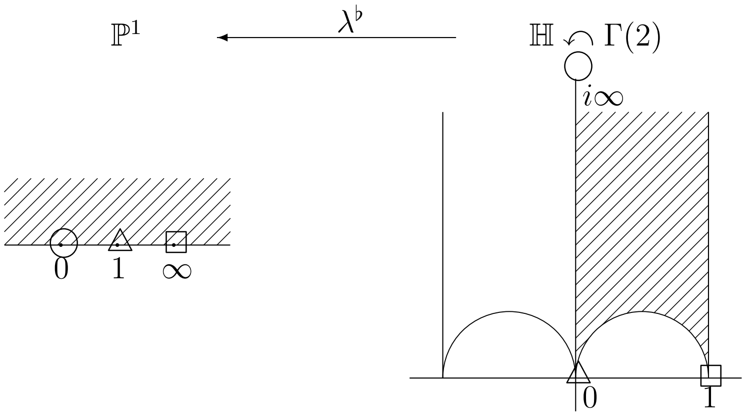


FIGURE 1.2

Here we have

$$\lambda^{\flat}(z) = \frac{\theta_2(z)^4}{\theta_3(z)^4}, \qquad \left(\theta_2(z) = \sum_{n\in\mathbb{Z}} q^{\frac{(2n+1)^2}{4}}, \theta_3(z) = \sum_{n\in\mathbb{Z}} q^{n^2}\right),$$

where $q = e^{\pi i z}$.

(3) The fundamental domain of $\lambda^{\sharp}_{1,2}$:

Let us make a biholomorphic map from the fundamental domain $F^{\sharp}_{1,2} = \{z \in \mathbb{H}; -1 < \mathrm{Re} z < 0, |z| >$ of $\Gamma_{1,2}$, that is indicated as a shaded part in Figure 1.3, onto the lower half complex plane $\mathbb{H}$. We assume the boundary points 0, 1 and ∞ of $\mathbb{H}$ correspond to i, $i\infty$ and -1, respectively. By the Schwarz reflection principle we obtain a modular function $\lambda^{\sharp}_{1,2}(z)$ defined on the whole $\mathbb{H}$.

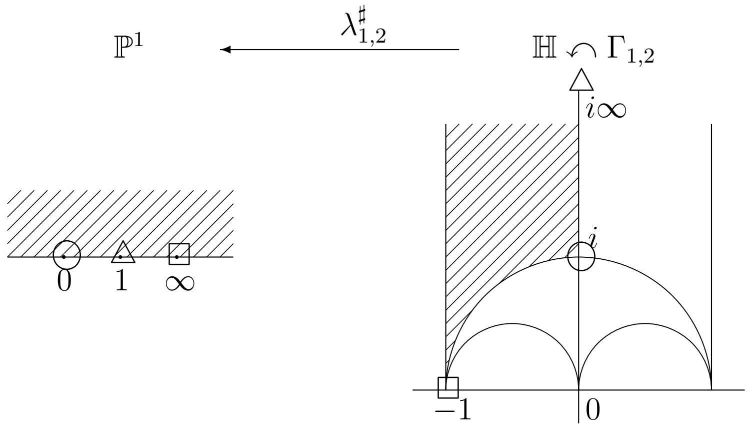


FIGURE 1.3

(4) The fundamental domain of $\lambda^{\flat}_{1,2}$:

Let us make a biholomorphic map from the fundamental domain $F^{\flat}_{1,2} = \{z \in \mathbb{H}; 0 < \mathrm{Re} z < 1, |z| > 1\}$ of $\Gamma_{1,2}$, that is indicated as a shaded part in Figure 1.4, onto the lower half complex plane $\mathbb{H}$. We assume the boundary points 0, 1 and ∞ of $\mathbb{H}$ correspond to $i\infty$, i and 1, respectively. By the Schwarz reflection principle we obtain a modular function $\lambda^{\flat}_{1,2}(z)$ defined on the whole $\mathbb{H}$.

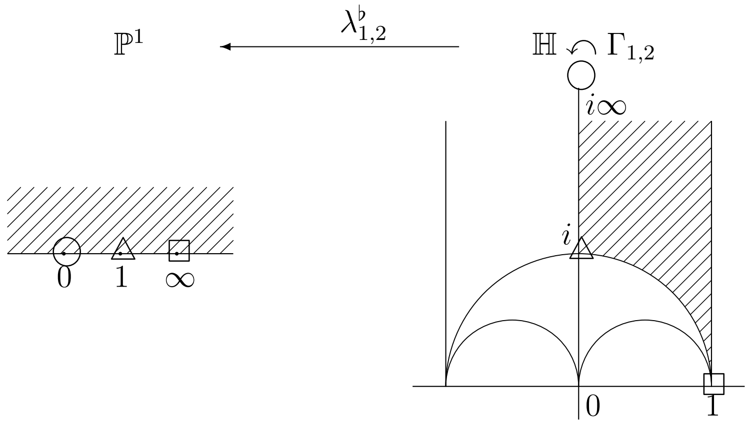


FIGURE 1.4

(5) The fundamental domain of J:

Let us make a biholomorphic map from the fundamental domain $F = \{z \in \mathbb{H}; -\frac{1}{2} < \mathrm{Re} z < 0, |z| > 1\}$ of Γ, that is indicated as a shaded part in Figure 1.5, onto the lower half complex plane $\mathbb{H}$. We assume the boundary points 0, 1 and ∞ of $\mathbb{H}$ correspond to ρ_2, i and $i\infty$, respectively. By the Schwarz reflection principle we obtain a modular function J defined on the whole $\mathbb{H}$.

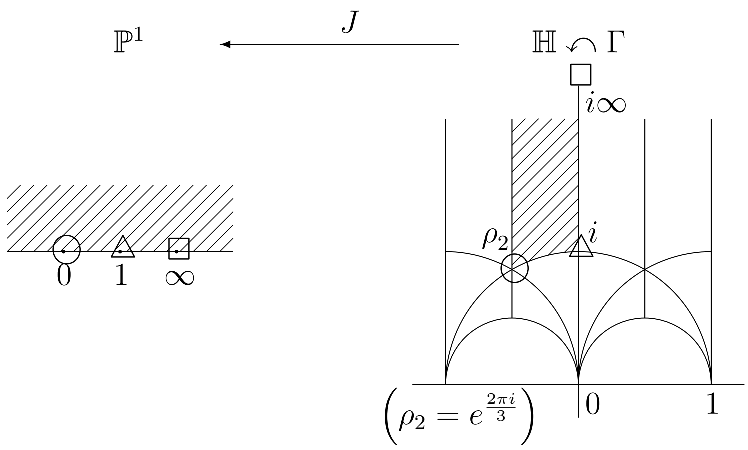


FIGURE 1.5

(6) The fundamental domain of $\lambda^*_{(2)}$:

Let us make a biholomorphic map from the fundamental domain $F^*(2) = \{z \in \mathbb{H}; 0 < \mathrm{Re} z < 1, |z| >$ of $\Gamma^*(2)$, that is indicated as a shaded part in Figure 1.6, onto the lower half complex plane $\mathbb{H}$. We assume the boundary points 0, 1 and ∞ of $\mathbb{H}$ correspond to $\sqrt{2}i$, $1+i$ and $i\infty$, respectively. By the Schwarz reflection principle we obtain a modular function $\lambda(2)^*$ defined on the whole $\mathbb{H}$.

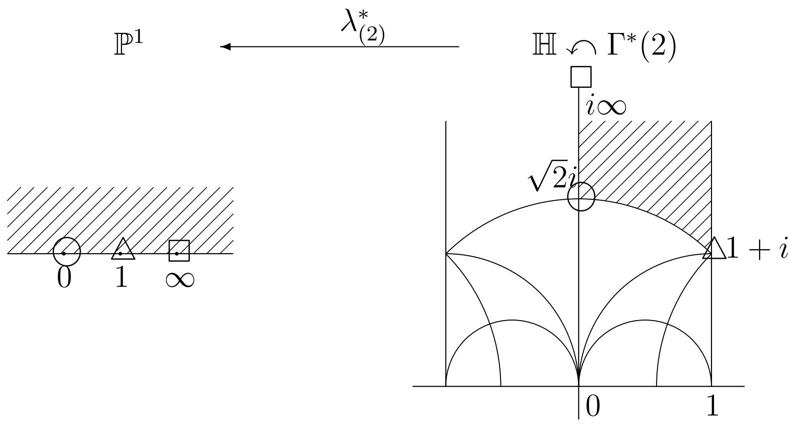


FIGURE 1.6

Proposition

We have the relations among the above modular functions as follows:

$$\begin{aligned}
J &= P(\lambda^\sharp) = P(\lambda^\flat), && \text{where } P(l) = \frac{4}{27}\frac{(1-l+l^2)^3}{l^2(1-l)^2},\\
\lambda^\sharp_{1,2} &= P^\sharp(\lambda^\sharp) = P^\sharp(\lambda^\flat), && \text{where } P^\sharp(l) = (1-2l)^2,\\
\lambda^\flat_{1,2} &= P^\flat(\lambda^\sharp) = P^\flat(\lambda^\flat), && \text{where } P^\flat(l) = 4l(1-l),\\
\lambda^*_{(2)} &= P^*(\lambda^\sharp) = P^*(\lambda^\flat), && \text{where } P^*(l) = -\frac{1}{16}\frac{(1-6l+l^2)^2}{l(1-l)^2},
\end{aligned}$$

and

$$\begin{aligned}
&\lambda^\sharp = 1-\lambda^\flat,\\
&\lambda^\sharp_{1,2} = 1-\lambda^\flat_{1,2},\\
&J = P^\sharp_{1,2}(\lambda^\sharp_{1,2}) = P^\flat_{1,2}(\lambda^\flat_{1,2}), \text{ where } P^\sharp_{1,2}(l) = \frac{1}{27}\frac{(3+l)^3}{(1-l)^2} \text{ and } P^\flat_{1,2} = \frac{1}{27}\frac{(4-l)^3}{l^2}.
\end{aligned}$$

We have some values of the above modular functions as following:

$z \in \mathbb{H}$	$\lambda^\sharp(z)$	$\lambda^\flat(z)$	$\lambda^\sharp_{1,2}(z)$	$\lambda^\flat_{1,2}(z)$	$J(z)$	$\lambda^*_{(2)}(z)$
$i\infty$	1	0	1	0	∞	∞
$\sqrt{2}i$	$-2-2\sqrt{2}$	$3-2\sqrt{2}$	$57-40\sqrt{2}$	$-8(7-5\sqrt{2})$	$\frac{125}{27}$	0
i	$\frac{1}{2}$	$\frac{1}{2}$	0	1	1	$-\frac{49}{32}$
0	0	1	1	0	∞	∞
$\frac{1+i}{2}$	-1	2	9	-8	1	$-\frac{49}{32}$
$\frac{2+\sqrt{2}i}{3}$	$-2-2\sqrt{2}$	$3+2\sqrt{2}$	$57+40\sqrt{2}$	$-8(7+5\sqrt{2})$	$\frac{125}{27}$	0
1	∞	∞	∞	∞	∞	∞
$1+i$	2	-1	9	-8	1	1
$\frac{1+\sqrt{3}i}{2}$	$\frac{1-\sqrt{3}i}{2}$	$\frac{1+\sqrt{3}i}{2}$	-3	4	0	$\frac{25}{16}$

2. Variants of the κ function

2.1 κ function

In this subsection we recall the κ function studied in [KY]. Because of $J(z) = P(\lambda^\sharp(z))$ we have the following diagram:

$$\begin{array}{ccc} \mathbb{P}^1 & \xleftarrow{J} & \mathbb{H} \curvearrowleft \Gamma \\ P \uparrow & & \| \\ \mathbb{P}^1 & \xleftarrow{\lambda^\sharp} & \mathbb{H} \curvearrowleft \Gamma(2) \end{array}$$

Let us make a biholomorphic map from the fundamental domain $F(2)$ of $\Gamma(2)$ to that of Γ that sends 0 $(1,\ i\infty)$ to $\rho_1 = e^{\frac{\pi i}{3}}$ $(i\infty,\ i)$, respectively. By the Schwarz reflection principle we obtain a mapping defined on the whole $\mathbb{H}$. We denote it by $\kappa^\sharp$.

$$\begin{array}{ccc} \mathbb{P}^1 & \xleftarrow{J} & \mathbb{H} \curvearrowleft \Gamma \\ \| & & \uparrow \kappa^\sharp \\ \mathbb{P}^1 & \xleftarrow{\lambda^\sharp} & \mathbb{H} \curvearrowleft \Gamma(2) \end{array}$$

We note that $\kappa^\sharp$ induces a 1:1 correspondence between two shaded parts in Figure 2.1.

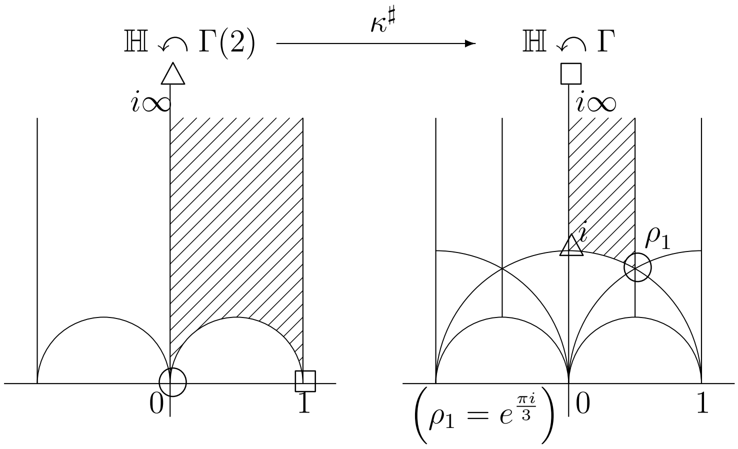


FIGURE 2.1

Now we use the symbol $\ll \cdot \gg$ to denote the smallest normal subgroup containing $\cdot$ of a given group. Then the isomorphism $\Gamma(2)/ \ll {\gamma_\infty}^3, {\gamma_0}^2 \gg \cong \Gamma$ is given by the following correspondence:

$$\begin{array}{ccc} \Gamma(2)/ \ll {\gamma_\infty}^3, {\gamma_0}^2 \gg & \longrightarrow & \Gamma \\ \gamma_\infty & \longmapsto & \gamma_{\rho_2} \\ \gamma_0 & \longmapsto & \gamma_i \end{array}$$

Thus the function $\kappa^\sharp$ satisfies

$$\kappa^\sharp\left(z+2\right) = -\frac{1}{\kappa^\sharp(z)} \quad \text{and} \quad \kappa^\sharp\left(\frac{z}{2z+1}\right) = 1 - \frac{1}{\kappa^\sharp(z)}.$$

2.2. Construction of new κ functions

By the same method, we obtain new κ functions using variants of λ functions. Starting from the correspondence of the fundamental domains of both modular groups, we obtain our variants of κ function by the Schwarz reflection principle.

(1) The function $\widetilde{\kappa}_{(2)} : \mathbb{H} \curvearrowleft \Gamma(2) \longrightarrow \mathbb{H} \curvearrowleft \Gamma(2)$.
That is defined by the equation $\lambda^\sharp(z) = \lambda^\flat(\widetilde{\kappa}_{(2)}(z))$.

$$\begin{array}{ccc} \mathbb{P}^1 & \xleftarrow{\lambda^\sharp} & \mathbb{H} \curvearrowleft \Gamma(2) \\ \Big\| & & \Big\uparrow \widetilde{\kappa}_{(2)} \\ \mathbb{P}^1 & \xleftarrow{\lambda^\flat} & \mathbb{H} \curvearrowleft \Gamma(2) \end{array}$$

The automorphism $\widetilde{\kappa}_{(2)} \in \mathrm{Aut}(\Gamma(2))$ is given by the following correspondence:

$$\begin{array}{ccc} \Gamma(2) & \longrightarrow & \Gamma(2) \\ \gamma_\infty & \longmapsto & \gamma_0 \\ \gamma_0 & \longmapsto & \gamma_\infty \end{array}$$

Thus the function $\widetilde{\kappa}_{(2)}$ satisfies

$$\widetilde{\kappa}_{(2)}\,(z+2) = \frac{\widetilde{\kappa}_{(2)}(z)}{-2\widetilde{\kappa}_{(2)}(z)+1} \quad \text{and} \quad \widetilde{\kappa}_{(2)}\left(\frac{z}{2z+1}\right) = \widetilde{\kappa}_{(2)}(z) - 2.$$

(2) The function $\kappa^\flat : \mathbb{H} \curvearrowleft \Gamma(2) \longrightarrow \mathbb{H} \curvearrowleft \Gamma$.
That is defined by the equation $\lambda^\flat(z) = J(\kappa^\flat(z))$.

$$\begin{array}{ccc} \mathbb{P}^1 & \xleftarrow{J} & \mathbb{H} \curvearrowleft \Gamma \\ \| & & \uparrow \kappa^\flat \\ \mathbb{P}^1 & \xleftarrow{\lambda^\flat} & \mathbb{H} \curvearrowleft \Gamma(2) \end{array}$$

The isomorphism $\Gamma(2)/ \ll {\gamma_\infty}^3, {\gamma_0}^2 \gg \cong \Gamma$ is given by the following correspondence:

$$\begin{array}{ccc} \Gamma(2)/ \ll {\gamma_\infty}^3, {\gamma_0}^2 \gg & \longrightarrow & \Gamma \\ \gamma_\infty & \longmapsto & \gamma_{\rho_2} \\ \gamma_0 & \longmapsto & \gamma_i \end{array}$$

Thus the function $\kappa^\flat$ satisfies

$$\kappa^\flat\,(z+2) = -\frac{1}{\kappa^\flat(z)} \quad \text{and} \quad \kappa^\flat\left(\frac{z}{2z+1}\right) = 1 - \frac{1}{\kappa^\flat(z)}.$$

(3) The function $\kappa_{1,2} : \mathbb{H} \curvearrowleft \Gamma(2) \longrightarrow \mathbb{H} \curvearrowleft \Gamma_{1,2}$.
That is defined by the equation $\lambda^\flat(z) = \lambda^\sharp_{1,2}(\kappa_{1,2}(z))$.

$$\begin{array}{ccc} \mathbb{P}^1 & \xleftarrow{\lambda^\sharp_{1,2}} & \mathbb{H} \curvearrowleft \Gamma_{1,2} \\ \| & & \uparrow \kappa_{1,2} \\ \mathbb{P}^1 & \xleftarrow{\lambda^\flat} & \mathbb{H} \curvearrowleft \Gamma(2) \end{array}$$

The isomorphism $\Gamma(2)/ \ll {\gamma_\infty}^2 \gg \cong \Gamma_{1,2}$ is given by the following correspondence:

$$\begin{array}{ccc} \Gamma(2)/ \ll {\gamma_\infty}^2 \gg & \longrightarrow & \Gamma_{1,2} \\ \gamma_\infty & \longmapsto & \gamma_i \\ \gamma_0 & \longmapsto & \gamma_0 \end{array}$$

Thus the function $\kappa_{1,2}$ satisfies

$$\kappa_{1,2}\,(z+2) = -\frac{1}{\kappa_{1,2}(z)} \quad \text{and} \quad \kappa_{1,2}\left(\frac{z}{2z+1}\right) = \frac{\kappa_{1,2}(z)}{2\kappa_{1,2}(z)+1}.$$

(4) The function $\widetilde{\kappa}_{1,2} : \mathbb{H} \curvearrowleft \Gamma_{1,2} \longrightarrow \mathbb{H} \curvearrowleft \Gamma_{1,2}$.
That is defined by the equation $\lambda^{\sharp}_{1,2}(z) = \lambda^{\flat}_{1,2}(\widetilde{\kappa}_{1,2}(z))$.

$$\begin{array}{ccc} \mathbb{P}^1 & \xleftarrow{\lambda^{\flat}_{1,2}} & \mathbb{H} \curvearrowleft \Gamma_{1,2} \\ \Big\| & & \uparrow \widetilde{\kappa}_{1,2} \\ \mathbb{P}^1 & \xleftarrow{\lambda^{\sharp}_{1,2}} & \mathbb{H} \curvearrowleft \Gamma_{1,2} \end{array}$$

The isomorphism $\langle \gamma_\infty \rangle \, / \, \langle {\gamma_\infty}^2 \rangle \cong \langle \gamma_i \rangle$ is given by the following correspondence:

$$\begin{array}{ccc} \langle \gamma_\infty \rangle \, / \, \langle {\gamma_\infty}^2 \rangle & \longrightarrow & \langle \gamma_i \rangle \\ \gamma_\infty & \longmapsto & \gamma_i \end{array}$$

Thus the function $\widetilde{\kappa}_{1,2}$ satisfies

$$\widetilde{\kappa}_{1,2}\,(z+2) = -\frac{1}{\widetilde{\kappa}_{1,2}(z)}.$$

(5) The function $\check{\kappa}_{1,2} : \mathbb{H} \curvearrowleft \Gamma_{1,2} \longrightarrow \mathbb{H} \curvearrowleft \Gamma$.
That is defined by the equation $\lambda^{\flat}_{1,2}(z) = J(\check{\kappa}_{1,2}(z))$.

$$\begin{array}{ccc} \mathbb{P}^1 & \xleftarrow{J} & \mathbb{H} \curvearrowleft \Gamma \\ \Big\| & & \uparrow \check{\kappa}_{1,2} \\ \mathbb{P}^1 & \xleftarrow{\lambda^{\flat}_{1,2}} & \mathbb{H} \curvearrowleft \Gamma_{1,2} \end{array}$$

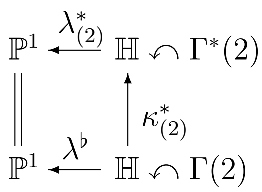
The isomorphism $\Gamma_{1,2} / \ll {\gamma_\infty}^3 \gg \cong \Gamma$ is given by the following correspondence:

$$\begin{array}{ccc} \Gamma_{1,2} / \ll {\gamma_\infty}^3 \gg & \longrightarrow & \Gamma \\ \gamma_\infty & \longmapsto & \gamma_{\rho_2} \\ \gamma_i & \longmapsto & \gamma_i \end{array}$$

Thus the function $\check{\kappa}_{1,2}$ satisfies

$$\check{\kappa}_{1,2}\,(z+2) = -1 - \frac{1}{\check{\kappa}_{1,2}(z)} \quad \text{and} \quad \check{\kappa}_{1,2}\left(-\frac{1}{z}\right) = -\frac{1}{\check{\kappa}_{1,2}(z)}.$$

(6) The function $\kappa^*_{(2)} : \mathbb{H} \curvearrowleft \Gamma(2) \longrightarrow \mathbb{H} \curvearrowleft \Gamma^*(2)$.
That is defined by the equation $\lambda^{\flat}(z) = \lambda^*_{(2)}(\kappa^*_{(2)}(z))$.



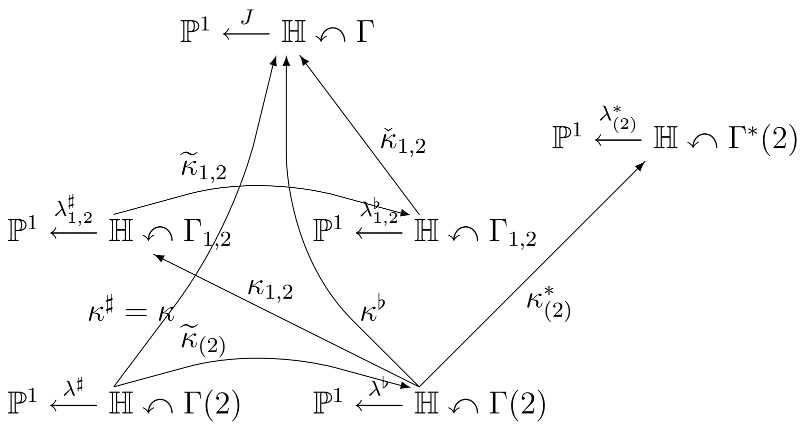
The isomorphism $\Gamma(2)/ \ll {\gamma_\infty}^2, {\gamma_0}^4 \gg \cong \Gamma^*(2)$ is given by the following correspondence:

$$\begin{array}{ccc} \Gamma(2)/ \ll {\gamma_\infty}^2, {\gamma_0}^4 \gg & \longrightarrow & \Gamma^*(2) \\ \gamma_\infty & \longmapsto & \gamma_{\sqrt{2}i} \\ \gamma_0 & \longmapsto & \gamma_{1+i} \end{array}$$

Thus the function $\kappa^*_{(2)}$ satisfies

$$\kappa^*_{(2)}(z+2) = -\frac{2}{\kappa^*_{(2)}(z)} \quad \text{and} \quad \kappa^*_{(2)}\left(\frac{z}{2z+1}\right) = -\frac{2}{\kappa^*_{(2)}(z) - 2}.$$

By the section 2.1 and (1), (2), $\cdots$, (6) we have the figure of the relations between the modular functions as following:



3. Fourier expansion

Let $q_n = e^{\frac{\pi i z}{n}}$ and

$$\sigma_k^-(n) = \sum_{d|n} (-1)^k d^k.$$

Each κ function, except $\kappa_{(2)}$, in the section 2 has the Fourier series expansion at $z = i\infty$ and $\kappa_{(2)}$ is a linear transformation as following:

Thorem 1

(0) The function $\kappa^\sharp$ ([KY])*:*

$$\kappa^\sharp(z) = i(1 + a_1 q_2 + a_2 {q_2}^2 + a_3 {q_2}^3 + \cdots),$$

where

$$a_1 = -i\frac{32}{\sqrt{3}}\frac{\pi^2}{\Gamma\left(\frac{1}{4}\right)^4} = -1.0552729262852\cdots \times i,$$

and the coefficients a_n satisfies the reccurent relation

$$
\begin{aligned}
&a_2 = \frac{{a_1}^2}{2} \quad \textit{and} \\
&2n(n-1)(n-2)a_1 a_n \\
&= -\sum_{m=2}^{n-1} i(n+1-m)\left(2(n+1)^2 - 7m(n+1) + 5m^2 + 1\right) a_m a_{n+1-m} \\
&\quad -\sum_{j=1}^{n-1} b_j \sum_{m=1}^{n-j} i(n+1-j-m) a_m a_{n+1-j-m}, \quad (n \geq 3),
\end{aligned}
$$

and the series b_n are

$$
b_n = \begin{cases}
1, & n = 0 \\
0, & \textit{for } n\textit{:odd}, \\
(-1)^{\frac{n}{2}} \frac{64}{9} \sigma_3^- \left(\frac{n}{2}\right), & \textit{for } n \equiv 2 \bmod 4, \\
(-1)^{\frac{n}{2}} \frac{64}{9} \sigma_3^- \left(\frac{n}{2}\right) + \frac{80}{9} \sigma_3^- \left(\frac{n}{4}\right), & \textit{for } n \equiv 0 \bmod 4,
\end{cases}
$$

i.e.

$$
\begin{aligned}
a_2 &= \frac{1}{2}{a_1}^2, \\
a_3 &= \frac{1}{4}{a_1}^3 - \frac{16}{27}a_1, \\
a_4 &= \frac{1}{8}{a_1}^4 - \frac{16}{27}{a_1}^2, \\
a_5 &= \frac{1}{16}{a_1}^5 - \frac{4}{9}{a_1}^3 + \frac{98}{1215}a_1, \\
a_6 &= \frac{1}{32}{a_1}^6 - \frac{8}{27}{a_1}^4 + \frac{934}{3645}{a_1}^2, \\
a_7 &= \frac{1}{64}{a_1}^7 - \frac{5}{27}{a_1}^5 + \frac{787}{2430}{a_1}^3 - \frac{1504}{6561}a_1, \\
a_8 &= \frac{1}{128}{a_1}^8 - \frac{1}{9}{a_1}^6 + \frac{41}{135}{a_1}^4 - \frac{9088}{32805}{a_1}^2, \\
&\cdots.
\end{aligned}
$$

(1) The function $\widetilde{\kappa}_{(2)}$:

$$
\widetilde{\kappa}_{(2)}(z) = -\frac{1}{z}
$$

(2) The function $\kappa^\flat$:

$$\kappa^\flat(z) = \rho_2(1 + a_1 q_3 + a_2 {q_3}^2 + a_3 {q_3}^3 + \cdots),$$

where

$$a_1 = -i\frac{3^2}{2^{\frac{4}{3}}}\frac{\Gamma\left(\frac{5}{6}\right)^2\Gamma\left(\frac{2}{3}\right)^2}{\pi^2} = -0.8454860396348\cdots\times i,$$

and the coefficients a_n *satisfies the reccurent relation*

$$a_2 = \frac{1}{3^{\frac{1}{2}}}e^{\frac{\pi i}{6}}{a_1}^2 \quad \textit{and}$$

$$\begin{aligned} &2n(n-1)(n+1)a_1a_{n+1} \\ &= -{a_1}^2 b_n + \sum_{m=1}^{n-1}\Bigg((m+1)(n-m+1)(3n+3mn-5m^2-4m)a_{m+1}a_{n-m+1} \\ &\quad -b_{n-m}\sum_{j=0}^{m}(j+1)(m-j+1)a_{j+1}a_{m-j+1}\Bigg), \quad (n \geq 2), \end{aligned}$$

and the series b_n *are*

$$b_n = \begin{cases} 1, & \textit{for } n = 0, \\ 0, & \textit{for } n \equiv 1,2,4,5 \bmod 6, \\ 36(-1)^n\sigma_3^-\left(\frac{n}{3}\right), & \textit{for } n \equiv 3 \bmod 6, \\ 36(-1)^n\sigma_3^-\left(\frac{n}{3}\right) - 20\sigma_3^-\left(\frac{n}{6}\right), & \textit{for } n \equiv 0 \bmod 6, \end{cases}$$

i.e.

$$\begin{aligned} a_2 &= \frac{1}{\sqrt{3}}e^{\frac{\pi i}{6}}{a_1}^2, \\ a_3 &= \frac{1}{3}e^{\frac{\pi i}{3}}{a_1}^3, \\ a_4 &= \frac{1}{3\sqrt{3}}e^{\frac{\pi i}{2}}{a_1}^4 - \frac{3}{4}a_1, \\ a_5 &= \frac{1}{9}e^{\frac{2\pi i}{3}}{a_1}^5 - \frac{\sqrt{3}}{2}e^{\frac{\pi i}{6}}{a_1}^2, \\ a_6 &= \frac{1}{9\sqrt{3}}e^{5\pi i}6{a_1}^6 - \frac{3}{4}e^{\frac{\pi i}{3}}{a_1}^3, \\ a_7 &= \frac{1}{27}e^{\pi i}{a_1}^7 - \frac{1}{\sqrt{3}}e^{\frac{\pi i}{2}}{a_1}^4 + \frac{79}{420}a_1, \\ a_8 &= \frac{1}{27\sqrt{3}}e^{\frac{7\pi i}{6}}{a_1}^8 - \frac{5}{12}e^{\frac{2\pi i}{3}}{a_1}^5 + \frac{1577}{1680\sqrt{3}}e^{\frac{\pi i}{6}}{a_1}^2, \\ &\cdots. \end{aligned}$$

(3) The function $\kappa_{1,2}$:

$$\kappa_{1,2}(z) = i(1 + a_1 q_2 + a_2 {q_2}^2 + a_3 {q_2}^3 + \cdots),$$

where

$$a_1 = -\frac{32\pi^2}{\Gamma\left(\frac{1}{4}\right)^4} = -1.8277863241779\cdots,$$

and the coefficients a_n satisfies the reccurent relation

$$\begin{aligned}
&a_2 = \frac{{a_1}^2}{2} \quad \textit{and} \\
&2n(n-1)(n+1)a_1 a_{n+1} \\
&= -{a_1}^2 b_n + \sum_{m=1}^{n-1}\Big((m+1)(n-m+1)(3n+3mn-5m^2-4m)a_{m+1}a_{n-m+1} \\
&\quad -b_{n-m}\sum_{j=0}^{m}(j+1)(m-j+1)a_{j+1}a_{m-j+1}\Big), \quad (n \geq 2),
\end{aligned}$$

and the series b_n are

$$b_n = \begin{cases} 1, & \textit{for } n = 0, \\ 0, & \textit{for } n \equiv 1,2,3 \bmod 4, \\ 16\sigma_3^-\left(\dfrac{n}{4}\right), & \textit{for } n \equiv 0 \bmod 4, \end{cases}$$

i.e.

$$\begin{aligned}
a_2 &= \frac{{a_1}^2}{2}, \\
a_3 &= \frac{{a_1}^3}{4}, \\
a_4 &= \frac{{a_1}^4}{8}, \\
a_5 &= \frac{{a_1}^5}{16} + \frac{2}{15}a_1, \\
a_6 &= \frac{{a_1}^6}{32} + \frac{2}{15}{a_1}^2, \\
a_7 &= \frac{{a_1}^7}{64} + \frac{{a_1}^3}{10}, \\
a_8 &= \frac{{a_1}^8}{128} + \frac{{a_1}^4}{15}, \\
&\cdots.
\end{aligned}$$

(4) The function $\widetilde{\kappa}_{1,2}$*:*

$$\widetilde{\kappa}_{1,2}(z) = i(1 + a_1 q_2 + a_2 {q_2}^2 + a_3 {q_2}^3 + \cdots),$$

where

$$a_1 = -\frac{64\pi^2}{\Gamma\left(\frac{1}{4}\right)^4} = -3.6555726483557\cdots,$$

and the coefficients a_n *satisfies the reccurent relation*

$$a_2 = \frac{{a_1}^2}{2} \quad \textit{and}$$

$$\begin{aligned}
&2n(n-1)(n+1)a_1 a_{n+1} \\
&= \sum_{m=1}^{n-1} \Bigg((m+1)(n-m+1)(3n+3mn-5m^2-4m) a_{m+1} a_{n-m+1} \\
&\quad + (4s_{n-m} - 4t_{n-m} + u_{n-m}) \left(\sum_{l=0}^{m} (l+1)^2 (m-l+1)(3m-5l+1) a_{l+1} a_{m-l+1} \right) \\
&\quad - \left(\sum_{l=0}^{n-m} (s_l - 8t_l + 8u_l) s_{n-m-l} \right) \left(\sum_{l=0}^{m} (l+1)(m-l+1) a_{l+1} a_{m-l+1} \right) \Bigg), \quad (n \geq 2),
\end{aligned}$$

and the series s_n*,* t_n *and* u_n *are*

$$s_n = \begin{cases} 1, & \textit{for } n = 0, \\ 0, & \textit{for } n : \textit{odd}, \\ 16(-1)^{\frac{n}{2}} \sigma_3^- \left(\frac{n}{2}\right), & \textit{for } n : \textit{even}, \end{cases}$$

$$t_n = \begin{cases} 1, & \textit{for } n = 0, \\ 0, & \textit{for } n \equiv 1, 2, 3 \bmod 4, \\ 16\sigma_3^- \left(\frac{n}{4}\right), & \textit{for } n \equiv 0 \bmod 4, \end{cases}$$

$$u_n = \begin{cases} 1, & \textit{for } n = 0, \\ 0, & \textit{for } n : \textit{odd}, \\ 16\sigma_3^- \left(\frac{n}{2}\right), & \textit{for } n : \textit{even}, \end{cases}$$

i.e.

$$\begin{aligned}
a_2 &= \frac{{a_1}^2}{2},\\
a_3 &= \frac{{a_1}^3}{4} - 12a_1,\\
a_4 &= \frac{{a_1}^4}{8} - 12{a_1}^2,\\
a_5 &= \frac{{a_1}^5}{16} - 9{a_1}^3 + \frac{1582}{15}a_1,\\
a_6 &= \frac{{a_1}^6}{32} - 6{a_1}^4 + \frac{2662}{15}{a_1}^2,\\
a_7 &= \frac{{a_1}^7}{64} - \frac{15}{4}{a_1}^5 + \frac{1871}{10}{a_1}^3 - \frac{153128}{105}a_1,\\
a_8 &= \frac{{a_1}^8}{128} - \frac{9}{4}{a_1}^6 + \frac{2411}{15}{a_1}^4 - \frac{286016}{105}{a_1}^2,\\
&\cdots.
\end{aligned}$$

(5) The function $\check{\kappa}_{1,2}$*:*

$$\check{\kappa}_{1,2}(z) = \rho_2(1 + a_1 q_3 + a_2 {q_3}^2 + a_3 {q_3}^3 + \cdots),$$

where

$$a_1 = -i\frac{3^2}{2^{\frac{4}{3}}}\frac{\Gamma\left(\frac{5}{6}\right)^2\Gamma\left(\frac{2}{3}\right)^2}{\pi^2} = -0.8454860396348\cdots \times i,$$

and the coefficients a_n *satisfies the reccurent relation*

$$a_2 = \frac{1}{3^{\frac{1}{2}}}e^{\frac{\pi i}{6}}{a_1}^2 \quad \textit{and}$$

$$\begin{aligned}
&2n(n-1)(n+1)a_1 a_{n+1}\\
&= -{a_1}^2 b_n + \sum_{m=1}^{n-1}\Big((m+1)(n-m+1)(3n+3mn-5m^2-4m)a_{m+1}a_{n-m+1}\\
&\quad -b_{n-m}\sum_{j=0}^{m}(j+1)(m-j+1)a_{j+1}a_{m-j+1}\Big), \quad (n \geq 2),
\end{aligned}$$

and the series b_n *are*

$$b_n = \begin{cases} 1, & \textit{for } n = 0,\\ 0, & \textit{for } n \equiv 1, 2 \bmod 3,\\ 16(-1)^n \sigma_3^-\left(\dfrac{n}{3}\right), & \textit{for } n \equiv 0 \bmod 3, \end{cases}$$

i.e.

$$
\begin{aligned}
a_2 &= \frac{1}{\sqrt{3}} e^{\frac{\pi i}{6}} {a_1}^2, \\
a_3 &= \frac{1}{3} e^{\frac{\pi i}{3}} {a_1}^3, \\
a_4 &= \frac{1}{3\sqrt{3}} e^{\frac{\pi i}{2}} {a_1}^4 - \frac{1}{3} a_1, \\
a_5 &= \frac{1}{9} e^{\frac{2\pi i}{3}} {a_1}^5 - \frac{2}{3\sqrt{3}} e^{\frac{\pi i}{6}} {a_1}^2, \\
a_6 &= \frac{1}{9\sqrt{3}} e^{5\pi i} 6 {a_1}^6 - \frac{1}{3} e^{\frac{\pi i}{3}} {a_1}^3, \\
a_7 &= \frac{1}{27} e^{\pi i} {a_1}^7 - \frac{4}{9\sqrt{3}} e^{\frac{\pi i}{2}} {a_1}^4 - \frac{32}{315} a_1, \\
a_8 &= \frac{1}{27\sqrt{3}} e^{\frac{7\pi i}{6}} {a_1}^8 - \frac{5}{27} e^{\frac{2\pi i}{3}} {a_1}^5 - \frac{29}{315\sqrt{3}} e^{\frac{\pi i}{6}} {a_1}^2, \\
&\cdots .
\end{aligned}
$$

(6) The function $\kappa^*_{(2)}$*:*

$$\kappa^*_{(2)}(z) = i(1 + a_1 q_2 + a_2 {q_2}^2 + a_3 {q_2}^3 + \cdots),$$

where

$$a_1 = -\sqrt{2}\, \frac{\Gamma\left(\frac{7}{8}\right)^2 \Gamma\left(\frac{5}{8}\right)^2}{\pi^2} i = -0.350109339053 \cdots \times i,$$

and the coefficients a_n *satisfies the reccurent relation*

$$
\begin{aligned}
& a_2 = \frac{{a_1}^2}{2} \quad \textit{and} \\
& 2n(n-1)(n+1) a_1 a_{n+1} \\
& = -{a_1}^2 b_n + \sum_{m=1}^{n-1} \Big((m+1)(n-m+1)(3n + 3mn - 5m^2 - 4m) a_{m+1} a_{n-m+1} \\
& \quad -b_{n-m} \sum_{j=0}^{m} (j+1)(m-j+1) a_{j+1} a_{m-j+1} \Big), \quad (n \geq 2),
\end{aligned}
$$

and the series b_n *are*

$$
b_n = \begin{cases}
1, & \textit{for } n = 0, \\
0, & \textit{for } n \equiv 1, 3 \pmod 4, \\
12\sigma_3^-\left(\frac{n}{2}\right), & \textit{for } n \equiv 2 \pmod 4, \\
12\sigma_3^-\left(\frac{n}{2}\right) + 4\sigma_3^-\left(\frac{n}{4}\right), & \textit{for } n \equiv 0 \pmod 4,
\end{cases}
$$

i.e.

$$
\begin{aligned}
a_2 &= \frac{{a_1}^2}{2}, \\
a_3 &= \frac{{a_1}^3}{4} - \frac{4}{3}a_1, \\
a_4 &= \frac{{a_1}^4}{8} - \frac{4}{3}{a_1}^2, \\
a_5 &= \frac{{a_1}^5}{16} - {a_1}^3 + \frac{11}{10}a_1, \\
a_6 &= \frac{{a_1}^6}{32} - \frac{2}{3}{a_1}^4 + \frac{179}{90}{a_1}^2, \\
a_7 &= \frac{{a_1}^7}{64} - \frac{5}{12}{a_1}^5 + \frac{259}{120}{a_1}^3 - \frac{82}{105}a_1, \\
a_8 &= \frac{{a_1}^8}{128} - \frac{{a_1}^6}{4} + \frac{113}{60}{a_1}^4 - \frac{236}{105}{a_1}^2, \\
&\cdots.
\end{aligned}
$$

Before proving Theorem 1, we show several lemmmas. We use $'$ as

$$
' = \frac{n}{\pi i}\frac{d}{dz} = q_n \frac{d}{dq_n}.
$$

<u>Lemma 1.</u>

Let

$$
f(z) = \rho \sum_{n=0}^{\infty} a_n {q_2}^n, \quad (a_0 = 1,\ \rho \in \mathbb{C}).
$$

If $f(z)$ satisfies the functional equation

$$
f(z+2) = \frac{\rho^2}{f(z)},
$$

then the recurrent relation among the coefficients a_1, a_2, $\cdots$, a_n $(n$: even, $n \geq 2)$ is

$$
a_n = \sum_{m=1}^{\frac{n}{2}-1} (-1)^{m-1} a_m a_{n-m} + \frac{(-1)^{\frac{n}{2}-1}}{2} {a_{\frac{n}{2}}}^2.
$$

In particular

$$
a_2 = \frac{{a_1}^2}{2}.
$$

Proof. Since $q_2 \mapsto -q_2$ as $z \mapsto z+2$, we have

$$
f(z+2) = \rho \sum_{n=0}^{\infty} (-1)^n a_n {q_2}^n.
$$

Thus

$$\begin{aligned}
\rho^2 &= f(z)f(z+2) \\
&= \left(\rho \sum_{n=0}^{\infty} a_n {q_2}^n\right) \left(\rho \sum_{n=0}^{\infty} (-1)^n a_n {q_2}^n\right) \\
&= \rho^2 \sum_{n=0}^{\infty} \left(\sum_{m=0}^{n} (-1)^m a_m a_{n-m} \right) {q_2}^n.
\end{aligned}$$

Then we have

$$\sum_{m=0}^{n} (-1)^m a_m a_{n-m} = 0$$

for $n \geq 1$.

In the case n is odd, the left-hand side of this equation is always equal to 0. In the case n is even ($n \geq 2$), we have

$$\begin{aligned}
&\sum_{m=0}^{n} (-1)^m a_m a_{n-m} \\
&= 2a_0 a_n + \sum_{m=1}^{\frac{n}{2}-1} (-1)^m a_m a_{n-m} + (-1)^{\frac{n}{2}} {a_{\frac{n}{2}}}^2 + \sum_{m=\frac{n}{2}-1}^{n-1} (-1)^m a_m a_{n-m} \\
&= 2a_n + 2\sum_{m=1}^{\frac{n}{2}-1} (-1)^m a_m a_{n-m} + (-1)^{\frac{n}{2}} {a_{\frac{n}{2}}}^2.
\end{aligned}$$

So we obtain Lemma 1. □

Lemma 2.

Let

$$f(z) = \rho_2 \sum_{n=0}^{\infty} a_n {q_2}^n, \quad (a_0 = 1,\ \rho_2 = e^{\frac{2\pi i}{3}}).$$

If $f(z)$ satisfies the functional equation

$$f(z+2) = -1 - \frac{1}{f(z)},$$

then the recurrent relation among the coefficients $a_1, a_2, \cdots, a_n$ ($n \equiv 0, 2 \pmod 3$, $n \geq 2$) is

$$a_n = -\frac{\rho_2}{1 + \rho_2 + {\rho_2}^{n+1}} \sum_{m=1}^{n-1} {\rho_2}^m a_m a_{n-m},$$

i.e.

$$a_n = \begin{cases} \dfrac{e^{\frac{7\pi i}{6}}}{\sqrt{3}} \left(\displaystyle\sum_{m=1}^{\frac{n}{2}-1} \delta_m a_m a_{n-m} + {a_{\frac{n}{2}}}^2 \right), & \textit{for } n \equiv 0 \pmod 6, \\ -\dfrac{i}{\sqrt{3}} \left(\displaystyle\sum_{m=1}^{\frac{n}{2}-1} \varepsilon_m a_m a_{n-m} + {a_{\frac{n}{2}}}^2 \right), & \textit{for } n \equiv 2 \pmod 6, \\ \dfrac{e^{\frac{7\pi i}{6}}}{\sqrt{3}} \displaystyle\sum_{m=1}^{\frac{n}{2}-1} \delta_m a_m a_{n-m}, & \textit{for } n \equiv 3 \pmod 6, \\ -\dfrac{i}{\sqrt{3}} \displaystyle\sum_{m=1}^{\frac{n}{2}-1} \varepsilon_m a_m a_{n-m}, & \textit{for } n \equiv 5 \pmod 6, \end{cases}$$

where

$$\delta_m = {\rho_2}^k + {\rho_2}^{-k} = \begin{cases} 2, & \textit{for } m \equiv 0 \ (\text{mod}3), \\ -1 & \textit{for } m \equiv 1, 2 \ (\text{mod}3), \end{cases}$$

$$\varepsilon_m = {\rho_2}^k + {\rho_2}^{2-k} = \begin{cases} 2\rho_2, & \textit{for } m \equiv 0, 2 \ (\text{mod}3), \\ -\rho_2 & \textit{for } m \equiv 1 \ (\text{mod}3). \end{cases}$$

In particular

$$a_2 = \frac{e^{\frac{\pi i}{6}}}{\sqrt{3}} \frac{{a_1}^2}{2}.$$

Proof. Since $q_3 \mapsto \rho_2 q_3$ as $z \mapsto z + 2$, we have

$$f(z+2) = \rho \sum_{n=0}^{\infty} {\rho_2}^n a_n {q_3}^n.$$

By transforming the equation $f(z+2) = -1 - 1/f(z)$ to $f(z)\,(1 + f(z+2)) = -1$, we have

$$\begin{aligned} -1 &= f(z)\,(1 + f(z+2)) \\ &= \left(\rho_2 \sum_{n=0}^{\infty} a_n {q_3}^n \right) \left(1 + \rho_2 \sum_{n=0}^{\infty} {\rho_2}^n a_n {q_3}^n \right) \\ &= \rho_2 \sum_{n=0}^{\infty} a_n {q_3}^n + {\rho_2}^2 \sum_{n=0}^{\infty} \left(\sum_{m=0}^{n} (-1)^m a_m a_{n-m} \right) {q_2}^n \\ &= \rho_2 + {\rho_2}^2 + \sum_{n=0}^{\infty} \left(\rho_2 a_n + {\rho_2}^2 \sum_{m=0}^{n} {\rho_2}^m a_m a_{n-m} \right) {q_3}^n. \end{aligned}$$

Then we have

$$\rho_2 a_n + {\rho_2}^2 \sum_{m=0}^{n} {\rho_2}^m a_m a_{n-m} = 0,$$

i.e.

$$\left(1 + \rho_2 + {\rho_2}^{n+1}\right) a_n = -\rho_2 \sum_{m=1}^{n-1} {\rho_2}^m a_m a_{n-m}$$

for $n \geq 1$. By transformaing this equation, we obtain the remaining part of Lemma 2. $\square$

Lemma 3.
Let $q = e^{\pi i z/k}$ $(k \in \mathbb{Z}^+)$, $\rho \in \mathbb{C}$ *and*

$$f(z) = \rho \sum_{n=0}^{\infty} a_n q^n \qquad (a_0 = 1).$$

Suppose

$$\{f; z\} = \frac{1}{4} \sum_{n=0}^{\infty} b_n q^n \quad (b_0 = 1),$$

where $\{f; z\}$ *is Schwarzian derivative. Then the recurrent relation among the coefficients* $a_1, a_2, \cdots, a_{n+1}$ *is*

$$\begin{aligned} & 2n(n-1)(n+1)a_1 a_{n+1} \\ = \; & -{a_1}^2 b_n + \sum_{m=1}^{n-1} \Bigg((m+1)(n-m+1)(3n+3mn-5m^2-4m) a_{m+1} a_{n-m+1} \\ & \qquad - b_{n-m} \sum_{l=0}^{m} (l+1)(m-l+1) a_{l+1} a_{m-l+1} \Bigg). \end{aligned}$$

Proof. By the definition of Schwarzian derivative we have

$$\begin{aligned} -4\{f; z\} &= \frac{2 \dfrac{df(z)}{dz} \dfrac{d^3 f(z)}{dz^3} - 3 \left(\dfrac{d^2 f(z)}{dz^2} \right)^2}{\left(\dfrac{df(z)}{dz} \right)^2} \\ &= \frac{2 f(z)' f(z)''' - 3 \left(f(z)'' \right)^2}{\left(f(z)' \right)^2} \\ &= -\sum_{n=0}^{\infty} b_n q^n. \end{aligned}$$

Hence we have

$$2 f(z)' f(z)''' - 3 \left(f(z)'' \right)^2 = \left(-\sum_{n=0}^{\infty} b_n q^n \right) \left(f(z)' \right)^2. \tag{3.1}$$

Since

$$\begin{aligned} f(z)' &= \rho q \sum_{n=0}^{\infty} (n+1) a_{n+1} q^n, \\ f(z)'' &= \rho q \sum_{n=0}^{\infty} (n+1)^2 a_{n+1} q^n, \\ f(z)''' &= \rho q \sum_{n=0}^{\infty} (n+1)^3 a_{n+1} q^n, \end{aligned}$$

we have the Fourier expansions of both sides of (3.1) as followings:

$$\begin{aligned}
&(\text{Left-hand side of } (3.1)) = 2f(z)'f(z)''' - 3\left(f(z)''\right)^2 \\
&= 2\left(\rho q \sum_{n=0}^{\infty}(n+1)a_{n+1}q^n\right)\left(\rho q \sum_{n=0}^{\infty}(n+1)^3 a_{n+1}q^n\right) \\
&\qquad -3\left(\rho q \sum_{n=0}^{\infty}(n+1)^2 a_{n+1}q^n\right)^2 \\
&= -\rho^2 q^2 \sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}(m+1)^2(n-m+1)(3n-5m+1)a_{m+1}a_{n-m+1}\right)q^n
\end{aligned}$$

and

$$\begin{aligned}
&(\text{Right-hand side of } (3.1)) = \left(-\sum_{n=0}^{\infty} b_n q^n\right)\left(f(z)'\right)^2 \\
&= -\left(\sum_{n=0}^{\infty} b_n q^n\right)\left(\rho q \sum_{n=0}^{\infty}(n+1)a_{n+1}q^n\right) \\
&= -\rho^2 q^2 \sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}\left(b_{n-m}\sum_{l=0}^{m}(l+1)(m-l+1)a_{l+1}a_{m-l+1}\right)\right)q^n.
\end{aligned}$$

By comparing the coefficients of q^n of the above two Fourier expansions we have the equation

$$\begin{aligned}
&\sum_{m=0}^{n}(m+1)^2(n-m+1)a_{m+1}a_{n-m+1} \\
&= \sum_{m=0}^{n}\left(b_{n-m}\sum_{l=0}^{m}(l+1)(m-l+1)a_{l+1}a_{m-l+1}\right).
\end{aligned}$$

This equation consists of a_1, a_2, $\cdots$, a_n and a_{n+1}. By rewriting the equation we obtain Lemmma 3. □

Lemma 4.
Let $q = e^{\pi i z/k}$ $(k \in \mathbb{Z}^+)$, $\rho \in \mathbb{C}$ *and*

$$f(z) = \rho \sum_{n=0}^{\infty} a_n q^n \qquad (a_0 = 1).$$

Suppose

$$\{f; z\} = \frac{1}{4}\frac{\sum_{n=0}^{\infty}\alpha_n q^n}{\sum_{n=0}^{\infty}\beta_n q^n} \quad (\alpha_0 = \beta_0 = 1),$$

where $\{f; z\}$ is Schwarzian derivative. Then the recurrent relation among the coefficients $a_1,\ a_2,\ \cdots,\ a_{n+1}$ is

$$
\begin{aligned}
& 2n(n-1)(n+1)a_1 a_{n+1} \\
= & \sum_{m=1}^{n-1}\Bigg((m+1)(n-m+1)(3n+3mn-5m^2-4m)a_{m+1}a_{n-m+1} \\
& + \sum_{l=0}^{m}\Big(-\alpha_{n-m}(l+1)(m-l+1)a_{l+1}a_{m-l+1} \\
& \qquad +\beta_{n-m}(l+1)^2(m-l+1)(3m-5l+1)\Big) a_{l+1}a_{m-l+1}\Bigg).
\end{aligned}
$$

Proof. By the definition of Schwarzian derivative we have

$$
-4\{f; z\} = \frac{2f(z)'f(z)''' - 3\left(f(z)''\right)^2}{\left(f(z)'\right)^2} = \frac{\sum_{n=0}^{\infty} \alpha_n q^n}{\sum_{n=0}^{\infty} \beta_n q^n}.
$$

Hence we have

$$
\left(\sum_{n=0}^{\infty} \beta_n q^n\right)\left(2f(z)'f(z)''' - 3\left(f(z)''\right)^2\right) = \left(-\sum_{n=0}^{\infty} \alpha_n q^n\right)\left(f(z)'\right)^2. \tag{3.1}
$$

Since

$$
\begin{aligned}
f(z)' &= \rho q \sum_{n=0}^{\infty}(n+1)a_{n+1}q^n, \\
f(z)'' &= \rho q \sum_{n=0}^{\infty}(n+1)^2 a_{n+1}q^n, \\
f(z)''' &= \rho q \sum_{n=0}^{\infty}(n+1)^3 a_{n+1}q^n,
\end{aligned}
$$

we have the Fourier expansions of both sides of (3.1) as followings:

$$\begin{aligned}
\text{(Left-hand side of (3.1))} &= \left(\sum_{n=0}^{\infty}\beta_n q^n\right)\left(2f(z)'f(z)''' - 3\left(f(z)''\right)^2\right) \\
&= -\rho^2 q^2\left(\sum_{n=0}^{\infty}\beta_n q^n\right) \\
&\quad\times\left(\sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}(m+1)^2(n-m+1)(3n-5m+1)a_{m+1}a_{n-m+1}\right)q^n\right) \\
&= -\rho^2 q^2\sum_{n=0}^{\infty}\sum_{m=0}^{n}\beta_{n-m}\left(\sum_{l=0}^{m}(l+1)^2(m-l+1)(3m-5l+1)a_{l+1}a_{m-l+1}\right)q^n
\end{aligned}$$

and

$$\begin{aligned}
\text{(Right-hand side of (3.1))} &= \left(-\sum_{n=0}^{\infty}\alpha_n q^n\right)\left(f(z)'\right)^2 \\
&= -\rho^2 q^2\sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}\left(\alpha_{n-m}\sum_{l=0}^{m}(l+1)(m-l+1)a_{l+1}a_{m-l+1}\right)\right)q^n.
\end{aligned}$$

By comparing the coefficients of q^n of the above two Fourier expansions we have the equation

$$\begin{aligned}
&\sum_{m=0}^{n}\beta_{n-m}\left(\sum_{l=0}^{m}(l+1)^2(m-l+1)(3m-5l+1)a_{l+1}a_{m-l+1}\right) \\
&= \sum_{m=0}^{n}\left(\alpha_{n-m}\sum_{l=0}^{m}(l+1)(m-l+1)a_{l+1}a_{m-l+1}\right).
\end{aligned}$$

This equation consists of a_1, a_2, $\cdots$, a_n and a_{n+1}. By rewriting the equation we obtain Lemmma 3. □

Now we recall the Eisenstein series

$$\begin{aligned}
E_2(z) &= 1 - 24\sum_{n=1}^{\infty}\sigma_1(n)e^{2\pi inz}, \\
E_4(z) &= 1 + 240\sum_{n=1}^{\infty}\sigma_3(n)e^{2\pi inz}, \\
E_6(z) &= 1 - 504\sum_{n=1}^{\infty}\sigma_5(n)e^{2\pi inz}.
\end{aligned}$$

Lemma 5.

We obtain following limit values:

$$\lim_{z\to i\infty}\frac{1}{q_2}\frac{d\lambda^\sharp(z)}{dq_2} = -32,$$

$$\lim_{z\to i\infty}\frac{1}{q_2}\frac{d\lambda^\flat(z)}{dq_2} = 32,$$

$$\lim_{z\to i\infty}\frac{1}{q_2}\frac{d}{dq_2}\left(q_2\frac{d\lambda^\flat(z)}{dq_2}\right) = 64,$$

$$\lim_{z\to i\infty}\frac{1}{q_2}\frac{d\lambda^\sharp_{1,2}(z)}{dq_2} = -256,$$

$$\lim_{z\to i\infty}\frac{1}{q_2}\frac{d\lambda^\sharp_{1,2}(\tau)}{d\tau} = -\frac{2}{3}\pi^2 ia_1E_4(i), \qquad \textit{for } \tau=\kappa_{1,2}(z),$$

$$\lim_{z\to i\infty}\frac{1}{q_2}\frac{d\lambda^\flat_{1,2}(\tau)}{d\tau} = \frac{2}{3}\pi^2 ia_1E_4(i), \qquad \textit{for } \tau=\widetilde{\kappa}_{1,2}(z),$$

$$\lim_{z\to i\infty}\frac{1}{q_2}\frac{dJ(\tau)}{d\tau} = -2\pi^2 ia_1E_4(i), \qquad \textit{for } \tau=i(1+a_1q_2+a_2{q_2}^2+a_3{q_2}^3+\cdots),$$

$$\lim_{z\to i\infty}\frac{1}{q_2}\frac{d\lambda^*_{(2)}(\tau)}{d\tau} = \frac{2^{\frac{5}{2}}}{(\sqrt{2}-1)^2}\pi^2 ia_1\theta_2(\sqrt{2}i)^8, \quad \textit{for } \tau=\kappa^*_{(2)}(z),$$

$$\lim_{z\to i\infty}\frac{d^2\lambda^*_{(2)}(\tau)}{d\tau^2} = \frac{2^2}{(\sqrt{2}-1)^2}\pi^2\theta_2(\sqrt{2}i)^8, \qquad \textit{for } \tau=\kappa^*_{(2)}(z).$$

Proof. Since

$$\frac{d\lambda^\sharp(z)}{dq_2} = \frac{d}{dq_2}\left(1-16{q_2}^2+128{q_2}^4-704{q_2}^6+\cdots\right) = -32q_2+512{q_2}^3-4224{q_2}^5+\cdots,$$

$$\frac{d}{dq_2}\left(q_2\frac{d\lambda^\flat(z)}{dq_2}\right) = -64q_2+2048{q_2}^3-25344{q_2}^5+\cdots$$

and $\lambda^\sharp(z)=1-\lambda^\flat(z)$, we obtain

$$\lim_{z\to i\infty}\frac{1}{q_2}\frac{d\lambda^\sharp(z)}{dq_2} = -32 \quad\text{and}\quad \lim_{z\to i\infty}\frac{1}{q_2}\frac{d\lambda^\flat(z)}{dq_2} = 32.$$

Let $\tau=\kappa_{1,2}(z)$. Since $J(\tau)=P^\sharp_{1,2}(\lambda^\sharp_{1,2}(\tau))$, we have

$$-2\pi i\frac{E_6(\tau)}{E_4(\tau)}J(\tau) = \frac{dJ(\tau)}{d\tau} = \frac{1}{27}\frac{(\lambda^\sharp_{1,2}(\tau)-9)(\lambda^\sharp_{1,2}(\tau)+3)^2}{(\lambda^\sharp_{1,2}(\tau)-1)^2}.$$

And we have $\lambda^\sharp_{1,2}(\kappa_{1,2}(z))=\lambda^\flat(z)$ and $\lambda^\flat(i\infty)=0$. So we obtain

$$\begin{aligned}\lim_{z\to i\infty}\frac{1}{q_2}\frac{d\lambda^\sharp_{1,2}(\tau)}{d\tau} &= \lim_{z\to i\infty}\frac{1}{q_2}\left(\frac{E_6(\kappa_{1,2}(z))}{E_4(\kappa_{1,2}(z))}\frac{(\lambda^\flat(z)+3)(\lambda^\flat(z)-1)}{\lambda^\flat(z)-9}\right)\\ &= -\frac{2}{3}\pi^2 ia_1E_4(i).\end{aligned}$$

Using same method we obtain

$$\lim_{z\to i\infty}\frac{1}{q_2}\frac{d\lambda^\flat_{1,2}(\tau)}{d\tau}=\frac{2}{3}\pi^2 ia_1E_4(i),$$

for $\tau=\widetilde{\kappa}_{1,2}(z)$.

Let $\tau=i(1+a_1q_2+a_2{q_2}^2+a_3{q_2}^3+\cdots)$. Since

$$\frac{dJ(\tau)}{d\tau}=-2\pi i\frac{E_6(\tau)}{E_4(\tau)}J(\tau)=-2\pi i\frac{E_4(\tau)^2E_6(\tau)}{E_4(\tau)^3-E_6(\tau)^2}$$

and

$$\lim_{z\to i\infty}\frac{E_6(\tau)}{q_2}=\pi a_1E_4(i)^2,$$

we obtain

$$\lim_{z\to i\infty}\frac{1}{q_2}\frac{dJ(\tau)}{d\tau}=-2\pi i\lim_{z\to i\infty}\frac{E_6(\tau)}{q_2}\frac{1}{E_4(\tau)}J(\tau)=-2\pi^2 ia_1E_4(i).$$

Let $\tau=\kappa^*_{(2)}(z)$, $N(z)=1-6\lambda^\flat_{(2)}(z)+\lambda^\flat_{(2)}(z)^2$ and $D(z)=16\lambda^\flat(z)\left(1-\lambda^\flat(z)\right)$. Then we can write

$$\lambda^*_{(2)}(z)=P^*\left(\lambda^\flat(z)\right)=-\frac{N(z)^2}{D(z)}.$$

Since $\lambda^\flat(z)'=2\theta_2(z)^8(1-\lambda^\flat(z))$ and $\lambda^\flat(\sqrt{2}i)=3-2\sqrt{2}$, we have

$$\lim_{z\to i\infty}\frac{d\lambda^\flat(\tau)}{\tau}=\lim_{z\to i\infty}\frac{\pi i}{2}\lambda^\flat(\tau)'=2(\sqrt{2}-1)\pi i\theta_2(\sqrt{2}i)^4.$$

Hence

$$\lim_{z\to i\infty}\frac{dN(\tau)}{d\tau}=\lim_{z\to i\infty}\left(-6+2\lambda^\flat(\tau)\right)\frac{d\lambda^\flat(\tau)}{d\tau}=-2^{\frac{7}{2}}(\sqrt{2}-1)\pi i\theta_2(\sqrt{2}i)^4,$$

$$\lim_{z\to i\infty}\frac{1}{q_2}N(\tau)=\lim_{z\to i\infty}\frac{\frac{dN(\tau)}{dz}}{\frac{\pi i}{2}q_2}=\lim_{z\to i\infty}\frac{\frac{\pi i}{2}\tau'\frac{dN(\tau)}{d\tau}}{\frac{\pi i}{2}q_2}=2^4(\sqrt{2}-1)\pi a_1\theta_2(\sqrt{2}i)^4.$$

So we obtain following two limit values with noting that $N(\sqrt{2}i)=0$:

$$\begin{aligned}\lim_{z\to i\infty}\frac{1}{q_2}\frac{d\lambda^*_{(2)}(\tau)}{d\tau} &= \lim_{z\to i\infty}\frac{1}{q_2}\frac{d}{d\tau}\left(N(\tau)^2\frac{1}{D(\tau)}\right)\\ &= \lim_{z\to i\infty}\left(-\frac{1}{q_2}2N(\tau)\frac{dN(\tau)}{d\tau}\frac{1}{D(\tau)}+\frac{1}{q_2}N(\tau)^2\frac{1}{D(\tau)^2}\frac{dD(\tau)}{d\tau}\right)\\ &= 2^{\frac{5}{2}}(\sqrt{2}-1)^{-2}\pi^2 ia_1\theta_2(\sqrt{2}i)^8\end{aligned}$$

and

$$\begin{aligned}\lim_{z\to i\infty}\frac{d^2\lambda^*_{(2)}(\tau)}{d\tau^2} &= \lim_{z\to i\infty}\left(N(\tau)\left(-\frac{d^2N(\tau)}{d\tau^2}\frac{1}{D(\tau)}+2\frac{dN(\tau)}{d\tau}\frac{1}{D(\tau)^2}\frac{dD(\tau)}{d\tau}\right.\right.\\ &\left.\left.+N(\tau)\frac{d}{d\tau}\left(\frac{1}{D(\tau)^2}\frac{dD(\tau)}{d\tau}\right)\right)-2\frac{1}{D(\tau)}\left(\frac{dN(\tau)}{d\tau}\right)^2\right)\\ &= 2^2(\sqrt{2}-1)^{-2}\pi^2\theta_2(\sqrt{2}i)^8.\end{aligned}$$

□

Lemma 6.
We obtain following limit values:

$$\lim_{z\to i\infty}\frac{1}{{q_3}^2}\frac{d\lambda^\flat(z)}{dq_3}=48,$$
$$\lim_{z\to i\infty}\frac{1}{{q_3}^2}\frac{d\lambda^\flat_{1,2}(z)}{dq_3}=192,$$
$$\lim_{z\to i\infty}\frac{1}{{q_3}^2}\frac{dJ(\tau)}{d\tau}=-\frac{8}{9}\pi^3 i{\rho_3}^2{a_1}^2E_6(\rho_2),\quad \textit{for } \tau=\rho_2(1+a_1q_3+a_2{q_3}^2+a_3{q_3}^3+\cdots).$$

Proof. Since

$$\lambda^\flat(z)=16{q_3}^3-128{q_3}^6+704{q_3}^9-\cdots,$$

we obtain

$$\lim_{z\to i\infty}\frac{1}{{q_3}^2}\frac{d\lambda^\flat(z)}{dq_3}=48.$$

Since

$$\lambda^\flat_{1,2}(z)=P^\flat(\lambda^\flat(z))=64{q_3}^3-1536{q_3}^6+19200{q_3}^{12}+\cdots,$$

we obtain

$$\lim_{z\to i\infty}\frac{1}{{q_3}^2}\frac{\lambda^\flat_{1,2}(z)}{dq_3}=192.$$

Let $\tau=\rho_2(1+a_1q_3+a_2{q_3}^2+a_3{q_3}^3+\cdots$.
Since

$$\frac{dJ(\tau)}{d\tau}=-2\pi i\frac{E_6(\tau)}{E_4(\tau)}J(\tau)=-2\pi i\frac{E_4(\tau)^2E_6(\tau)}{E_4(\tau)^3-E_6(\tau)^2}$$

and

$$\lim_{z\to i\infty}\frac{E_4(\tau)}{q_3}=-\frac{2}{3}\pi i\rho_3a_1E_6(\rho_2),$$

we obtain

$$\lim_{z\to i\infty}\frac{1}{{q_3}^2}\frac{dJ(\tau)}{d\tau}=-2\pi i\lim_{z\to i\infty}\left(\frac{E_4(\tau)}{q_3}\right)^2\frac{E_6(\tau)}{E_4(\tau)^3-E_6(\tau)^2}=-\frac{8}{9}\pi^3 i{\rho_3}^2{a_1}^2E_6(\rho_2).$$

□

Proof of Theorem 1.
(0) Since

$$\kappa^\sharp(z+2) = -\frac{1}{\kappa^\sharp(z)} \quad , \quad \kappa^\sharp(z+4) = \kappa^\sharp(z),$$

and $\kappa^\sharp(i\infty) = i$, we can identify $\kappa^\sharp(z)$ with the Fourier expansion

$$i(1 + a_1 q_2 + a_2 {q_2}^2 + a_3 {q_2}^3 + \cdots).$$

Let $x = \lambda^\sharp(z)$. Since the proparties of Schwarzian derivative, we have

$$\begin{aligned} -4\{(\lambda^\sharp)^{-1}; x\} &= \frac{1}{x^2} + \frac{1}{(1-x)^2} + \frac{1}{x(1-x)}, \\ -4\{J^{-1}; x\} &= \frac{1-\left(\frac{1}{3}\right)^2}{x^2} + \frac{1-\left(\frac{1}{2}\right)^2}{(1-x)^2} + \frac{1-\left(\frac{1}{3}\right)^2-\left(\frac{1}{2}\right)^2}{x(1-x)} \end{aligned}$$

and

$$\begin{aligned} -4\{\kappa^\sharp; z\} &= -4\{\lambda^\sharp; z\} - 4\{J^{-1}; x\}\left(\frac{dx}{dz}\right)^2 \\ &= \left(-\left(-4\{(\lambda^\sharp)^{-1}; x\}\right) + \left(-4\{J^{-1}; x\}\right)\right)\left(\frac{dx}{dz}\right)^2 \\ &= -\frac{1}{36}\frac{4+5x}{x^2(1-x)^2}\left(\frac{dx}{dz}\right)^2. \end{aligned}$$

Multiplying both sides by $\left(\frac{2}{\pi i}\right)^2$, we have

$$\begin{aligned} \frac{2\kappa^\sharp(z)'\kappa^\sharp(z)''' - 3(\kappa^\sharp(z)'')}{(\kappa^\sharp(z)')^2} &= -\frac{1}{36}\frac{4+5\lambda^\sharp(z)}{\lambda^\sharp(z)^2(1-\lambda^\sharp(z))^2}\left(\lambda^\sharp(z)'\right)^2 \\ &= -\frac{1}{9}\left(5\theta_0(z)^4\theta_3(z)^4 + 4\theta_3(z)^8\right). \end{aligned}$$

(Note that $\lambda^\sharp(z)' = 2\theta_2(z)(1-\lambda^\sharp(z))$ for q_2.)
Put

$$\frac{1}{9}\left(5\theta_0(z)^4\theta_3(z)^4 + 4\theta_3(z)^4\right) = \sum_{n=0}^{\infty} b_n {q_2}^n.$$

By the fomulas

$$\begin{aligned} \theta_0(z)^4\theta_3(z)^4 &= 1 + 16\sum_{n=1}^{\infty}\sigma_3^-(n){q_2}^{4n}, \\ \theta_3(z)^8 &= 1 + 16\sum_{n=1}^{\infty}\sigma_3^-(n){q_2}^{2n}, \end{aligned}$$

we have

$$b_n = \begin{cases} 1, & n=0 \\ 0, & \text{for } n\text{:odd}, \\ (-1)^{\frac{n}{2}}\dfrac{64}{9}\sigma_3^-\left(\dfrac{n}{2}\right), & \text{for } n \equiv 2 \bmod 4, \\ (-1)^{\frac{n}{2}}\dfrac{64}{9}\sigma_3^-\left(\dfrac{n}{2}\right) + \dfrac{80}{9}\sigma_3^-\left(\dfrac{n}{4}\right), & \text{for } n \equiv 0 \bmod 4, \end{cases}$$

By Lemma 1 and 3, we obtain the recurrent relation of $\{a_n\}$ in Theorem 1 (0).

Now we calculate a_1. Differentiating the identity $J(\kappa^\sharp(z)) = \lambda^\sharp(z)$ and multiplying both sides by $\frac{2}{\pi i}$, we have

$$\kappa^\sharp(z)'\frac{dJ(\tau)}{d\tau} = \lambda^\sharp(z)'$$

where $\tau = \kappa^\sharp(z)$. Multiplying both sides by $1/{q_2}^2$ and using Lemma 5, we have

$$ia_1 \times \left(-2\pi^2 i a_1 E_4(i)\right) = -32$$

as $z \to i\infty$, and since we have

$$E_4(i) = \frac{3\Gamma\left(\frac{1}{4}\right)^8}{2^6\pi^6},$$

we obtain (0) in Theorem 1. This result is obtained by KANEKO and YOSHIDA [KY].

(1) $\widetilde{\kappa}_{(2)}$ is bijection and the automorphism on $\mathbb{H} \curvearrowleft \Gamma(2)$. So it is linear transformation. Since $\widetilde{\kappa}_{(2)}$ maps 0, 1 and $i\infty$ to $i\infty$, 1 and 0, respectively, we obtain

$$\widetilde{\kappa}_{(2)}(z) = -\frac{1}{z}.$$

(2) Since

$$\kappa^\flat(z+2) = -1 - \frac{1}{\kappa^\flat(z)} \quad , \quad \kappa^\flat(z+6) = \kappa^\flat(z),$$

and $\kappa^\flat(i\infty) = \rho_2$, we can identify $\kappa^\flat(z)$ with the Fourier expansion

$$\rho_2(1 + a_1q_3 + a_2{q_3}^2 + a_3{q_3}^3 + \cdots).$$

Let $x = \lambda^\flat(z)$. Since the proparties of Schwarzian derivative, we have

$$\begin{aligned} -4\{(\lambda^\flat)^{-1};x\} &= \frac{1}{x^2} + \frac{1}{(1-x)^2} + \frac{1}{x(1-x)}, \\ -4\{J^{-1};x\} &= \frac{1-\left(\frac{1}{3}\right)^2}{x^2} + \frac{1-\left(\frac{1}{2}\right)^2}{(1-x)^2} + \frac{1-\left(\frac{1}{3}\right)^2-\left(\frac{1}{2}\right)^2}{x(1-x)} \end{aligned}$$

and

$$
\begin{aligned}
-4\{\kappa^\flat; z\} &= -4\{\lambda^\flat; z\} - 4\{J^{-1}; x\}\left(\frac{dx}{dz}\right)^2 \\
&= \left(-\left(-4\{(\lambda^\flat)^{-1}; x\}\right) + \left(-4\{J^{-1}; x\}\right)\right)\left(\frac{dx}{dz}\right)^2 \\
&= -\frac{1}{36}\frac{4+5x}{x^2(1-x)^2}\left(\frac{dx}{dz}\right)^2.
\end{aligned}
$$

Multiplying both sides by $\left(\frac{3}{\pi i}\right)^2$, we have

$$
\begin{aligned}
\frac{2\kappa^\flat(z)'\kappa^\flat(z)''' - 3(\kappa^\flat(z)'')}{(\kappa^\flat(z)')^2} &= -\frac{1}{36}\frac{4+5\lambda^\flat(z)}{\lambda^\flat(z)^2(1-\lambda^\flat(z))^2}\left(\lambda^\flat(z)'\right)^2 \\
&= -\frac{1}{4}\left(9\theta_3(z)^8 - 5\theta_0(z)^4\theta_3(z)^4\right).
\end{aligned}
$$

(Note that $\lambda^\flat(z)' = 3\theta_2(z)(1-\lambda^\flat(z))$ for q_3.)
Put

$$
\frac{1}{4}\left(9\theta_3(z)^8 - 5\theta_0(z)^4\theta_3(z)^4\right) = \sum_{n=0}^{\infty} b_n {q_3}^n.
$$

By the fomulas

$$
\begin{aligned}
\theta_0(z)^4\theta_3(z)^4 &= 1 + 16\sum_{n=1}^{\infty}\sigma_3^-(n){q_3}^{6n}, \\
\theta_3(z)^8 &= 1 + 16\sum_{n=1}^{\infty}\sigma_3^-(n){q_3}^{3n},
\end{aligned}
$$

we have

$$
b_n = \begin{cases}
1, & \text{for } n = 0, \\
0, & \text{for } n \equiv 1,2,4,5 \bmod 6, \\
36(-1)^n\sigma_3^-(\frac{n}{3}), & \text{for } n \equiv 3 \bmod 6, \\
36(-1)^n\sigma_3^-(\frac{n}{3}) - 20\sigma_3^-(\frac{n}{6}), & \text{for } n \equiv 0 \bmod 6,
\end{cases}
$$

By Lemma 2 and 3, we obtain the recurrent relation of $\{a_n\}$ in Theorem 1 (2).

Now we calculate a_1. Differentiating the identity $J(\kappa^\flat(z)) = \lambda^\flat(z)$ and multiplying both sides by $\frac{3}{\pi i}$, we have

$$
\kappa^\flat(z)'\frac{dJ(\tau)}{d\tau} = \lambda^\flat(z)'
$$

where $\tau = \kappa^\flat(z)$. Multiplying both sides by $1/{q_3}^3$ and using Lemma 6, we have

$$
\rho_2 a_1 \times \left(-\frac{8}{9}\pi^3 i {\rho_3}^2 {a_1}^2 E_6(\rho_2)\right) = 48
$$

as $z \to i\infty$, and since we have

$$E_6(\rho_2) = \frac{2^5}{3^3} \frac{\pi^3}{\Gamma\left(\frac{5}{6}\right)\Gamma\left(\frac{2}{3}\right)},$$

we obtain (2) in Theorem 1.

(3) Since

$$\kappa_{1,2}(z+2) = -\frac{1}{\kappa_{1,2}(z)} \quad , \quad \kappa_{1,2}(z+4) = \kappa_{1,2}(z),$$

and $\kappa_{1,2}(i\infty) = i$, we can identify $\kappa_{1,2}(z)$ with the Fourier expansion

$$i(1 + a_1 q_2 + a_2 {q_2}^2 + a_3 {q_2}^3 + \cdots).$$

Let $x = \lambda^\flat(z)$. Since the proparties of Schwarzian derivative, we have

$$\begin{aligned} -4\{(\lambda^\flat)^{-1}; x\} &= \frac{1}{x^2} + \frac{1}{(1-x)^2} + \frac{1}{x(1-x)}, \\ -4\{(\lambda^\sharp_{1,2})^{-1}; x\} &= \frac{1 - \left(\frac{1}{2}\right)^2}{x^2} + \frac{1}{(1-x)^2} + \frac{1 - \left(\frac{1}{2}\right)^2}{x(1-x)} \end{aligned}$$

and

$$\begin{aligned} -4\{\kappa_{1,2}; z\} &= -4\{\lambda^\flat; z\} - 4\{(\lambda^\sharp_{1,2})^{-1}; x\} \left(\frac{dx}{dz}\right)^2 \\ &= \left(-\left(-4\{(\lambda^\flat)^{-1}; x\}\right) + \left(-4\{(\lambda^\sharp_{1,2})^{-1}; x\}\right)\right) \left(\frac{dx}{dz}\right)^2 \\ &= -\frac{1}{4} \frac{1}{x^2(1-x)} \left(\frac{dx}{dz}\right)^2. \end{aligned}$$

Multiplying both sides by $\left(\frac{2}{\pi i}\right)^2$, we have

$$\begin{aligned} \frac{2\kappa_{1,2}(z)'\kappa_{1,2}(z)''' - 3(\kappa_{1,2}(z)'')}{(\kappa_{1,2}(z)')^2} &= -\frac{1}{4} \frac{1}{\lambda^\flat(z)^2(1 - \lambda^\flat(z))} \left(\lambda^\flat(z)'\right)^2 \\ &= -\theta_0(z)^4 \theta_3(z)^4. \end{aligned}$$

(Note that $\lambda^\flat(z)' = 2\theta_2(z)(1 - \lambda^\flat(z))$ for q_2.)
Put

$$\theta_0(z)^4 \theta_3(z)^4 = \sum_{n=0}^{\infty} b_n {q_2}^n.$$

By the fomula

$$\theta_0(z)^4 \theta_3(z)^4 = 1 + 16 \sum_{n=1}^{\infty} \sigma_3^-(n) {q_2}^{4n},$$

we have

$$b_n = \begin{cases} 1, & \text{for } n = 0, \\ 0, & \text{for } n \equiv 1, 2, 3 \bmod 4, \\ 16\sigma_3^-\left(\frac{n}{4}\right), & \text{for } n \equiv 0 \bmod 4, \end{cases}$$

By Lemma 1 and 3, we obtain the recurrent relation of $\{a_n\}$ in Theorem 1 (3).

Now we calculate a_1. Differentiating the identity $\lambda^\sharp_{1,2}(\kappa_{1,2}(z)) = \lambda^\flat(z)$ and multiplying both sides by $\frac{2}{\pi i}$, we have

$$\kappa_{1,2}(z)' \frac{d\lambda^\sharp_{1,2}(\tau)}{d\tau} = \lambda^\flat(z)'$$

where $\tau = \kappa_{1,2}(z)$. Multiplying both sides by $1/{q_2}^2$ and using Lemma 5, we have

$$ia_1 \times \left(-\frac{2}{3}\pi^2 i a_1 E_4(i)\right) = 32$$

as $z \to i\infty$, and since we have

$$E_4(i) = \frac{3\Gamma\left(\frac{1}{4}\right)^8}{2^6\pi^6},$$

we obtain (3) in Theorem 1.

(4) Since

$$\widetilde{\kappa}_{1,2}(z+2) = -\frac{1}{\widetilde{\kappa}_{1,2}(z)} \quad , \quad \widetilde{\kappa}_{1,2}(z+4) = \widetilde{\kappa}_{1,2}(z),$$

and $\widetilde{\kappa}_{1,2}(i\infty) = i$, we can identify $\widetilde{\kappa}_{1,2}(z)$ with the Fourier expansion

$$i(1 + a_1 q_2 + a_2 {q_2}^2 + a_3 {q_2}^3 + \cdots).$$

Let $x = \lambda^\sharp_{1,2}(z) = P^\sharp(\lambda^\flat(z))$. Since the proparties of Schwarzian derivative, we have

$$\begin{aligned} -4\{(\lambda^\sharp_{1,2})^{-1}; x\} &= \frac{1-\left(\frac{1}{2}\right)^2}{x^2} + \frac{1}{(1-x)^2} + \frac{1-\left(\frac{1}{2}\right)^2}{x(1-x)}, \\ -4\{(\lambda^\flat_{1,2})^{-1}; x\} &= \frac{1}{x^2} + \frac{1-\left(\frac{1}{2}\right)^2}{(1-x)^2} + \frac{1-\left(\frac{1}{2}\right)^2}{x(1-x)} \end{aligned}$$

and

$$\begin{aligned} -4\{\widetilde{\kappa}_{1,2}; z\} &= -4\{\lambda^\sharp_{1,2}; z\} - 4\{(\lambda^\flat_{1,2})^{-1}; x\}\left(\frac{dx}{dz}\right)^2 \\ &= \left(-\left(-4\{(\lambda^\sharp_{1,2})^{-1}; x\}\right) + \left(-4\{(\lambda^\flat_{1,2})^{-1}; x\}\right)\right)\left(\frac{dx}{dz}\right)^2 \\ &= \frac{1}{4}\frac{1-2x}{x^2(1-x)^2}\left(\frac{dx}{dz}\right)^2. \end{aligned}$$

Multiplying both sides by $\left(\frac{2}{\pi i}\right)^2$, we have

$$\begin{aligned}
&\frac{2\widetilde{\kappa}_{1,2}(z)'\widetilde{\kappa}_{1,2}(z)''' - 3(\widetilde{\kappa}_{1,2}(z)'')}{(\widetilde{\kappa}_{1,2}(z)')^2} \\
= &\ -\frac{1}{4}\frac{1}{\lambda^{\sharp}_{1,2}(z)^2(1-\lambda^{\sharp}_{1,2}(z))}\left(\lambda^{\sharp}_{1,2}(z)'\right)^2 \\
= &\ \frac{1}{4}\frac{1-2(2\lambda^{\flat}(z)-1)^2}{(2\lambda^{\flat}(z)-1)^2\left(1-(2\lambda^{\flat}(z)-1)\right)^2}\left(4(2\lambda^{\flat}(z)-1)\lambda^{\flat}(z)'\right) \\
= &\ -\frac{\theta_3(z)^8(8\theta_0(z)^8 - 8\theta_0(z)^4\theta_3(z)^4 + \theta_3(z)^8)}{(\theta_0(z)^4 - 2\theta_3(z)^4)^2}.
\end{aligned}$$

(Note that $\lambda^{\flat}(z)' = 2\theta_2(z)(1-\lambda^{\flat}(z))$ for q_2.)
Put

$$\theta_3(z)^8(8\theta_0(z)^8 - 8\theta_0(z)^4\theta_3(z)^4 + \theta_3(z)^8) = \sum_{n=0}^{\infty}\alpha_n {q_2}^n,$$
$$(\theta_0(z)^4 - 2\theta_3(z)^4)^2 = \sum_{n=0}^{\infty}\beta_n {q_2}^n.$$

By the fomula

$$\begin{aligned}
\theta_0(z)^4\theta_3(z)^4 &= 1 + 16\sum_{n=1}^{\infty}\sigma_3^-(n){q_2}^{4n}, \\
\theta_3(z)^8 &= 1 + 16\sum_{n=1}^{\infty}\sigma_3^-(n){q_2}^{2n},
\end{aligned}$$

we have

$$\alpha_n = \sum_{m=0}^{n} s_{n-m}(s_m - 8t_m - 8u_m),$$
$$\beta_n = 4s_n - 4t_n + u_n,$$

where s_n, t_n and u_n satisfy

$$\begin{aligned}
\theta_3(z)^8 &= \sum_{n=0}^{\infty} s_n {q_2}^n, \\
\theta_0(z)^4\theta_3(z)^4 &= \sum_{n=0}^{\infty} t_n {q_2}^n, \\
\theta_0(z)^8 &= \sum_{n=0}^{\infty} u_n {q_2}^n,
\end{aligned}$$

i.e.

$$s_n = \begin{cases} 1, & \text{for } n = 0, \\ 0, & \text{for } n : \text{odd}, \\ 16(-1)^{\frac{n}{2}}\sigma_3^-\left(\dfrac{n}{2}\right), & \text{for } n : \text{even}, \end{cases}$$

$$t_n = \begin{cases} 1, & \text{for } n = 0, \\ 0, & \text{for } n \equiv 1, 2, 3 \bmod 4, \\ 16\sigma_3^-\left(\dfrac{n}{4}\right), & \text{for } n \equiv 0 \bmod 4, \end{cases}$$

$$u_n = \begin{cases} 1, & \text{for } n = 0, \\ 0, & \text{for } n : \text{odd}, \\ 16\sigma_3^-\left(\dfrac{n}{2}\right), & \text{for } n : \text{even}, \end{cases}$$

By Lemma 1 and 4, we obtain the recurrent relation of $\{a_n\}$ in Theorem 1 (4).

Now we calculate a_1. Differentiating the identity $\lambda^\flat_{1,2}(\widetilde{\kappa}_{1,2}(z)) = \lambda^\sharp_{1,2}(z)$ and multiplying both sides by $\left(\frac{2}{\pi i}\right)$, we have

$$\widetilde{\kappa}_{1,2}(z)' \frac{d\lambda^\flat_{1,2}(\tau)}{d\tau} = \lambda^\sharp_{1,2}(z)'$$

where $\tau = \widetilde{\kappa}_{1,2}(z)$. Multiplying both sides by $1/{q_2}^2$ and using Lemma 5, we have

$$ia_1 \times \frac{2}{3}\pi^2 i a_1 E_4(i) = -256$$

as $z \to i\infty$, and since we have

$$E_4(i) = \frac{3\Gamma\left(\frac{1}{4}\right)^8}{2^6\pi^6},$$

we obtain (4) in Theorem 1.

(5) Since

$$\check{\kappa}_{1,2}(z+2) = -1 - \frac{1}{\check{\kappa}_{1,2}(z)} \quad , \quad \check{\kappa}_{1,2}(z+6) = \check{\kappa}_{1,2}(z),$$

and $\check{\kappa}_{1,2}(i\infty) = \rho_2$, we can identify $\check{\kappa}_{1,2}(z)$ with the Fourier expansion

$$\rho_2(1 + a_1 q_3 + a_2 {q_3}^2 + a_3 {q_3}^3 + \cdots).$$

Let $x = \lambda^\flat_{1,2}(z) = P^\flat(\lambda^\flat(z))$. Since the proparties of Schwarzian derivative, we have

$$\begin{aligned} -4\{(\lambda^\flat_{1,2})^{-1}; x\} &= \frac{1}{x^2} + \frac{1-\left(\frac{1}{2}\right)^2}{(1-x)^2} + \frac{1-\left(\frac{1}{2}\right)^2}{x(1-x)}, \\ -4\{J^{-1}; x\} &= \frac{1-\left(\frac{1}{3}\right)^2}{x^2} + \frac{1-\left(\frac{1}{2}\right)^2}{(1-x)^2} + \frac{1-\left(\frac{1}{3}\right)^2-\left(\frac{1}{2}\right)^2}{x(1-x)} \end{aligned}$$

and

$$\begin{aligned}
-4\{\check{\kappa}_{1,2};z\} &= -4\{\lambda^{\flat}_{1,2};z\} - 4\{J^{-1};x\}\left(\frac{dx}{dz}\right)^2 \\
&= \left(-\left(-4\{(\lambda^{\flat}_{1,2})^{-1};x\}\right) + \left(-4\{J^{-1};x\}\right)\right)\left(\frac{dx}{dz}\right)^2 \\
&= -\frac{1}{9}\frac{1}{x^2(1-x)}\left(\frac{dx}{dz}\right)^2.
\end{aligned}$$

Multiplying both sides by $\left(\frac{3}{\pi i}\right)^2$, we have

$$\begin{aligned}
\frac{2\check{\kappa}_{1,2}(z)'\check{\kappa}_{1,2}(z)''' - 3(\check{\kappa}_{1,2}(z)'')}{(\check{\kappa}_{1,2}(z)')^2} &= -\frac{1}{9}\frac{1}{\lambda^{\flat}_{1,2}(z)^2(1-\lambda^{\flat}_{1,2}(z))}\left(\lambda^{\flat}_{1,2}(z)'\right)^2 \\
&= -\frac{1}{9}\frac{1}{\lambda^{\flat}(z)^2\,(1-\lambda^{\flat}(z))}\left(4(1-2\lambda^{\flat}(z))\lambda^{\flat}(z)'\right)^2 \\
&= -\theta_3(z)^8.
\end{aligned}$$

(Note that $\lambda^{\flat}(z)' = 3\theta_2(z)(1-\lambda^{\flat}(z))$ for q_3.)
Put

$$\frac{1}{4}\left(9\theta_3(z)^8 - 5\theta_0(z)^4\theta_3(z)^4\right) = \sum_{n=0}^{\infty} b_n {q_3}^n.$$

By the fomula

$$\theta_3(z)^8 = 1 + 16\sum_{n=1}^{\infty}\sigma_3^-(n){q_3}^{3n},$$

we have

$$b_n = \begin{cases} 1, & \text{for } n=0, \\ 0, & \text{for } n \equiv 1,2 \bmod 3, \\ 16(-1)^n\sigma_3^-\left(\dfrac{n}{3}\right), & \text{for } n \equiv 0 \bmod 3, \end{cases}$$

By Lemma 2 and 3, we obtain the recurrent relation of $\{a_n\}$ in Theorem 1 (5).

Now we calculate a_1. Differentiating the identity $J(\check{\kappa}_{1,2}(z)) = \lambda^{\flat}_{1,2}(z)$ and multiplying both sides by $\frac{3}{\pi i}$, we have

$$\check{\kappa}_{1,2}(z)'\frac{dJ(\tau)}{d\tau} = \lambda^{\flat}_{1,2}(z)'$$

where $\tau = \check{\kappa}_{1,2}(z)$. Multiplying both sides by $1/{q_3}^3$ and using Lemma 6, we have

$$\rho_2 a_1 \times \left(-\frac{8}{9}\pi^3 i{\rho_3}^2{a_1}^2E_6(\rho_2)\right) = 192$$

as $z \to i\infty$, and since we have

$$E_6(\rho_2) = \frac{2^5}{3^3}\frac{\pi^3}{\Gamma\left(\frac{5}{6}\right)\Gamma\left(\frac{2}{3}\right)},$$

we obtain (5) in Theorem 1.

(6) Since

$$\kappa^*_{(2)}(z+2) = -\frac{2}{\kappa^*_{(2)}(z)} \quad , \quad \kappa^*_{(2)}(z+4) = \kappa^*_{(2)}(z),$$

and $\kappa^*_{(2)}(i\infty) = \sqrt{2}i$, we can identify $\kappa^*_{(2)}(z)$ with the Fourier expansion

$$\sqrt{2}i(1 + a_1 q_2 + a_2 {q_2}^2 + a_3 {q_2}^3 + \cdots).$$

Let $x = \lambda^\flat(z)$. Since the proparties of Schwarzian derivative, we have

$$\begin{aligned} -4\{(\lambda^\flat)^{-1}; x\} &= \frac{1}{x^2} + \frac{1}{(1-x)^2} + \frac{1}{x(1-x)}, \\ -4\{(\lambda^*_{(2)})^{-1}; x\} &= \frac{1-\left(\frac{1}{2}\right)^2}{x^2} + \frac{1-\left(\frac{1}{4}\right)}{(1-x)^2} + \frac{1-\left(\frac{1}{2}\right)^2-\left(\frac{1}{4}\right)}{x(1-x)} \end{aligned}$$

and

$$\begin{aligned} -4\{\kappa^*_{(2)}; z\} &= -4\{\lambda^\flat; z\} - 4\{(\lambda^*_{(2)})^{-1}; x\}\left(\frac{dx}{dz}\right)^2 \\ &= \left(-\left(-4\{(\lambda^\flat)^{-1}; x\}\right) + \left(-4\{(\lambda^*_{(2)})^{-1}; x\}\right)\right)\left(\frac{dx}{dz}\right)^2 \\ &= -\frac{1}{16}\frac{4-3x}{x^2(1-x)^2}\left(\frac{dx}{dz}\right)^2. \end{aligned}$$

Multiplying both sides by $\left(\frac{2}{\pi i}\right)^2$, we have

$$\begin{aligned} \frac{2\kappa^*_{(2)}(z)'\kappa^*_{(2)}(z)''' - 3(\kappa^*_{(2)}(z)'')^2}{(\kappa^*_{(2)}(z)')^2} &= -\frac{1}{16}\frac{4-3\lambda^\flat(z)}{\lambda^\flat(z)^2(1-\lambda^\flat(z))^2}\left(\lambda^\flat(z)'\right)^2 \\ &= -\frac{1}{4}\left(3\theta_0(z)^4\theta_3(z)^4 + \theta_3(z)^8\right). \end{aligned}$$

(Note that $\lambda^\flat(z)' = 2\theta_2(z)(1-\lambda^\flat(z))$ for q_2.)
Put

$$\frac{1}{4}\left(3\theta_0(z)^4\theta_3(z)^4 + \theta_3(z)^8\right) = \sum_{n=0}^{\infty} b_n {q_2}^n.$$

By the fomula

$$\begin{aligned} \theta_0(z)^4\theta_3(z)^4 &= 1 + 16\sum_{n=1}^{\infty}\sigma_3^-(n){q_2}^{4n}, \\ \theta_3(z)^8 &= 1 + 16\sum_{n=1}^{\infty}\sigma_3^-(n){q_2}^{2n}, \end{aligned}$$

we have

$$b_n = \begin{cases} 1, & \text{for } n = 0, \\ 0, & \text{for } n \equiv 1,3 \pmod 4, \\ 12\sigma_3^-\left(\frac{n}{2}\right), & \text{for } n \equiv 2 \pmod 4, \\ 12\sigma_3^-\left(\frac{n}{2}\right) + 4\sigma_3^-\left(\frac{n}{4}\right), & \text{for } n \equiv 0 \pmod 4, \end{cases}$$

By Lemma 1 and 3, we obtain the recurrent relation of $\{a_n\}$ in Theorem 1 (6).

Now we calculate a_1. Differentiating the identity $\lambda^*_{(2)}(\kappa^*_{(2)}(z)) = \lambda^\flat(z)$ twice and multiplying both sides by $\left(\frac{2}{\pi i}\right)^2$, we have

$$\kappa^*_{(2)}(z)'' \frac{d\lambda^*_{(2)}(\tau)}{d\tau} + \left(\kappa^*_{(2)}(z)'\right)^2 \frac{d^2\lambda^*_{(2)}(\tau)}{d\tau^2} = \lambda^\flat(z)''$$

where $\tau = \kappa^*_{(2)}(z)$. Multiplying both sides by $1/{q_2}^2$ and using Lemma 5, we have

$$\sqrt{2}ia_1 \times \frac{2^{\frac{5}{2}}}{(\sqrt{2}-1)^2}\pi^2 i a_1 \theta_2(\sqrt{2}i)^8 + (\sqrt{2}ia_1)^2 \times \frac{2^2}{(\sqrt{2}-1)^2}\pi^2\theta_2(\sqrt{2}i)^8 = 64$$

as $z \to i\infty$, and since we have

$$\theta_2(\sqrt{2}i)^8 = \frac{(\sqrt{2}-1)^2}{2^3} \frac{\pi^2}{\Gamma\left(\frac{5}{8}\right)^4 \Gamma\left(\frac{7}{8}\right)^4},$$

we obtain (6) in Theorem 1.

□

4. Appendix

In this section we give the special values of the elliptic theta functions θ_0, θ_2 and θ_3, and the Eisenstein series E_4 and E_6 used in this paper.

4.1. Elliptic theta function

Let $k = k(z) = \theta_2(z)^2/\theta_3(z)^2$. Since

$$K(k) = \int_0^1 \frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}} = \frac{\pi}{2}\theta_3(z)^2$$

and

$$\int_0^1 \frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}} = \frac{\pi}{2}\, {}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; k^2\right),$$

we have the identity

$$\theta_3(z)^2 = {}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; \lambda^\flat(z)\right).$$

In the case $z=i$:

Since the inversion fourmula of gamma function and the equation

$$ {}_2F_1\left(2a, 1-2a; 2c; \frac{1}{2}\right) = \frac{2^{1-2c}\Gamma(2c)\Gamma\left(\frac{1}{2}\right)}{\Gamma(a+c)\Gamma\left(c-a+\frac{1}{2}\right)}, \quad \text{([EMOT], 2.1.5, p68]),} $$

we have

$$ {}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; \frac{1}{2}\right) = \frac{\sqrt{\pi}}{\Gamma\left(\frac{3}{4}\right)^2} = \frac{\Gamma\left(\frac{1}{4}\right)^2}{2\pi^{\frac{3}{2}}} $$

as $a=\frac{1}{4}$ and $c=\frac{1}{2}$. So we obtain

$$ \begin{aligned} \theta_3(i)^4 &= {}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; \lambda^\flat(i)\right)^2 \\ &= {}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; \frac{1}{2}\right)^2 \\ &= \frac{\Gamma\left(\frac{1}{4}\right)^4}{2^2\pi^3}. \end{aligned} $$

Since we have

$$ \begin{aligned} \frac{\theta_2(i)^4}{\theta_3(i)^4} &= \lambda^\flat(i) = \frac{1}{2}, \\ \frac{\theta_0(i)^4}{\theta_3(i)^4} &= 1-\lambda^\flat(i) = \frac{1}{2}, \end{aligned} $$

we obtain

$$ \theta_0(i)^4 = \theta_2(i)^4 = \frac{\Gamma\left(\frac{1}{4}\right)^4}{2^3\pi^3}. $$

In the case $z=\rho_1=e^{\frac{\pi i}{3}}$:

Since the equation

$$ {}_2F_1\left(a+\frac{1}{3}, 3a; 2a+\frac{2}{3}; e^{\frac{\pi i}{3}}\right) = 2\pi e^{\frac{\pi i a}{2}} 3^{-\frac{3a+1}{2}} \frac{\Gamma\left(2a+\frac{2}{3}\right)}{\Gamma\left(a+\frac{1}{3}\right)\Gamma\left(a+\frac{2}{3}\right)\Gamma\left(\frac{2}{3}\right)}, $$

([EMOT], 2.9 (55), p105),

we obtain

$$ \theta_3(\rho_1)^4 = {}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; e^{\frac{\pi i}{3}}\right)^2 = 2^2 3^{-\frac{3}{2}} \pi e^{\frac{\pi i}{6}} \frac{1}{\Gamma\left(\frac{2}{3}\right)^2\Gamma\left(\frac{5}{6}\right)^2} $$

as $a=\frac{1}{6}$. Since we have

$$\frac{\theta_2(\rho_1)^4}{\theta_3(\rho_1)^4} = \lambda^\flat(\rho_1) = \rho_1,$$
$$\frac{\theta_0(\rho_1)^4}{\theta_3(\rho_1)^4} = 1-\lambda^\flat(\rho_1) = e^{\frac{5\pi i}{3}},$$

we obtain

$$\begin{aligned} \theta_0(\rho_1)^4 &= 2^2 3^{-\frac{3}{2}}\pi e^{\frac{11\pi i}{6}}\frac{1}{\Gamma\left(\frac{2}{3}\right)^2\Gamma\left(\frac{5}{6}\right)^2}, \\ \theta_2(\rho_1)^4 &= 2^2 3^{-\frac{3}{2}}\pi e^{\frac{\pi i}{2}}\frac{1}{\Gamma\left(\frac{2}{3}\right)^2\Gamma\left(\frac{5}{6}\right)^2}. \end{aligned}$$

In the case $z=\rho_2=e^{\frac{2\pi i}{3}}$:

Since the properties of elliptic theta functions

$$\begin{aligned} \theta_0(z) &= \theta_3(1+z), \\ e^{\frac{\pi i}{4}}\theta_2(z) &= \theta_2(1+z), \\ \theta_3(z) &= \theta_0(1+z), \end{aligned}$$

we obtain

$$\begin{aligned} \theta_0(\rho_2)^4 &= \theta_3(\rho_1)^4 &&= 2^2 3^{-\frac{3}{2}}\pi e^{\frac{\pi i}{6}}\frac{1}{\Gamma\left(\frac{2}{3}\right)^2\Gamma\left(\frac{5}{6}\right)^2}, \\ \theta_2(\rho_2)^4 &= e^{-\pi i}\theta_2(\rho_1)^4 &&= 2^2 3^{-\frac{3}{2}}\pi e^{-\frac{\pi i}{2}}\frac{1}{\Gamma\left(\frac{2}{3}\right)^2\Gamma\left(\frac{5}{6}\right)^2}, \\ \theta_3(\rho_2)^4 &= \theta_0(\rho_1)^4 &&= 2^2 3^{-\frac{3}{2}}\pi e^{\frac{11\pi i}{6}}\frac{1}{\Gamma\left(\frac{2}{3}\right)^2\Gamma\left(\frac{5}{6}\right)^2}. \end{aligned}$$

In the case $z=\sqrt{2}i$:

Since the equation

$${}_2F_1(a,b;a-b+1;z) = (1-z)^{-a}\ {}_2F_1\left(\frac{a}{2},\frac{a+1-2b}{2};a-b+1;-\frac{4z}{(1-z)^2}\right),$$
([EMOT], 2.11 (32), p113),

we have

$${}_2F_1\left(\frac{1}{2},\frac{1}{2};1;3-2\sqrt{2}\right) = 2^{-\frac{1}{2}}(\sqrt{2}-1)^{-\frac{1}{2}}\ {}_2F_1\left(\frac{1}{4},\frac{1}{4};1;-1\right)$$

as $a=b=\frac{1}{2}$. And since the equation

$${}_2F_1(a,b;1+a-b;-1) = 2^{-a}\frac{\Gamma(1+a-b)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(1-b+\frac{a}{2}\right)\Gamma\left(\frac{1}{2}+\frac{a}{2}\right)}, \quad \text{([EMOT], 2.8 (47), p104)},$$

we have

$${}_2F_1\left(\frac{1}{2},\frac{1}{2};1;-1\right) = 2^{-\frac{1}{4}}\frac{\pi^{\frac{1}{2}}}{\Gamma\left(\frac{5}{8}\right)\Gamma\left(\frac{7}{8}\right)}.$$

By the above two equations we obtain

$$\begin{aligned}
\theta_3(\sqrt{2}i)^4 &= {}_2F_1\left(\frac{1}{2},\frac{1}{2};1;\lambda^\flat(\sqrt{2}i)\right)^2 \\
&= {}_2F_1\left(\frac{1}{2},\frac{1}{2};1;3-2\sqrt{2}\right)^2 \\
&= 2^{-1}(\sqrt{2}-1)^{-1}\,{}_2F_1\left(\frac{1}{4},\frac{1}{4};1;-1\right)^2 \\
&= 2^{-\frac{3}{2}}(\sqrt{2}-1)^{-1}\pi\frac{1}{\Gamma\left(\frac{5}{8}\right)^2\Gamma\left(\frac{7}{8}\right)^2}.
\end{aligned}$$

Since we have

$$\begin{aligned}
\frac{\theta_2(\sqrt{2}i)^4}{\theta_3(\sqrt{2}i)^4} &= \lambda^\flat(\sqrt{2}i) = 3-2\sqrt{2} = (\sqrt{2}-1)^2, \\
\frac{\theta_0(\sqrt{2}i)^4}{\theta_3(\sqrt{2}i)^4} &= 1-\lambda^\flat(\rho_1) = 2(\sqrt{2}-1),
\end{aligned}$$

we obtain

$$\begin{aligned}
\theta_0(\sqrt{2}i)^4 &= 2^{-\frac{1}{2}}\pi\frac{1}{\Gamma\left(\frac{5}{8}\right)^2\Gamma\left(\frac{7}{8}\right)^2}, \\
\theta_2(\sqrt{2}i)^4 &= 2^{-\frac{3}{2}}(\sqrt{2}-1)\pi\frac{1}{\Gamma\left(\frac{5}{8}\right)^2\Gamma\left(\frac{7}{8}\right)^2}.
\end{aligned}$$

4.2. Eisenstein series

Let $\tau \in \mathbb{H}$, $L = \mathbb{Z}\oplus\mathbb{Z}\tau$ and $\mathcal{P}(z)$ is the Wierstrass $\mathcal{P}$ function with the lattice L. By the equation

$$\mathcal{P}'(z)^2 = 4\mathcal{P}^3(z) - g_2(\tau)\mathcal{P}(z) + g_3(\tau),$$

we define $g_2(\tau)$ and $g_3(\tau)$. Since we know the equations

$$g_2(\tau) = \frac{4}{3}\pi^4E_4(\tau), \qquad g_3 = \frac{8}{27}\pi^6E_6(\tau),$$

we have

$$\Delta(\tau) = g_2(\tau)^3 - 27g_3(\tau)^2 = \frac{2^6}{3^3}\pi^{12}\left(E_4(\tau)^3 - E_6(\tau)^2\right).$$

And we know the equation

$$\begin{aligned}
\Delta(\tau) &= (2\pi)^{12}e^{2\pi i\tau}\prod_{n=1}^{\infty}\left(1-e^{\pi in\tau}\right)^{24} \\
&= (2\pi)^4\theta_1'(\tau)^8 \\
&= 2^4\pi^{12}\left(\theta_0(\tau)\theta_2(\tau)\theta_3(\tau)\right)^8 \\
&= 2^4\pi^{12}\theta_2(\tau)^8\theta_3(\tau)^8\left(\theta_3(\tau)^4-\theta_2(\tau)^4\right)^2.
\end{aligned}$$

So we obtain

$$\frac{2^6}{3^3}\pi^{12}\left(E_4(\tau)^3 - E_6(\tau)^2\right) = 2^4\pi^{12}\left(\theta_0(\tau)\theta_2(\tau)\theta_3(\tau)\right)^8.$$

On the otherhand we know two expressions for J as followings:

$$\begin{aligned} J(\tau) &= \frac{4}{27}\frac{(1-\lambda^\flat(\tau)+\lambda^\flat(\tau)^2)^3}{\lambda^\flat(\tau)^2(1-\lambda^\flat(\tau))^2} \\ &= \frac{4}{27}\frac{\theta_2(\tau)^8-\theta_2(\tau)^4\theta_3(\tau)^4+\theta_3(\tau)^8}{\theta_2(\tau)^8\theta_3(\tau)^8\left(\theta_3(\tau)^4-\theta_2(\tau)^4\right)^2} \end{aligned}$$

and

$$J(\tau) = \frac{E_4(\tau)^3}{E_4(\tau)^3 - E_6(\tau)^2}.$$

Thus we have

$$\begin{cases} E_4(\tau) = \theta_2(\tau)^8-\theta_2(\tau)^4\theta_3(\tau)^4+\theta_3(\tau)^8, \\ E_6(\tau) = \frac{1}{2}\left(\theta_2(\tau)^4-2\theta_3(\tau)^4\right)\left(2\theta_2(\tau)^4-\theta_3(\tau)^4\right)\left(\theta_2(\tau)^4+\theta_3(\tau)^4\right). \end{cases}$$

Using 4.1, we obtain

$$\begin{aligned} E_4(i) &= \frac{3}{2^6}\frac{\Gamma\left(\frac{1}{4}\right)^8}{\pi^6}, \\ E_4(\rho_1) &= 0, \\ E_4(\rho_2) &= 0, \\ E_4(\sqrt{2}i) &= \frac{5}{2^3}\frac{\pi^2}{\Gamma\left(\frac{5}{8}\right)^4\Gamma\left(\frac{7}{8}\right)^4}, \\ E_6(i) &= 0, \\ E_6(\rho_1) &= \frac{2^5}{3^3}\frac{\pi^3}{\Gamma\left(\frac{2}{3}\right)^6\Gamma\left(\frac{5}{6}\right)^6}, \\ E_6(\rho_2) &= \frac{2^5}{3^3}\frac{\pi^3}{\Gamma\left(\frac{2}{3}\right)^6\Gamma\left(\frac{5}{6}\right)^6}, \\ E_6(\sqrt{2}i) &= \frac{7}{2^4}\frac{\pi^3}{\Gamma\left(\frac{5}{8}\right)^6\Gamma\left(\frac{7}{8}\right)^6}. \end{aligned}$$

References.

[EMOT] A. Erdélyi, W. Magnus, F. Oberhettinger and F. G. Tricomi, Higher transcendental functions, McGraw-Hill, New York-Toronto-London, 1953.

[KY] M. KANEKO and M. YOSHIDA, The kappa function, Internat. J. Math. 14 (2003), no. 9, 1003-1013.

[NS] N. NARUMIYA and H. SHIGA, The mirror map for a family of $K3$ surfaces induced from the simplest 3-dimensional reflexive polytope, Proceedings on Moonshine and related topics (Montréal, QC, 1999), 139-161.

Technical Reports of Mathematical Sciences
Chiba University, vol. 21 (2005)

[1] *A three terms Arithmetic-Geometric mean*, by K. Koike and H. Shiga.

[2] *Isogeny formulas for Picard modular forms and a three terms AGM*, by K. Koike and H. Shiga.

[3] *Iteration of the covariant function for Gauss hypergeometric differential equation*, by Y. Tanaka.

[4] *Commutativity of operators*, by M. Nagisa, M. Ueda and S. Wada.

[5] *Finite groups with three classes lengths*, by K. Kanke and S. Nozawa.

[6] *The variants of kappa function*, by Y. Tanaka.

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