COMMUTATIVITY OF OPERATORS

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Abstract. For two bounded positive linear operators $a, b$ on a Hilbert space, we give conditions which imply the commutativity of $a, b$. Some of them are related to well-known formulas for indefinite elements, e.g., $(a + b)^n = \sum_k \binom{n}{k} a^{n-k} b^k$ etc. and others are related to the property of operator monotone functions. We also give a condition which implies the commutativity of a C*-algebra.

1. Introduction

Ji and Tomiyama ([3]) give a characterization of commutativity of C*-algebra, where they also give a condition that two positive operators commute. For bounded linear operators on a Hilbert space $\mathcal{H}$, we slightly generalize their result as follows:

**Theorem 1.** Let $a$ and $b$ be self-adjoint operators on $\mathcal{H}$. Then the following are equivalent.

1. $ab = ba$.
2. $\exp(a + b) = \exp(a) \exp(b)$.
3. There exist a positive integer $n \geq 2$ and distinct non-zero real numbers $t_1, t_2, \ldots, t_{n-1}$ such that
   \[(a + t_ib)^n = \sum_{k=0}^{n} \binom{n}{k} t_i^k a^{n-k} b^k\]
   for $i = 1, 2, \ldots, n - 1$.
4. There exist a positive integer $n \geq 2$ and distinct non-zero real numbers $t_1, t_2, \ldots, t_{n-1}$ such that
   \[a^n - (t_i b)^n = (a - t_i b) \sum_{k=0}^{n-1} a^{n-k-1} (t_i b)^k\]
   for $i = 1, 2, \ldots, n - 1$. 
DePrima and Richard([2]), and Uchiyama([9],[10]) independently prove that, for any positive operators $a$ and $b$, the following conditions are equivalent:

1. $ab = ba$.
2. $ab^n + b^na$ is positive for all $n \in \mathbb{N}$.

We give a little weakened condition for two operators commuting.

Ji and Tomiyama, and Wu([12]) use a commutativity condition of two operators and a gap of monotonicity and operator monotonicity of functions to characterize commutativity of C*-algebras. With a similar point of view, we can get the following result:

**Theorem 2.** Let $A$ be a unital C*-algebras. Then the following are equivalent:

1. $A$ is commutative.
2. There exists a continuous, increasing functions $f$ on $[0, \infty)$ such that $f$ is not concave and operator monotone for $A$.
3. Whenever positive operators $a$ and $b$ satisfy $ab + ba \geq 0$, $ab^2 + b^2a \geq 0$.

## 2. Proof of Theorem 1

**Lemma 3.** Let $a$ and $b$ be self-adjoint operators on $\mathcal{H}$, and $f$ a continuous function on the spectrum $\text{Sp}(a)$ of $a$. Then $ab = ba$ implies that $f(a)b = bf(a)$.

**Proof.** We can choose a sequence $\{p_n\}$ of polynomials which converges to $f$ uniformly on $\text{Sp}(a)$. So we have

$$f(a)b = \lim_{n \to \infty} p_n(a)b = \lim_{n \to \infty} bp_n(a) = bf(a).$$

\square

**Lemma 4.** Let $a, b$ be self-adjoint operators on $\mathcal{H}$ and $k$ a positive integer. If $a^kba = a^{k+1}b$, then $ab = ba$.

**Proof.** We put $p$ the orthogonal projection of $\mathcal{H}$ onto $\ker(a)$. We remark that

$$\ker a = \ker a^2 = \cdots = \ker a^{k+1}, \quad pa = ap = 0.$$  

Since

$$0 = a^k bap = a^{k+1} bp = a^{k+1} (1 - p) bp,$$
we have \((1 - p)bp = 0\). The self-adjointness of \(b\) implies
\[
b = pbp + (1 - p)b(1 - p).
\]
So we have
\[
ab - ba = (p + (1 - p))(ab - ba) = (1 - p)(ab - ba) - pba
= (1 - p)(ab - ba) - pbpa = (1 - p)(ab - ba).
\]
Since \(a^k(ab - ba) = 0\), we can get \(ab = ba\). □

**Proof of Theorem 1.** (1)⇒(2), (1)⇒(3) and (1)⇒(4) are trivial.
(2)⇒(1) The element \(\exp(a + b)\) is self-adjoint, so we have
\[
\exp(a)\exp(b) = \exp(b)\exp(a).
\]
We apply Lemma 3 for the function \(f(x) = \log x\) on \(\text{Sp}(a)\). Since
\[
\log(\exp(a)) = a,
\]
we have \(a \exp(b) = \exp(b) a\). Repeated the same argument, we can show \(ab = ba\).
(3)⇒(1) Since \((a + t_ib)^n\) is self-adjoint, we have
\[
\sum_{k=0}^{n} \binom{n}{k} t_i^k a^{n-k} b^k = \sum_{k=0}^{n} \binom{n}{k} t_i^k b^k a^{n-k}, \quad (i = 1, 2, \ldots, n - 1).
\]
This means that
\[
\begin{pmatrix}
1 & t_1 & \cdots & t_1^{n-2} \\
1 & t_2 & \cdots & t_2^{n-2} \\
\vdots & \vdots & \ddots & \vdots \\
1 & t_{n-1} & \cdots & t_{n-1}^{n-2}
\end{pmatrix}
\begin{pmatrix}
\binom{n}{1}(a^{n-1}b - ba^{n-1}) \\
\binom{n}{2}(a^{n-2}b^2 - b^2a^{n-2}) \\
\vdots \\
\binom{n}{n-1}(ab^{n-1} - b^{n-1}a)
\end{pmatrix}
= \begin{pmatrix}0 \\
0 \\
\vdots \\
0\end{pmatrix}.
\]
So we have \(a^{n-1}b = ba^{n-1}\). When \(n\) is even, we have \(ab = ba\), by using Lemma 3 and the fact \(a = (a^{n-1})^{1/n-1}\).

We assume that \(n\) is odd. Then we have
\[
a^2b = (a^{n-1})^{2/n-1}b = b(a^{n-1})^{2/n-1} = ba^{2}.
\]
If we apply the same argument for the relation
\[
(a + t_ib)^n
\]
\[
= a^n + t_i(a^{n-1}b + a^{n-2}ba + \cdots + ba^{n-1}) + t_i^2(\cdots) = \sum_{k=0}^{n} \binom{n}{k} t_i^k a^{n-k} b^k,
\]
then we can get
\[
a^{n-1}b + a^{n-2}ba + \cdots + ba^{n-1} = na^{n-1}b.
\]
Using the commutativity of $a^2$ and $b$, we have
\[ a^{n-1}b = a^{n-2}ba. \]

By Lemma 4, it follows that $ab = ba$.

(4) ⇒ (1) By using the same argument as (3) ⇒ (1), we can get that a coefficient of $t_i^{n-1}$ vanishes, that is,
\[ ab^{n-1} - bab^{n-2} = 0. \]

By Lemma 4, we can get $ab = ba$. □

**Remark 5.** On the implication (2) ⇒ (1), the following stronger result is known for self-adjoint matrices (see [7] and [8]). If self-adjoint matrices $a$, $b$ satisfy the condition
\[ \text{Trace}(\exp(a + b)) = \text{Trace}(\exp(a) \exp(b)), \]
then $ab = ba$.

### 3. Operator monotone functions

Let $f$ be a continuous function on $[0, \infty)$. We call $f$ a matrix monotone (resp. matrix concave) function of order $n$ if it satisfies the following condition:

\[ a, b \in M_n(\mathbb{C}), 0 \leq a \leq b \Rightarrow f(a) \leq f(b) \]

(resp. \ $a, b \in M_n(\mathbb{C}), 0 \leq a \leq b, 0 \leq t \leq 1$
\[ \Rightarrow f(ta + (1 - t)b) \leq tf(a) + (1 - t)f(b). \]

When $f$ is matrix monotone of order $n$ for any $n$, $f$ is called operator monotone. We call a function $f$ operator monotone for a C*-algebra $A$ if, for $a, b \in A$, $0 \leq a \leq b$ implies $0 \leq f(a) \leq f(b)$. The following fact is well-known ([5]: Theorem 2.1). Here we give a different proof of this.

**Lemma 6.** If $f : [0, \infty) \longrightarrow [0, \infty)$ is continuous and matrix monotone of order $2n$, then $f$ is matrix concave of order $n$.

**Proof.** For $a, b \in M_n(\mathbb{C})^+$ and $0 \leq t \leq 1$, we put
\[ X = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}, \quad Y = \begin{pmatrix} \sqrt{t} & -\sqrt{1-t} \\ \sqrt{1-t} & \sqrt{t} \end{pmatrix} \in M_{2n}(\mathbb{C}). \]
Then we have
\[
Y^*XY = \begin{pmatrix}
ta + (1-t)b & \sqrt{t(1-t)}(b-a) \\
\sqrt{t(1-t)}(b-a) & (1-t)a + tb
\end{pmatrix}
\leq \begin{pmatrix}
ta + (1-t)b + \epsilon & 0 \\
0 & (1-t)a + tb + \frac{t(1-t)}{\epsilon}(a-b)^2
\end{pmatrix}
\]
for any positive number \(\epsilon\). By the assumption for \(f\), we can get
\[
Y^*f(X)Y = f(Y^*XY)
\leq \begin{pmatrix}
f(ta + (1-t)b + \epsilon) & 0 \\
0 & f((1-t)a + tb + \frac{t(1-t)}{\epsilon}(a-b)^2)
\end{pmatrix}.
\]
Since \(\epsilon\) is arbitrary, we have
\[
tf(a) + (1-t)f(b) \geq f(ta + (1-t)b).
\]

As an application of this lemma, we can see that the exponential function \(\exp(\cdot)\) is increasing and convex but not matrix monotone of order 2. By Theorem 2, we can get another proof of Wu’s result [12].

Let \(f\) be an operator monotone function on \((0, \infty)\), that is, \(f\) is a matrix monotone function on \((0, \infty)\) of order \(n\) for any \(n \in \mathbb{N}\). Then \(f\) has the analytic continuation on the upper half plane \(H_+ = \{z \in \mathbb{C} \mid \text{Im}z > 0\}\) and also has the analytic continuation on the lower half plane \(H_-\) by the reflection across \((0, \infty)\). By Pick function theory, it is known that \(f\) is represented as follows:
\[
f(z) = f(0) + \beta z + \int_0^\infty \frac{\lambda z}{\lambda + z} dw(\lambda),
\]
where \(\beta \geq 0\) and \(w\) is a positive measure with
\[
\int_0^\infty \frac{\lambda}{1+\lambda} dw(\lambda) < +\infty
\]
(see [1]:page 144). We denote by \(P_+\) the closed right half plane \(\{z \in \mathbb{C} \mid \text{Re}z \geq 0\}\) and by \(\overline{C}(S)\) the closed convex hull of a subset \(S\) of \(\mathbb{C}\). We consider the case that \(f(0) \geq 0\). Then we can easily check \(f(P_+) \subset P_+\).

For \(a \in B(\mathcal{H})\), we denote by \(W(a)\) its numerical range
\[
\{(a\xi, \xi) \mid \|\xi\| = 1\} \subset \mathbb{C}.
\]
By Kato’s theorem ([4]:Theorem 7), if \(W(a)\) is contained in \(P_+\), then we have
\[
W(f(A)) \subset \overline{C}(f(P_+)).
\]
Proposition 7. Let $a, b \in B(\mathcal{H})$ be positive and $f, f_n$ be operator monotone functions from $[0, \infty)$ to $[0, \infty)$.

1. If $ab + ba \geq 0$, then $af(b) + f(b)a \geq 0$.
2. If $\text{Sp}(b) \subset f_n([0, \infty))$, $af_n^{-1}(b) + f_n^{-1}(b)a \geq 0$ for all $n$ and $\bigcap_n \overline{C(f_n(P_+))} \subset \mathbb{R}$, then $ab = ba$.

Proof. (1) We may assume that $a$ is invertible, replacing $a$ by $a + \epsilon$ ($\epsilon > 0$). Then we can define the new inner product on $\mathcal{H}$ by $\langle \xi, \eta \rangle = (a\xi, \eta)$, $\xi, \eta \in \mathcal{H}$.

It suffices to show that the positivity of $\text{Re} b$ with respect to $\langle \cdot, \cdot \rangle$ implies the positivity of $\text{Re} f(b)$ with respect to $\langle \cdot, \cdot \rangle$. Since $\text{Re} b \geq 0$ is equivalent to $W(b) = \{ \langle b\xi, \xi \rangle | \langle \xi, \xi \rangle = 1 \} \subset P_+$ and $W(f(b)) \subset \overline{C(f(P_+))} \subset P_+$, we have $\text{Re} f(b) \geq 0$.

(2) In the same setting in (1), if we get $W(b) \subset \mathbb{R}$, this implies $ab = ba$. By the argument of (1) and the assumption, we have $W(f_n^{-1}(b)) \subset P_+$ and $W(b) = W(f_n(f_n^{-1}(b)) \subset \overline{C(f_n(P_+))}$ for any $n$. So we have $W(b) \subset \bigcap_n \overline{C(f_n(P_+))} \subset \mathbb{R}$.

□

In [11], Uchiyama defines the function $u(t)$ on $[-a_1, \infty)$ as follows:

$$ u(t) = (t + a_1)^{\gamma_1}(t + a_2)^{\gamma_2}\cdots(t + a_k)^{\gamma_k}, $$

where $a_1 < a_2 < \ldots < a_k$, $\gamma_j > 0$, and he shows that the inverse function $f(x) = u^{-1}(x)$ becomes operator monotone on $[0, \infty)$ if $\gamma_1 \geq 1$. We assume that $f(0) \geq 0$ (i.e., $a_1 \leq 0$) and

$$ \gamma = \sum_{j: a_j \leq 0} \gamma_j > 1. $$

Then $f(z)$ is a holomorphic function from $D$ into $D$, where $D = \mathbb{C} \setminus (-\infty, 0] = \{ z \in \mathbb{C} \setminus \{0\} | -\pi < \arg z < \pi \}$. For $z = re^{i\theta}$ ($0 < \theta < \pi/2$), we set $z + a_j = r_je^{i\theta_j}$ ($j = 1, 2, \ldots, k$). Then we have

$$ 0 < \theta_k < \cdots < \theta_1 < \pi \text{ and } \arg(z) = \sum_{j=1}^{k} \gamma_j \theta_j \geq \gamma \theta. $$
This means that $|\arg f(z)| < \frac{1}{\gamma}|\arg z|$ if $0 < |\arg z| < \pi/2$. Since

$$
\overline{C}(f(P_+)) \subset \overline{C}(\{ z \in D \mid |\arg z| < \frac{\pi}{2\gamma} \}) \subset \{ z \in D \mid |\arg z| \leq \frac{\pi}{2\gamma} \}
$$

$$
\overline{C}(f^2(P_+)) \subset \overline{C}(f(\{ z \in D \mid |\arg z| \leq \frac{\pi}{2\gamma} \})) \subset \{ z \in D \mid |\arg z| \leq \frac{\pi}{2\gamma^2} \}
$$

... 

$$
\overline{C}(f^n(P_+)) \subset \overline{C}(f(\overline{C}(f^{n-1}(P_+)))) \subset \{ z \in D \mid |\arg z| \leq \frac{\pi}{2\gamma^n} \},
$$

we can get

$$
\bigcap_{n=1}^{\infty} \overline{C}(f^n(P_+)) \subset \mathbb{R}.
$$

**Corollary 8.** Let $a, b \in B(\mathcal{H})$ be positive and the function $u$ have the following form:

$$
u(t) = (t + a_1)^{\gamma_1} (t + a_2)^{\gamma_2} \cdots (t + a_k)^{\gamma_k},$$

where $a_1 < a_2 < \ldots < a_k$, $\gamma_j > 0$, $a_1 \leq 0$, $\gamma_1 \geq 1$ and $\sum_{j:a_j<0} \gamma_j > 1$. If $au^n(b) + u^n(b)a \geq 0$ for all $n \in \mathbb{N}$, then we have $ab = ba$.

**Proof of Theorem 2.** (1)$\Rightarrow$(2) and (1)$\Rightarrow$(3) are trivial.

(2)$\Rightarrow$(1) If $A$ is not commutative, then there exists an irreducible representation $\pi$ of $A$ on a Hilbert space $\mathcal{H}$ with dim $\mathcal{H} > 1$. Let $\mathcal{K}$ be a 2-dimensional subspace of $\mathcal{H}$. By Kadison’s transitivity theorem (see [6]), for any positive operator $T \in B(\mathcal{K})(\cong M_2(\mathbb{C}))$, we can choose a positive element $a \in A$ such that $\pi(a)|_{\mathcal{K}} = T$. By the assumption and Lemma 5, $f$ is not matrix monotone of order 2. This means that we can choose $S, T \in B(\mathcal{K})$ such that

$$
0 \leq S \leq T \text{ and } f(S) \not\succ f(T).
$$

So there exist $a, b \in A$ such that

$$
0 \leq a \leq b \text{ and } \pi(a) = S, \pi(b) = T.
$$

Since $f(S) = f(\pi(a)) = \pi(f(a))$ and $f(T) = f(\pi(b)) = \pi(f(b))$, this contradicts the operator monotonicity of $f$ for $A$.

(3)$\Rightarrow$(1) Let $a, b$ be positive in $A$. For a sufficiently large positive number $t$, $(a + t)b + b(a + t)$ becomes positive. By the assumption, we have

$$
(a + t)^2^n b + b(a + t)^2^n \geq 0 \quad \text{for all } n \in \mathbb{N}.
$$

By Corollary 7, we have $(a + t)b = b(a + t)$, i.e., $ab = ba$. Therefore $A$ is commutative. $\square$
Using the same method as the proof of (3)⇒(1), we can see the following condition (4) also becomes an equivalent condition in Theorem 2:

(4) Whenever positive operators \(a\) and \(b\) satisfy \(au(b) + u(b)a \geq 0\) for a function \(u\) as in Corollary 7, \(au^2(b) + u^2(b)a \geq 0\).

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