A three terms Arithmetic-Geometric mean

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0 Introduction

After the discovery of arithmetic geometric mean of Gauss in 1796, there has been proposed several its
variants and generalizations. There are some interesting studies among them, like the cubic-AGM of
Borweins in two terms case [B-B]. Also it has been aimed to find a new three terms AGM, for example
the trial of Richelot ([R], also [B-M]). But up to present time there was no nicely settled theory for it.
In this article we show a new AGM of three terms that has an expression by the Appell hypergeometric
function. Our AGM is related to the Picard modular form appeared in [S], this story will be published
elsewhere.

1 Definition

Definition 1.1 Let $a, b, c$ be real positive numbers satisfying $a \geq b \geq c$. Set

$$\Psi(a, b, c) = (\alpha, \beta, \gamma) = \left(\frac{a+b+c}{3}, \sqrt[3]{A}, \sqrt[3]{B}\right),$$

(1.1)

with

$$A = \frac{1}{6} (a^2 b + b^2 c + c^2 a + ab^2 + bc^2 + ca^2) + \sqrt[6]{\frac{4}{3}} (a-c)(a-b)(b-c),$$

and

$$B = \frac{1}{6} (a^2 b + b^2 c + c^2 a + ab^2 + bc^2 + ca^2) - \sqrt[6]{\frac{4}{3}} (a-c)(a-b)(b-c).$$

Here we choose the arguments of $\beta = \sqrt[3]{A}$ and $\gamma = \sqrt[3]{B}$ such that

$$0 \leq \arg \sqrt[3]{A} < \frac{\pi}{6}, \ 0 < \beta + \gamma.$$

Remark 1.1 If we make a permutation of $a, b, c$, it causes only the difference of the choice of complex
conjugates.

Lemma 1.1 For a triple $(\alpha, \beta, \gamma) = \Psi(a, b, c)$, we find uniquely determined real triple $(x, y, z) = T(\alpha, \beta, \gamma)$
such that

$$(\alpha, \beta, \gamma) = (x, y, z) \begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega^2 & \omega \\ 1 & \omega & \omega^2 \end{pmatrix}, \quad (\omega = \exp(2\pi i/3))$$

and

$$x > |y|, x > |z|.$$
We have Proposition 1.2. So we obtain the second inequality.

It holds

\[
\begin{align*}
\text{and} \\
&\text{Because of our choice of the argument the first inequality holds.}
\end{align*}
\]

For a real triple \((a, b, c)\) of complex numbers and \((x, y, z) = T(a, b, c)\), we have

\[
x = \alpha, \ x^3 - y^3 = \beta^3, \ x^3 - z^3 = \gamma^3
\]

with \((\alpha, \beta, \gamma) = \Psi(a, b, c)\), and we don’t mind the branches of \(\beta, \gamma\).

By using the above notations, we have

Proposition 1.1 For a real triple \((a, b, c)\) with \(a \geq b \geq c\) we can determine

\[
\Psi(\Psi(a, b, c)) = (a', b', c') = (x, \sqrt[3]{x^3 - y^3}, \sqrt[3]{x^3 - z^3}).
\]

Especially the twice composite of \(\Psi\) induces a real positive triple again.

Lemma 1.2 We have

\[
\frac{\beta^3 - \gamma^3}{\beta^3 - \alpha^3} \geq \frac{\beta - \gamma}{\beta - \alpha}\]

proof].

Put \(\beta = s + it\). Then

\[
|\beta^2 + \beta \gamma + \gamma^2| \geq |\beta - \gamma|^2 \iff s^2 \geq t^2/3.
\]

Because of our choice of the argument the first inequality holds.

We have

\[
|\alpha^3 - \beta^3||\alpha^3 - \gamma^3| = 27(x^2 + xy + y^2)(x^2 + xz + z^2)(y^2 + yz + z^2)
\]

and

\[
|\alpha - \beta|^3|\alpha - \gamma|^3 = 27(y^2 + yz + z^2)^3.
\]

It holds

\[
\begin{align*}
&\frac{(x^2 + xy + y^2) - (y^2 + yz + z^2)}{x - z}(x + y + z) = \alpha(x - z) > 0 \\
&\frac{(x^2 + xz + z^2) - (y^2 + yz + z^2)}{x - y}(x + y + z) = \alpha(x - z) > 0.
\end{align*}
\]

So we obtain the second inequality.

q.e.d.

Proposition 1.2 We have

\[
\begin{align*}
\frac{\beta^3 - \alpha^3}{\beta^3 - \gamma^3} &= \frac{1}{18\sqrt[3]{3}}((a - b)^2 + (b - c)^2 + (a - c)^2), \\
\frac{|\beta^3 - \gamma^3|}{|\beta - \gamma|} &= \frac{1}{3\sqrt[3]{3}}(a - c)(b - c)(a - b)
\end{align*}
\]

and

\[
\begin{align*}
&\frac{|\beta - \gamma|}{|\alpha - \beta|} \leq \frac{1}{\sqrt[3]{3\sqrt[3]{3}}}(a - c), \\
&\frac{|\alpha - \gamma|}{|\alpha - \beta|} \leq \frac{1}{4}(a - c), \ |\alpha - \gamma| \leq \frac{1}{4}(a - c).
\end{align*}
\]

proof].

We have

\[
\begin{align*}
\frac{|\beta^3 - \alpha^3|^2}{|\beta^3 - \gamma^3|^2} &= \frac{1}{27}\left((a^2 - ab + b^2 - ac - bc + c^2)^3\right) = \frac{1}{27\sqrt[3]{27}}((a - b)^2 + (b - c)^2 + (a - c)^2)^3.
\end{align*}
\]

So we get the equalities. By using the equalities and the above Lemma together with the inequalities

\[
(a - b)^2 + (b - c)^2 \leq (a - c)^2, \ (b - c)(a - b) \leq \frac{1}{4}(a - c)^2,
\]

we obtain the required inequalities.

q.e.d.
Proposition 1.3 We have

\[ |b' - c'| < 0.1982(a - c), \quad |d' - b'| < 0.2255(a - c), \quad |a' - c'| < 0.2255(a - c). \]  

[proof]. According to the above Proposition we have:

\[ |b' - c'| \leq |b' - c'| = |y^3 - z^3| = \frac{1}{3\sqrt{3}} |\beta - \gamma| |\alpha - \beta|^2 \]
\[ \leq \frac{1}{3\sqrt{3}} \frac{1}{\sqrt{4\sqrt{3}}} (a - c), \]
\[ |a' - c'| \leq 0.2255(a - c). \]

So it holds

\[ |y' - z'| \leq 2^{(-2/3)} 3^{(-4/3)} (a - c) < 0.1982(a - c), |y - z| \leq 2^{(-2/3)} 3^{(-4/3)} (a - c) < 0.1982(a - c). \]

\[ \sqrt{3}|y - \omega^2 z| = |\alpha - \beta| \leq \frac{1}{3} (a - c). \]

Hence

\[ |b' - a'|^3 \leq |b' - a'|^3 = |y|^3 \leq (\frac{1}{\sqrt{3}} (|y - z| + |z - \omega y|))^3 \leq \frac{1}{\sqrt{3}} (2^{(-2/3)} 3^{(-4/3)} + 3^{(-3/2)}) (a - c)^3. \]

So we have

\[ |a' - b'| < 0.2255(a - c). \]

By changing the roles of \( \beta \) and \( \gamma \) we have

\[ |a' - c'| < 0.2255(a - c). \]

By Proposition 1.2 and Proposition 1.3 we have the following.

Theorem 1.1 Let \( a, b, c \) be real positive numbers satisfying the condition \( a \geq b \geq c \). Set \( \Psi^n(a, b, c) = (a_n, b_n, c_n) = \Psi(a_{n-1}, b_{n-1}, c_{n-1}) \). Then there is a common limit

\[ M3(a, b, c) = \lim_{n \to \infty} a_n = \lim_{n \to \infty} b_n = \lim_{n \to \infty} c_n. \]

2 Functional equation and Appell’s HGDE

2.1 Functional equation

Let \( x, y \) be real positive numbers. By the limit process of

\[ \Psi(1, x, y) = \left( \frac{1 + x + y}{3}, \beta(1, x, y), \gamma(1, x, y) \right) = \left( \frac{1 + x + y}{3}, \frac{3\beta}{1 + x + y}, \frac{3\gamma}{1 + x + y} \right) \]

we have

\[ M3(1, x, y) = \frac{1 + x + y}{3} M3(1, \frac{3\beta}{1 + x + y}, \frac{3\gamma}{1 + x + y}). \]  

(2.1)

Set

\[ H(x, y) = \frac{1}{M3(1, x, y)}. \]

Proposition 2.1 We have

\[ H(x, y) = \frac{3}{1 + x + y} H(\frac{3\beta}{1 + x + y}, \frac{3\gamma}{1 + x + y}). \]
2.2 Appell’s Hypergeometric function $F_1$

According to Appell [App] we define the hypergeometric function $F_1$ with parameters $a, b, b', c$ (see also [Y]):

**Definition 2.1**

$$F_1(a, b, b', c; x, y) = \sum_{m, n \geq 0} \frac{(a, m + n)(b, m)(b', n)}{(c, m + n)m!n!} x^m y^n,$$

(2.2)

, where we use the conventional notation

$$(\lambda, k) = \begin{cases} \lambda(\lambda + 1) \cdots (\lambda + k - 1), & (k \geq 1) \\ 0, & (k = 0) \end{cases}$$

$f = F_1(a, b, b', c; x, y)$ satisfies the following linear partial differential equation

$$\begin{cases} x(1 - x)f_{xx} + (1 - x) yf_{xy} + (c - (a + b + 1)x)f_x - byf_y - abf = 0 \\ y(1 - y)f_{yy} + (1 - y)x f_{xy} + (c - (a + b' + 1)y)f_y - b'f_x - ab'f = 0 \\ (x - y)f_{xy} - b'f_x + bf_y = 0. \end{cases}$$

(2.3)

It has a three-dimensional space of solutions at a regular point, and it has unique holomorphic solution up to constant at the origin. The third equation is derived from the first and the second.

3 Differential equation

**Proposition 3.1** We have

$$F_1(1/3, 1/3, 1/3, 1; 1 - x^3, 1 - y^3) = \frac{3}{1 + x + y} F_1\left(1/3, 1/3, 1/3, 1; \left(\frac{1 + \omega x + \omega^2 y}{1 + x + y}\right)^3, \left(\frac{1 + \omega^2 x + \omega y}{1 + x + y}\right)^3\right).$$

We use the abbreviation

$F_1(x, y) = F_1(1/3, 1/3, 1/3, 1; x, y),$

and set

$$z(x, y) = F_1(1/3, 1/3, 1/3, 1; 1 - x^3, 1 - y^3).$$

We describe the differential equation for $z(x, y)$ and deform it to an equivalent system under a change of variables. At first we obtain the system

$$\begin{cases} \delta_1 = \frac{1}{x}(1 - x^3)\partial_{xx} + \frac{1}{y}(1 - y^3)\partial_{xy} - 3x\partial_x + \frac{1}{y}(1 - y^3)\partial_y - 1 \\ \delta_2 = \frac{1}{y}(1 - y^3)\partial_{yy} + \frac{1}{x}(1 - x^3)\partial_{xy} - 3y\partial_y + \frac{1}{x}(1 - x^3)\partial_x - 1 \\ \delta_3 = (x^3 - y^3)\partial_{xy} - y^2\partial_x + x^2\partial_y \end{cases}$$

(3.1)

for $z(x, y)$, where we use the convention $\partial_x = \frac{\partial}{\partial x}$ etc. Put

$$\begin{cases} X = \frac{1 + \omega x + \omega^2 y}{1 + x + y}, \\ Y = \frac{1 + \omega^2 x + \omega y}{1 + x + y}, \\ P = 1 + X + Y \\ Q = 1 + \omega X + \omega Y \\ R = 1 + \omega^2 X + \omega^2 Y \end{cases}$$
and
\[
\begin{align*}
D_X &= (\omega - X)\partial_X + (\omega^2 - Y)\partial_Y \\
D_Y &= (\omega^2 - X)\partial_X + (\omega - Y)\partial_Y.
\end{align*}
\]
Then we have
\[(1 + x + y)(1 + X + Y) = 3\]
and
\[
\begin{align*}
\partial_x &= \frac{1 + X + Y}{3} D_X \\
\partial_y &= \frac{1 + X + Y}{3} D_Y.
\end{align*}
\]
We can rewrite the system (3.1) in terms of \(X, Y, D_X, D_Y:\)
\[
\begin{align*}
\delta_1^z &= \frac{1}{3\sqrt{q}}(P^3 - Q^3)(D_X D_X - 3D_X + 2) + \frac{Q}{3\sqrt{p}}(P^3 - R^3)(D_X D_Y - D_Y - 2D_Y + 2) \\
- Q(D_X - 1) + \frac{1}{\sqrt{p}}(P^3 - R^3)(D_Y - 1) - 1
\end{align*}
\]
\[
\begin{align*}
\delta_2^z &= \frac{1}{3\sqrt{r}}(P^3 - R^3)(D_Y D_Y - 3D_Y + 2) + \frac{R}{3\sqrt{q}}(P^3 - Q^3)(D_Y D_X - D_X - 2D_X + 2) \\
- R(D_Y - 1) + \frac{1}{\sqrt{q}}(P^3 - Q^3)(D_X - 1) - 1,
\end{align*}
\]
for \(\tilde{Z} = \frac{Z}{1 + x + y} = \frac{Z(x, y)}{1 + x + y} = Z(X, Y) = Z.\) We obtain the system
\[
\begin{align*}
\delta_1^z Z(X, Y) &= \delta_2^z Z(X, Y) = 0 \iff \delta_1^z \tilde{Z}(X, Y) = \delta_2^z \tilde{Z}(X, Y) = 0.
\end{align*}
\]
On the other hand, we can rewrite the system (2.3) for the function \(F_1(X^3, Y^3)\) that is coming from the right hand side. Namely we get
\[
\begin{align*}
\delta_1^z &= \frac{1}{X}(1 - X^3)\partial_{XX} + \frac{Y}{X^2}(1 - X^3)\partial_{XY} + \frac{X}{1 - 3X^2}(1 - 3X^3)\partial_X - Y\partial_Y - 1 \\
\delta_2^z &= \frac{1}{Y}(1 - Y^3)\partial_{YY} + \frac{X}{Y^2}(1 - Y^3)\partial_{XY} + \frac{Y}{1 - 3Y^2}(1 - 3Y^3)\partial_Y - X\partial_X - 1 \\
\delta_3^z &= \frac{1}{XY^2}(Y^3 - X^3)\partial_{XY} + \frac{1}{X^2}\partial_X - \frac{1}{Y^2}\partial_Y.
\end{align*}
\]
By direct calculation we get the following equality of the differential operators:
\[
\begin{align*}
\frac{QR}{\sqrt{q}p}(R\delta_1^z + Q\delta_2^z) &= X(1 + 2X)(X - Y^2)\delta_1^z + Y(1 + 2Y)(Y - X^2)\delta_2^z + XY(Y - X)(XY - 1)\delta_3^z \\
\frac{QR}{\sqrt{q}p}(R\delta_1^z - Q\delta_2^z) &= -X(X - Y^2)\delta_1^z + Y(Y - X^2)\delta_2^z + XY(X + Y)(XY - 1)\delta_3^z.
\end{align*}
\]
So the function \(F_1(X^3, Y^3)\) in the right hand side satisfies the same hypergeometric differential equation for \(\tilde{Z}(X, Y)\). Because \(F_1(1 - x^3, 1 - y^3)_{|x=y=1} = F_1(0, 0) = 1\), we obtain the required equality.
q.e.d.

4 Main theorem

Theorem 4.1 We have
\[
\frac{1}{M3(1, x, y)} = H(x, y) = F_1(1/3, 1/3, 1/3, 1; 1 - x^3, 1 - y^3).
\]
in a neighborhood of \((x, y) = (1, 1)\).
Theorem 5.1
According to Appell we have integral representations for $F_1(a, b, b', c; x, y)$ as follows:

(1) If we have $\Re(a) > 0, \Re(c - a) > 0$, it holds

$$F_1(a, b, b', c; x, y) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(c - a)} \int_0^1 u^{a-1}(1 - u)^{c-a-1}(1 - xu)^{-b}(1 - yu)^{-b'} du$$

(2) If we have $\Re(b) > 0, \Re(b') > 0, \Re(c - b - b') > 0, |x| < 1, |y| < 1$, it holds

$$F_1(a, b, b', c; x, y) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(b')\Gamma(c - b - b')} \int_{u,v,1-u-v \geq 0} u^{b-1}v^{b'-1}(1 - u - v)^{-a}(1 - xu - yv)^{-a} dudv.$$ 

So our $M3(1, x, y)$ has expressions as period integrals for a family of Picard curves and at the same time for a family of certain elliptic K3 surfaces (see [S]).

Theorem 5.2 We have

$$\frac{1}{M3(1, x, y)} = \frac{1}{\Gamma(\frac{3}{4})^2} \int_1 ^{\infty} \frac{dv}{v} = \frac{1}{(\Gamma(\frac{3}{4}))^2} \int_{u,v,1-u-v \geq 0} \frac{dudv}{w},$$

where $w^3 = uv(1 - u - v)(1 - (1 - x^3) - v(1 - y^3))$

for $|x^3 - 1| < 1, |y^3 - 1| < 1$.

Remark 5.1 If we put $x = y$, our $M3$ coincides with the "cubic AGM" discovered by Borweins [B-B].

References


