Algebraic values of triangle Schwarz functions

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November 9, 2004

The classical triangle Schwarz functions $D(\nu_0, \nu_1, \nu_\infty; z)$ are defined as quotients of two linearly independent solutions of Gauss' hypergeometric differential equations. If their *angular parameters* ν_0, ν_1, ν_∞ are real and have absolute value in the open interval $]0, 1[$, they define biholomorphic mappings of the complex upper half plane $\mathcal{H}$ onto triangles in the Riemann sphere bounded by circular arcs. The singular points $0, 1, \infty$ of the differential equation are sent by D to the vertices of the triangle including there angles $\pi|\nu_0|, \pi|\nu_1|, \pi|\nu_\infty|$, respectively. Particularly interesting special cases are those where ν_0, ν_1, ν_∞ are the inverses of positive integers p, q, r because then D is the inverse function of an automorphic function for the triangle group with signature $\langle p, q, r \rangle$, isomorphic to the (projective) monodromy group of the hypergeometric differential equation.

The present paper considers the question if triangle Schwarz functions can have algebraic values at algebraic arguments. The problem has its origins in the natural general question if or under which conditions (suitably normalized) transcendental functions have transcendental values at algebraic arguments, and in this special context it is related to automorphic functions and periods of abelian varieties. For a general survey about algebraic and transcendental periods in number theory see Waldschmidt's recent article [Wa]. In the cases related to automorphic functions mentioned above the problem is treated already in our previous paper [SW] (Cor. 5). It turned out that a positive answer is directly related to the condition if certain Prym varieties are of complex multiplication (CM) type, the Pryms being defined in a natural way via the integral representation of the associated hypergeometric functions. Now we generalize the setting and consider arbitrary rational angular parameters ν_0, ν_1, ν_∞, restricted only by some mild technical condition excluding logarithmic singularities and some other very special situations. The main results will show that we have still 'CM' as necessary condition for 'algebraic values at algebraic arguments', but that even under the CM condition this algebraicity is rather exceptional. However, we will give examples that such exceptions occur.

This more general type of triangle functions has still images of $\mathcal{H}$ bounded by parts of circles but they are in general not globally biholomorphic — the image domains may overlap. We treated in [STW] an analogous problem admitting apparent singularities in the associate Fuchsian differential equations. The triangle functions of the present paper may in fact be considered as limit cases of those of [STW], and many techniques developed there are useful also for the problem treated in the present paper. Therefore we collect in Section 1 some known material mainly from [STW], [SW], [Wo]. In Section 2 we state and prove the main results. The methods rely in part on the classical theory of hypergeometric functions, in part on the consideration of families of abelian varieties, and in part on Wüstholz' transcendence techniques [Wü]. Section 3 presents instructive examples.

1 Families of Prym varieties and associate functions

1.1 Integral representation by the periods on curves

Throughout this paper we will suppose that the angular parameters satisfy

$$\nu_0,\ \nu_1,\ \nu_\infty \in \mathbb{Q} - \mathbb{Z}\,, \quad \nu_0 \pm \nu_1 \pm \nu_\infty \notin \mathbb{Z}\,. \tag{1.1}$$

We will use the integral representation of the Gauss hypergeometric function $F(a,b,c;z)$ — omitting the usual normalizing Beta factor and some algebraic nonzero factors, see Section 5 of [STW] for a careful discussion — in the form

$$\int_\gamma u^{a-c}(u-1)^{c-b-1}(u-z)^{-a}du \;=\; \int_\gamma u^{-\mu_0}(u-1)^{-\mu_1}(u-z)^{-\mu_z}du \;=\; \int_\gamma \eta(z)$$

with the (rational) exponents

$$\begin{aligned}
\mu_0 &= \frac{1}{2}(1-\nu_0+\nu_1-\nu_\infty)\\
\mu_1 &= \frac{1}{2}(1+\nu_0-\nu_1-\nu_\infty)\\
\mu_z &= \frac{1}{2}(1-\nu_0-\nu_1+\nu_\infty)\\
\mu_\infty &= \frac{1}{2}(1+\nu_0+\nu_1+\nu_\infty)\\
\mu_0+\mu_1+\mu_z+\mu_\infty &= 2
\end{aligned}$$

for some Pochhammer cycle γ around two of the singularities $0, 1, z, \infty$. As already remarked by Klein [K], analytic continuation of $F(a,b,c;z)$ means only to replace γ by another cycle of integration, and a basis of solutions of the corresponding hypergeometric differential equation will be obtained by taking two Pochhammer cycles around different pairs of singularities: remark that our hypothesis on the sums of the angular parameters guarantees that no exponent μ_j is an integer, whence all singularities are nontrivial. For fixed arguments $z \neq 0, 1, \infty$ this integral representation can be seen as a period integral on a nonsingular projective model $X(k,z) = X(\mu_0, \mu_1, \mu_z; z)$ of the algebraic curve

$$y^k \;=\; u^{k\mu_0}\,(u-1)^{k\mu_1}\,(u-z)^{k\mu_z} \tag{1.2}$$

where k is the least common denominator of the μ_j , γ some homology cycle on $X(k,z)$, and η a differential given on the singular model as

$$\eta \;=\; \eta(z) \;=\; \frac{du}{y}$$

of the second kind what can be seen using appropriate local variables ([Wo]; N. Archinard [Ar] explains in more detail the desingularization procedure). Our triangle Schwarz map is a multivalued analytic function on $\mathbb{C} - \{0,1\}$ defined by

$$D(\nu_0,\nu_1,\nu_\infty;z) \;=\; D(\eta;z) \;=\; D(z) \;=\; \frac{\int_{\gamma_1}\eta(z)}{\int_{\gamma_2}\eta(z)}$$

for some *independent* cycles γ_1,γ_2 on $X(k,z)$.

In the next subsection we will give a precise definition of *independence* for these cycles, for the moment we can assume that they come from Pochhammer cycles around different pairs of singularities and are locally independent of $z \neq 0,1,\infty$. The triangle functions extend continuously to the arguments excluded here, and our normalization guarantees that $D(0), D(1), D(\infty)$ become algebraic or ∞ , see [STW], Section 3.1. For later use recall the relation between angular and exponential parameters and a,b,c .

$$\begin{array}{rcl} \nu_0 = & 1-c & = 1-\mu_0-\mu_z = \mu_1+\mu_\infty-1 \\ \nu_1 & = c-a-b = & 1-\mu_1-\mu_z = \mu_0+\mu_\infty-1 \\ \nu_\infty = & b-a & = \mu_z+\mu_\infty-1 = 1-\mu_0-\mu_1\,. \end{array} \tag{1.3}$$

1.2 The family of Prym varieties

The family of Prym varieties in question can be described as follows. For all proper divisors d of k there is an obvious morphism of the curve $X(k,z)$ onto the curve $X(d,z)$ in whose definition 1.2 we keep fixed the exponential parameters $k\mu_i$ on the right hand side and replace k by d as exponent of y . These morphisms induce epimorphisms

$$\mathrm{Jac}\, X(k,z) \;\to\; \mathrm{Jac}\, X(d,z)\;.$$

Let $T(k,z)$ be the connected component of 0 in the intersection of all kernels of these epimorphisms. Then it is known by [Wo], [Ar] that $T(k,z)$ is an abelian variety of dimension $\varphi(k)$ where φ denotes Euler's function. $T(k,z)$ has generalized complex multiplication by the cyclotomic field

$$\mathbb{Q}(\zeta_k) \;\subseteq\; \mathrm{End}_0 T(k,z) \;:=\; \mathbb{Q}\otimes_{\mathbb{Z}} \mathrm{End}\, T(k,z)$$

induced by an automorphism of the curve $X(k,z)$ described on its singular model by

$$\alpha \;:\; (y,u) \;\mapsto\; (\zeta_k^{-1}y,u)\;, \quad \zeta_k \;=\; e^{\frac{2\pi i}{k}}\;.$$

If $\langle s\rangle$ denotes the fractional part $s-[s]$ of $s\in\mathbb{Q}$, the CM type of $T(k,z)$ can be easily calculated in terms of the μ_j by

$$r_n \;=\; \dim W_n \;=\; -1+\sum_j \langle \mu_j n\rangle\ ,$$

where W_n denotes the eigenspace for the eigenvalue ζ_k^n for the action of α on the vector space $H^0(T(k,z),\Omega)$ of the first kind differentials, see e.g. [SW] (on p.23 use formula (4) with $N=2$). Note that r_n can take the values $0,1,2$ only and satisfies $r_n+r_{-n}=2$ for all n.

In the following we will consider the second kind differentials η always as differentials on $T(k,z)$ and the cycles γ_1,γ_2 as cycles of the homology in $T(k,z)$. This homology $H_1(T(k,z),\mathbb{Z})$ is a $\mathbb{Z}[\zeta_k]$–module of rank two, and *independence* of the cycles in the definition of the normalized Schwarz triangle function $D(z)=\int_{\gamma_1}\eta(z)/\int_{\gamma_2}\eta(z)$ means now $\mathbb{Q}(\zeta_k)$–linear independence in the $\mathbb{Q}(\zeta_k)$–module $H_1(T(k,z),\mathbb{Q})=\mathbb{Q}\otimes_{\mathbb{Z}}H_1(T(k,z),\mathbb{Z})$. Note that for algebraic z the curve, its Jacobian, its Prym variety $T(k,z)$ and the differential $\eta(z)$ are all defined over number fields.

1.3 Associate functions

As common in the literature about hypergeometric functions, we call two hypergeometric functions $F(a,b,c;z)$, $F(a',b',c';z)$ *associate* if

$$a\equiv a'\ ,\ b\equiv b'\ ,\ c\equiv c' \bmod \mathbb{Z}$$

or equivalently, if the respective angular parameters satisfy

$$\nu_0\equiv\nu_0'\ ,\ \nu_1\equiv\nu_1'\ ,\ \nu_\infty\equiv\nu_\infty' \bmod \mathbb{Z}\quad\text{and}\quad \nu_0+\nu_1+\nu_\infty\equiv\nu_0'+\nu_1'+\nu_\infty' \bmod 2\mathbb{Z}$$

or if the respective exponential parameters satisfy

$$\mu_j\equiv\mu_j' \bmod \mathbb{Z}\quad\text{for all}\quad j=0,1,z,\infty\quad\text{and}\quad \sum_j\mu_j=\sum_j\mu_j'=2\ .$$

All functions associate to $F(a,b,c;z)$ generate a vector space of dimension two over the field of rational functions $\mathbb{C}(z)$, and since our parameters are supposed to be rational, between any three associate functions there is a linear relation with coefficients in $\mathbb{Q}(z)$. These relations can explicitely produced by means of Gauss' relations between contiguous functions, see [EMOT]. Any two associate hypergeometric functions generate the vector space over $\mathbb{C}(z)$ (obvious exceptions like $F(a,a+1,c;z)$, $F(a+1,a,c;z)$ are excluded by our assumptions about the angular parameters). The congruences for the exponential parameters imply that the differentials η,η' differ only by factors which are rational functions $R(u,z)\in\mathbb{Q}(u,z)$. As second kind differentials on the Prym variety $T(k,z)$ they belong therefore to the same $\mathbb{Q}(\zeta_k)$–eigenspace V_1 in its de Rham cohomology. In our normalization, the differentials of this eigenspace are characterized by $\eta\circ\alpha=\zeta_k\eta$. The intersection of V_1 with $H^0(T(k,z),\Omega)$ gives the eigenspace W_1 mentioned in the definition of the CM type. This observation extends to the other eigenspaces V_n, $n \in (\mathbb{Z}/k\mathbb{Z})^*$, and the fact that all associate hypergeometric functions generate a 2–dimensional vector space over $\mathbb{C}(z)$ has an obvious interpretation for the eigenspaces V_n in the de Rham cohomology:

Lemma 1.1 $\dim V_n = 2$ *for all* $n \in (\mathbb{Z}/k\mathbb{Z})^*$.

Dimension means here the dimension over $\mathbb{C}$, but for algebraic z we can give another useful interpretation: as already mentioned, $T(k,z)$ is then defined over $\overline{\mathbb{Q}}$, all differentials η in the integral representation are defined over $\overline{\mathbb{Q}}$ as well whence we consider the vector spaces $H^0(T(k,z),\Omega)$, $H^1_{DR}(T(k,z)$, V_n of differentials of the first and second kind defined over $\overline{\mathbb{Q}}$ as vector spaces over $\overline{\mathbb{Q}}$. In this sense, the Lemma remains true as a statement about $\overline{\mathbb{Q}}$–dimensions. For $z \in \overline{\mathbb{Q}}$ we will follow this interpretation.
In the proof of Lemma 1 there is only one point which is not obvious: even if associate differentials $\eta(z)$ generate a 2–dimensional $\mathbb{C}(z)$–vector space modulo exact differentials, it could be possible that for some fixed value $z = \tau$ the $\mathbb{C}$–dimension would be smaller if e.g. all differentials in question vanish for $z = \tau$. This breakdown of the dimension can be seen to be impossible for $\tau \neq 0, 1, \infty$ either by a careful analysis of the possible relations between contiguous functions or by the fact that the genus of $X(k,z)$ is the same for all $z \neq 0, 1, \infty$.

1.4 Tools from transcendence

Now we suppose $z = \tau \in \overline{\mathbb{Q}}$ and consider all eigenspaces V_n as vector spaces over $\overline{\mathbb{Q}}$. The consequences of Wüstholz' analytic subgroup theorem [Wü] for abelian varieties have been worked out by Paula Cohen in Section 6 of [STW]. They imply in particular

Lemma 1.2 *Suppose* $\tau \in \overline{\mathbb{Q}}, \neq 0, 1$ *and that* $T(k,\tau)$ *is a simple abelian variety with* $\mathbb{Q}(\zeta_k) = \mathrm{End}_0 T(k,\tau)$ *. Then all periods*

$$\int_\gamma \eta\,, \quad \gamma \in H_1(T(k,\tau),\mathbb{Z})$$

of a fixed nonzero $\eta \in V_n \subset H^1_{DR}(T(k,\tau))$ *generate a* $\overline{\mathbb{Q}}$*–vector space* Π_η *of dimension* 2 *,*

(see [STW], Prop. 4.1) and more general

Lemma 1.3 *Suppose* $\tau \in \overline{\mathbb{Q}}, \neq 0, 1$ *and that* $T(k,\tau)$ *is a simple abelian variety with* $\mathbb{Q}(\zeta_k) = \mathrm{End}_0 T(k,\tau)$ *. Then for every fixed* $n \in (\mathbb{Z}/n\mathbb{Z})^*$ *all periods*

$$\int_\gamma \eta\,, \quad \gamma \in H_1(T(k,\tau),\mathbb{Z})$$

of all second kind eigendifferentials $\eta \in V_n \subset H^1_{DR}(T(k,\tau))$ *generate a* $\overline{\mathbb{Q}}$*–vector space* Π_n *of dimension* 4 *.*

The proof results from the fact that 4 is an obvious upper bound for the dimension, coming from Lemma 1 and the action of the endomorphism algebra $\mathbb{Z}[\zeta_k]$ on the differentials and the homology. On the other hand this upper bound is attained because the space generated by all periods of the second kind on $T(k,\tau)$ has $\overline{\mathbb{Q}}$–dimension $4\varphi(k)$, see [STW], Thm. 6.1.

1.5 Splitting Prym varieties

There are cases in which $T(k,\tau)$ is not simple or $\mathrm{End}_0 T(k,\tau)$ is strictly larger than $\mathbb{Q}(\zeta_k)$ by other reasons. By Bertrand [Be], Section 1, Example 3, we know which kind of splitting is possible. Lemma 4 and Lemma 5 treat the two possibilities. All these remaining cases have in common that $T(k,\tau)$ is of *CM type*, i.e. it is isogenous to a product of simple abelian varieties with complex multiplication.

Lemma 1.4 *Suppose $\tau \in \overline{\mathbb{Q}}, \neq 0,1$ and that $T(k,\tau)$ is an abelian variety whose $\mathrm{End}_0 T(k,\tau)$ is strictly larger than $\mathbb{Q}(\zeta_k)$ but not containing zero divisors commuting with $\mathbb{Q}(\zeta_k)$. Then $K \subseteq \mathrm{End}_0 T(k,\tau)$ for a CM field K, $[K:\mathbb{Q}(\zeta_k)] = 2$. All periods*

$$\int_\gamma \eta\,, \quad \gamma \in H_1(T(k,\tau),\mathbb{Z})$$

of a nonzero second kind $\mathbb{Q}(\zeta_k)$–eigendifferential $\eta \in V_n \subset H^1_{DR}(T(k,\tau))$ generate a $\overline{\mathbb{Q}}$–vector space Π_η of dimension

- 1 *if η is a K–eigendifferential,*
- 2 *if not.*

The first case happens in precisely two onedimensional subspaces of V_n. For the period space Π_n of Lemma 3 we have in both cases $\dim \Pi_n = 2$.

For the proof compare the arguments of [STW], Prop. 4.2 : $T(k,\tau)$ is isogenous to a power B^m of a simple abelian variety with complex multiplication by some subfield L of K, and V_n splits into two L–eigenspaces for factors B but for different eigenvalues. Then the result follows again from Wüstholz' subgroup theorem. The other alternative is

Lemma 1.5 *Suppose $\tau \in \overline{\mathbb{Q}}, \neq 0,1$ and that $T(k,\tau)$ is an abelian variety with $\mathbb{Q}(\zeta_k) \subset \mathrm{End}_0 T(k,\tau)$, isogenous to $A_1 \oplus A_2$ for two abelian varieties of A_i dimension $\frac{1}{2}\varphi(k)$ and with endomorphism algebra $\mathrm{End}_0 A_i \subseteq \mathbb{Q}(\zeta_k)$. Then both A_i are of CM type.*

1. *If A_1 and A_2 have the same CM type, all periods*

$$\int_\gamma \eta\,, \quad \gamma \in H_1(T(k,\tau),\mathbb{Z})$$

of a fixed nonzero $\eta \in V_n \subset H^1_{DR}(T(k,\tau))$ generate a $\overline{\mathbb{Q}}$–vector space Π_η of dimension 1, and $\Pi_n = \Pi_\eta$.

2. *If A_1 and A_2 have different CM types, we have* $\dim \Pi_n = 2$ *, and the periods of of every fixed* $0 \neq \eta \in V_n$ *generate a 2–dimensional vector space* Π_η *over* $\overline{\mathbb{Q}}$ *, except in the case that* η *belongs to one of the factors in the decomposition*

$$H^1_{DR}(T(k,\tau)) \;=\; H^1_{DR}(k,A_1) \oplus H^1_{DR}(k,A_2)\;.$$

In this case (happening in precisely two onedimensional subspaces of V_n *)* Π_η *is of dimension* 1 *.*

In both cases the A_i are isogenous to pure powers $B_i^{m_i}$ of simple abelian varieties B_i with complex multiplication. In the first case, B_1 and B_2 are isogenous, and in the second case not. Then the result follows again by Wüstholz' analytic subgroup theorem in the version of [STW], Thm. 6.1.
Finally we give precise conditions under which the first case of Lemma 5 can occur. Note that these conditions do not depend on the algebraicity of z .

Lemma 1.6 *The following statements are equivalent.*

- *For one (hence for all)* $z \neq 0,1,$ *the CM type of* $T(k,z)$ *satisfies*

$$r_n \;=\; 0 \quad or \quad 2 \qquad for\ all \quad n \in (\mathbb{Z}/k\mathbb{Z})^*\;.$$

- *For one (hence for all)* $z \neq 0,1$ *, the abelian variety* $T(k,z)$ *is isogenous to* $A_1 \oplus A_2$ *, both* A_i *have dimension* $\varphi(k)$ *with* $\mathrm{End}_0 A_i \subseteq \mathbb{Q}(\zeta_k)$ *and have equal CM type.*

- *The monodromy group of the corresponding hypergeometric differential equation is finite.*

- *The corresponding triangle function* $D(\nu_0,\nu_1,\nu_\infty;z)$ *is an algebraic function of* z *.*

The equivalence between the first and the second point is known by work of Shimura [Sh], for the equivalence between the second and the third point see [STW], Prop. 4.3, or [Wo], and the equivalence between the third and the fourth point are classical, see e.g. [K].

2 Special values of triangle Schwarz functions

2.1 The role of complex multiplication

We work still under the hypothesis $z = \tau \in \overline{\mathbb{Q}}$ and recall that the cycles γ_1, γ_2 in the definition $D(\nu_0,\nu_1,\nu_\infty;\tau) = D(\tau) = \int_{\gamma_1} \eta(\tau)/\int_{\gamma_2} \eta(\tau)$ are generators of the 2–dimensional $\mathbb{Q}(\zeta_k)$–module $H_1(T(k,\tau),\mathbb{Q}) = \mathbb{Q} \otimes_{\mathbb{Z}} H_1(T(k,\tau),\mathbb{Z})$. Numerator and denominator generate the period vector space Π_η discussed in the last section. We conclude from Lemmata 1.2, 1.4 and 1.5

Theorem 2.1 *Suppose* $\tau \in \overline{\mathbb{Q}}, \neq 0, 1$.

$$D(\nu_0, \nu_1, \nu_\infty; \tau) \;=\; D(\tau) \;=\; \frac{\int_{\gamma_1} \eta(\tau)}{\int_{\gamma_2} \eta(\tau)}$$

is algebraic or ∞ *if and only if* $T(k, \tau)$ *is of CM type and* $\dim_{\overline{\mathbb{Q}}} \Pi_{\eta(\tau)} = 1$, *i.e. if* $\eta(\tau)$ *is a*

- K*–eigendifferential under the hypotheses of Lemma 1.4, or a*
- $\mathbb{Q}(\zeta_k)$*–eigendifferential on one of the factors* A_1, A_2 *under the hypotheses of Lemma 1.5.*

In two special situations we can give more concrete conditions. The first is obvious by Lemma 1.6.

Theorem 2.2 *If the monodromy group* Δ *of the corresponding differential equation is finite, all values* $D(\tau)$ *of the triangle function at algebraic arguments* τ *are algebraic or* ∞ .

In the following we will therefore restrict our attention to infinite monodromy groups Δ . In these cases, we know by Lemma 6 that at least one $r_n = 1$, in other words one $W_n = V_n \cap H^0(T(k, \tau), \Omega)$ contains a nonzero differential $\eta = \omega$ of the first kind, unique up to multiples. For periods of the first kind we can apply a sharper version of Wüstholz' theorem giving a period vector space Π_ω of dimension 1 if the abelian variety has CM type. Another way to prove $\dim \Pi_\omega = 1$ is a second look on Lemma 1.4 and Lemma 1.5 : In Lemma 1.4, $H^0(T(k, \tau), \Omega)$ is K–invariant, therefore W_n is one of the onedimensional subspaces of K–eigendifferentials. In Lemma 1.5, W_n belongs to precisely one of the homology factors $H^0(A_i, \Omega)$ since only one of them contains eigendifferentials ω with $\omega \circ \alpha = \zeta_k^n \omega$, otherwise we would have $\dim W_n = 2$. Summing up we get (see also [SW], Cor. 5 for a different argument)

Theorem 2.3 *Suppose* $\tau \in \overline{\mathbb{Q}}, \neq 0, 1$ *, and that* $T(k, \tau)$ *is of CM type, let* W_n *be a onedimensional* $\mathbb{Q}(\zeta_k)$*–eigenspace in* $H^0(T(k, \tau), \Omega)$ *. If* $0 \neq \omega = \eta(\tau) \in W_n$ *, the value of the corresponding triangle function* $D(\tau) = \int_{\gamma_1} \eta(\tau) / \int_{\gamma_2} \eta(\tau)$ *is algebraic.*

The first natural question is now: how to control that η is of first kind? For simplicity, take $n = 1$. There $\eta = du/y$ — see Section 1.1 — is of first kind if and only if the exponential parameters μ_j are all < 1 . The second question is already much more difficult: for which $\tau \in \overline{\mathbb{Q}}$ is $T(k, \tau)$ of CM type? The answer depends on the nature of the monodromy group Δ and, unfortunately, does not give a general explicit criterion for the distinction between CM and non–CM cases.

1. If Δ is finite, $T(k, \tau)$ is of CM type for every $\tau \in \overline{\mathbb{Q}}$, see Lemma 1.6.

2. If Δ is an arithmetic group, there is an infinity of $T(k,\tau)$ of CM type and an infinity of $T(k,\tau)$ not of CM type. In these cases — classified by Takeuchi [Ta] — Δ is commensurable to the modular group for a complex onedimensional family of polarized abelian varieties with a certain endomorphism structure. Our $T(k,z)$, $z \neq 0,1$, form a dense subset of this family, and the Schwarz triangle function D is the inverse function of an arithmetic automorphic function for this modular group, possibly up to composition with an algebraic function.

3. If Δ is infinite and non–arithmetic, the $T(k,z)$ form a subfamily not of Hodge type in the Shimura variety of all polarized abelian varieties of their endomorphism structure. In this case, the André–Oort conjecture predicts that there are only finitely many $T(k,\tau)$ of CM type. This conjecture is proven by Edixhoven and Yafeev [EY] for those CM types discussed in Lemma 1.5.2, but it is open in general. For more information and applications to other hypergeometric questions see [CWü].

2.2 Other algebraic values at algebraic arguments

The aim of this part is to show that Theorems 2.2 and 2.3 describe very exceptional situations, i.e. that in general for $\tau \in \overline{\mathbb{Q}} - \{0,1\}$

$$D(\nu;\tau) \ = \ D(\nu_0,\nu_1,\nu_\infty;\tau) \notin \overline{\mathbb{Q}}$$

even if the necessary condition given by Theorem 1 is satisfied that $T(k,\tau)$ is of CM type. We used here an abbreviated notation $\nu := (\nu_0,\nu_1,\nu_\infty)$ for the rational triples of angular parameters (always under the restriction (1.1)). We call two such triples ν,ν' *associate* if they belong to associate hypergeometric functions, see the conditions on their components given in section 1.3. Observe that triangle functions with associate angular parameters belong to the same monodromy group.

Theorem 2.4 *Let P be a finite set of associate rational angular parameter triples ν, belonging to an infinite monodromy group Δ. There is a finite set $E_P \subset \overline{\mathbb{Q}}$ of exceptional arguments such that for all $\tau \in \overline{\mathbb{Q}} - E_P$ at most two of the values $D(\nu;\tau)$, $\nu \in P$, are algebraic or ∞.*

For the notation, we may assume $0,1 \in E_P$, and for the proof we may assume that $T(k,\tau)$ is of CM type since we know by Theorem 2.1 that otherwise all values in question are transcendental. Theorem 2.4 uses Lemma 1.1 and classical facts about associate hypergeometric functions: denote the differentials in the integral representation of $D(\nu,z), \nu \in P$ by $\eta(\nu;z)$ and observe that all these $\eta(\nu;z), \nu \in P$, belong to one eigenspace V_n. By Gauss' relations among contiguous hypergeometric functions, any two of them generate V_n as a $\mathbb{C}(z)$–vector space. The only obstacle is mentioned already in Section 1.3 that for a fixed value $z=\tau$ they may fail to be a basis over $\mathbb{C}$ or $\overline{\mathbb{Q}}$. A closer look into Gauss' relations [EMOT] shows that this can happen only at finitely many algebraic points since the relations always have coefficients in $\mathbb{Q}(z)$: For any two different fixed $\eta(\nu';z),\eta(\nu'';z)$, $\nu',\nu''\in P$ and every other $\nu\in P$ we get a representation

$$\eta(\nu;z) \;=\; r'(z)\eta(\nu';z) \;+\; r''(z)\eta(\nu'';z)$$

with nonvanishing rational functions $r',r''\in\mathbb{Q}(z)$. We can use these relations in all special points $\tau\in\overline{\mathbb{Q}}$ as relations over $\overline{\mathbb{Q}}$, except for the finitely many algebraic poles of r',r''. If we include these finitely many exceptions in our exceptional set E_P, in all other points $\tau\in\overline{\mathbb{Q}}$ the $\eta(\nu,\tau),\nu\in P$, generate pairwise different onedimensional subspaces of V_n. Lemma 1.4 and Lemma 1.5.2 show that in only two such onedimensional subspaces the period vector spaces Π_η are of dimension 1, and this is equivalent to the algebraicity of the period quotient $D(\nu;\tau)$.

As an example why these exceptional arguments can occur take relation (28) on p. 103 of [EMOT]

$$(c-a)F(a-1,b,c;z)+(2a-c-az+bz)F(a,b,c;z)+a(z-1)F(a+1,b,c;z) \;=\; 0\,.$$

Translated to the language of differentials and angular parameters it says that in the special point $\tau=(2a-c)/(a-b)=(\nu_1+\nu_\infty)/\nu_\infty$ the associate differentials

$$\eta(\nu_0,\nu_1+1,\nu_\infty-1;\tau)\;,\quad \eta(\nu_0,\nu_1-1,\nu_\infty+1;\tau)$$

are $\overline{\mathbb{Q}}$–linearly dependent and give there the same period quotient $D(\tau)$. Whether or not this value is algebraic depends of course on two further conditions, i.e.

- if $T(k,\tau)$ is of CM type and
- if the $\eta(\tau)$ generate one of the two onedimensional eigenspaces mentioned in Lemma 1.4 or Lemma 1.5.2.

In the case treated in Theorem 2.3 we can better localize at least one of these onedimensional eigenspaces — it is *the* subspace of differentials of the first kind — whence we get a sharper result.

Theorem 2.5 *Let P be a finite set of associate rational angular parameter triples ν, belonging to an infinite monodromy group Δ, and suppose further that there is precisely one first kind differential $\omega=\eta(\nu';z)$ associate to these $\eta(\nu;z)$, $\nu\in P$, but with $\nu'\notin P$. Then there is a finite set $E_P\subset\overline{\mathbb{Q}}$ of exceptional arguments such that for all $\tau\in\overline{\mathbb{Q}}-E_P$ at most one of the values $D(\nu;\tau)$, $\nu\in P$, is algebraic or ∞.*

3 Examples of the hypergeometric Schwarz map

As for the special values of the Schwarz map $D(z)$ for a differential $\eta(z)$ of a family of hypergeometric curves, we have established the general properties in preceding sections. Here we show some concrete examples which explain the situation in question. They all arise from specialisations of a family of curves studied in the framework of ball quotients and Appell–Lauricella hypergeometric functions in two variables.

3.1 Pentagonal curves and their degeneration

Let us consider a family of hypergeometric curves given by (1.2):

$$X(\mu_0, \mu_1, \mu_z; z) = X(p, z) = X(z) :$$

$$y^p = x^{p\mu_0}(x-1)^{p\mu_1}(x-z)^{p\mu_z} \quad (z \in \mathbb{C} - \{0, 1\}), \tag{3.1}$$

where we suppose p is a prime integer and μ_0, μ_1 and μ_∞ satisfy the non-integrality condition (1.1). We defined the Prym variety $T(k, z)$ for $X(z)$ that is induced from the Jacobi variety $\mathrm{Jac}(X(z))$. If k is a prime number, then $T(k, z)$ coincides with $\mathrm{Jac}(X(z))$. So in our case the field $\mathbb{Q}(\zeta_p)$ acts on the space of holomorphic differentials $H^0(\mathrm{Jac}(X(z)), \Omega) \cong H^0(X(z), \Omega)$ with parameter z. We note also that the $\mathbb{Q}(\zeta_p)$ - action on $X(z)$ induces a $\mathbb{Q}(\zeta_p)$ module structure on $H_1(X(z), \mathbb{Z})$ of rank two. Let γ_1, γ_2 be two 1-cycles on $X(z)$ independent over $\mathbb{Q}(\zeta_p)$ and let

$$\eta(\ell, m, n; z) = \eta(z) = x^{-\mu_0+\ell}(x-1)^{-\mu_1+m}(x-z)^{-\mu_z+n}dx \quad (\ell, m, n \in \mathbb{Z})$$

be a differential of second kind on $X(z)$. Then the correspnding Schwarz map is defined by

$$D(\eta, z) = D(z) = \frac{\int_{\gamma_1} \eta(z)}{\int_{\gamma_2} \eta(z)}. \tag{3.2}$$

Let $P(\lambda_1, \lambda_2)$ be the compact nonsingular model of the affine curve

$$y^5 = x(x-1)(x-\lambda_1)(x-\lambda_2), \quad (\lambda_1, \lambda_2, \lambda_1/\lambda_2, \in \mathbb{C} - \{0, 1\}). \tag{3.3}$$

$P(\lambda_1, \lambda_2)$ is a curve of genus 6 and is called a pentagonal curve. There are many articles concerned with this family. We cite here just one by K. Koike [Ko]. We have a basis of $H^0(P(\lambda_1, \lambda_2), \Omega)$:

$$\varphi_1 = \frac{dx}{y^2}, \varphi_2 = \frac{dx}{y^3}, \varphi_3 = \frac{xdx}{y^3}, \varphi_4 = \frac{dx}{y^4}, \varphi_5 = \frac{xdx}{y^4}, \varphi_6 = \frac{x^2dx}{y^4}. \tag{3.4}$$

Let $DegP(z)$ be the compact nonsingular model of

$$y^5 = x^2(x-1)(x-z) \quad (z \in \mathbb{C} - \{0, 1\}). \tag{3.5}$$

It is a degenerate pentagonal curve of genus 4. There is a natural ζ_5–action

$$(x, y) \mapsto (x, \zeta_5^{-1} y).$$

So we have

$$\mathbb{Q}(\zeta_5) \subset \mathrm{End}_0(\mathrm{Jac}\ (DegP(z))).$$

We have a basis of $H^0(DegP(z), \Omega)$:

$$\omega_1 = \frac{dx}{y^2}, \omega_2 = \frac{xdx}{y^3}, \omega_3 = \frac{xdx}{y^4}, \omega_4 = \frac{x^2dx}{y^4}. \tag{3.6}$$

They are eigendifferentials for the action of $\mathbb{Q}(\zeta_5)$.

Remark 3.1 *We note that ω_3 and ω_4 are mutually associate.*

In general we have a solution for the Gauss hypergeometric differential equation

$$E(a,b,c) : z(1-z)f'' + (c-(1+a+b)z)f' - abf = 0$$

given by the integrals

$$\begin{aligned}
&e^{-\pi i(-c+b+1-a)} \int_1^\infty x^{a-c}(x-1)^{c-b-1}(x-z)^{-a}dx \\
&= e^{-\pi i(-c+b+1-a)} \int_0^1 u^{b-1}(1-u)^{c-b-1}(1-zu)^{-a}du \\
&= \int_1^\infty x^{a-c}(1-x)^{c-b-1}(z-x)^{-a}dx = F_{1\infty}(a,b,c;z)
\end{aligned}$$

with

$$ux = 1, 1-x = e^{-\pi i}(x-1), z-x = e^{\pi i}(x-z).$$

That is single valued holomorphic at $z=0$ and

$$F(a,b,c;z) = e^{\pi i(1-c+b-a)} \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} F_{1\infty}(a,b,c;x).$$

Then the integral

$$\int_1^\infty \omega_1(z)$$

is a holomorphic solution of $E(2/5, 3/5, 6/5)$ at $z=0$. The absolute values of the angular parameters are given by

$$(|1-c|, |c-a-b|, |a-b|) = (\frac{1}{5}, \frac{1}{5}, \frac{1}{5}).$$

By putting $x = 1/x_1, y = z^{1/5}y_1/x_1, z = 1/z_1$ in (3.5) we obtain an isomorphic nonsingular curve given by

$$y_1^5 = x_1(x_1-1)(x_1-z_1),$$

the integral $\int dx_1/y_1^2$ gives a solution for $E(2/5, 1/5, 4/5)$ with the same angular parameters in absolute values. So the inverse of the Schwarz map becomes to be an automorphic function on the upper half plane with respect to a co-compact arithmetic triangle group $\Delta(5,5,5)$. An explicit expression of this automorphic function is given by Koike [Ko], Theorem 6.3.

3.2 First example

Define

$$\Sigma : w^5 = t^2(t^3 - 1). \tag{3.7}$$

It is a singular model of a curve of genus 4 and we have on this model a basis of the space of holomorphic differentials:

$$\varphi_1 = \frac{dt}{w^2}, \varphi_2 = \frac{tdt}{w^3}, \varphi_3 = \frac{tdt}{w^4}, \varphi_4 = \frac{t^2 dt}{w^4}. \tag{3.8}$$

There are actions of ζ_3 and ζ_5:

$$t' = \zeta_3 t, w' = \zeta_3 w,$$

$$t' = t, w' = \zeta_5 w$$

on Σ. They generate a group of automorphisms on Σ. It is generated by single action

$$t' = \zeta_{15}^5 t, w' = \zeta_{15}^2 w.$$

It induces an action of $\mathbb{Q}(\zeta_{15})$ on the space of holomorphic differentials. Any φ_i $(i = 1, 2, 3, 4)$ is an eigendifferential for this action, and $\mathbb{Q}(\zeta_{15})$ acts faithfully on the space of holomorphic differentials. We have

$$[\mathbb{Q}(\zeta_{15}) : \mathbb{Q}] = 8 = 2 \cdot \text{ genus of } \Sigma.$$

It means $\mathrm{End}_0\mathrm{Jac}(\Sigma) = \mathbb{Q}(\zeta_{15})$ and $\mathrm{Jac}(\Sigma)$ is a simple CM abelian variety.

As easily shown every φ_i $(i = 1, 2, 3, 4)$ is an eigendifferential for the action of the CM field $\mathbb{Q}(\zeta_{15})$ with different eigenvalues.

Defining

$$T : t(x) = \frac{x}{e^{\frac{2\pi i}{3}\pi}\left(-1 + e^{\frac{2\pi i}{3}\pi} + x\right)}, \quad w(x, y) = \frac{(-1)^{\frac{1}{10}} 3^{\frac{3}{10}} y}{-1 + e^{\frac{2\pi i}{3}\pi} + x}, \tag{3.9}$$

the CM curve Σ is transformed to the degenerated pentagonal hypergeometric curve

$$DegP(-e^{2\pi i/3}) : y^5 = x^2(x - 1)(x + e^{2\pi i/3}). \tag{3.10}$$

The converse transformation T^{-1} is given by

$$x(t) = \frac{\left(1 - e^{\frac{2\pi i}{3}\pi}\right) t}{-e^{\frac{-2\pi i}{3}\pi} + t}, \quad y(t, w) = \frac{3^{\frac{1}{5}} \left(e^{\frac{2\pi i}{3}\pi}\right)^{\frac{1}{5}} w}{-1 + e^{\frac{2\pi i}{3}\pi} t}. \tag{3.11}$$

We have the shifting of the differentials via the transformation T :

$$T^*(\omega_1) = \frac{\left(-\left(\frac{1}{3}\right)\right)^{\frac{2}{5}}\left(-1+(-1)^{\frac{2}{3}}\right)}{w^2}dt$$
$$T^*(\omega_2) = \frac{(-1)^{\frac{23}{30}}\left(-1+(-1)^{\frac{2}{3}}\right)t}{3^{\frac{1}{10}}\,w^3}dt$$
$$T^*(\omega_3) = \frac{(-1)^{\frac{19}{30}}\left(-1+(-1)^{\frac{2}{3}}\right)t\left(-1+(-1)^{\frac{2}{3}}t\right)}{3^{\frac{3}{10}}\,w^4}dt$$
$$T^*(\omega_4) = -\left(\frac{(-1)^{\frac{2}{15}}\,3^{\frac{1}{5}}\left(-1+(-1)^{\frac{2}{3}}\right)t^2}{w^4}\right)dt.$$

So via the transformation T, ω_1, ω_2 and ω_4 are equal to φ_1, φ_2 and φ_4 up to a constant factor, respectively. But ω_3 is a linear combination of φ_3 and φ_4 and it is not an eigendifferential for the action of the CM field $\mathbb{Q}(\zeta_{15})$.

If we consider the Schwarz map

$$D(\omega_j, z) = \frac{\int_{\gamma_1}\omega_i}{\int_{\gamma_2}\omega_j} \quad (j = 1, 2, 3, 4)$$

for the family $\{DegP(z)\}$ with respect to the differential ω_i, according to Theorem 2.1 and Lemma 1.4 we can discuss the algebraicity of $D(\omega_j, -e^{2\pi i/3})$. Namely

Proposition 3.1 *$D(\omega_1, z), D(\omega_2, z), D(\omega_4, z)$ are algebraic at $z = -e^{2\pi i/3}$. But $D(\omega_3, -e^{2\pi i/3})$ is not an algebraic number.*

3.3 Second example

Let us study
$$DegP(-1) = \Sigma' : y^5 = x^2(x^2 - 1).$$
$\mathrm{Jac}(\Sigma')$ is a non–simple abelian variety of CM type. We have a direct sum decomposition into two abelian varieties with complex multiplication:

$$\mathrm{Jac}(\Sigma') = T(5, -1) \sim T_1 \oplus T_2$$

with $\mathrm{End}_0(T_i) = \mathbb{Q}(\zeta_5)$. We shall see this fact later in a direct way.

Set
$$HypE : y^5 = u(u - 1).$$
We have a natural map
$$\Sigma' \to HypE$$
by $x \mapsto u = x^2$. It shows that $\mathrm{Jac}(\Sigma')$ is not simple and $T_1 = \mathrm{Jac}(HypE)$ is a component. The differentials

$$\omega_2 = \frac{xdx}{y^3}, \omega_3 = \frac{xdx}{y^4}$$

are the lift from those on $HypE$. The action of ζ_5 is given by $\sigma : (x, y) \mapsto (x, \zeta^{-1}y)$. So

$$\sigma(\omega_2) = \sigma(\frac{xdx}{y^3}) = \zeta_5^3\omega_2,$$

$$\sigma(\omega_3) = \sigma(\frac{xdx}{y^4}) = \zeta_5^4\omega_3.$$

Hence T_1 is an abelian variety of CM type with the field $\mathbb{Q}(\zeta_5)$ and simple CM type $(3, 4)$. As we see later the rest T_2 is of CM type $(4, 2)$. By the change of a primitive 5-th root of unity T_1 and T_2 are isogenous . We will see below by a period matrix calculation that we have even an isomorphism.

Let us consider the special values of the Schwarz map $D(\omega_2, -1)$ and $D(\omega_3, -1)$. They are reduced to consider the periods

$$\int_1^\infty \frac{du}{y^3}, \int_0^1 \frac{du}{y^3}$$

and

$$\int_1^\infty \frac{du}{y^4}, \int_0^1 \frac{du}{y^4}$$

on the CM hyperelliptic curve $y^5 = u(u-1)$. The differentials du/y^3 and du/y^4 are eigendifferentials for the action of the corresponding CM field $\mathbb{Q}(\zeta_5)$ on the factor T_1 . According to Theorems 2.1, 2.3 and Lemma 1.5.2 the values $D(\omega_2, -1)$ and $D(\omega_3, -1)$ are algebraic. Theorem 2.3 shows the algebraicity of $D(\omega_1, -1)$ as well. Only ω_4 cannot be seen directly to be a differential on T_2. But this is true by showing explicit period matrices. So we obtain

Proposition 3.2 *The special values of the Schwarz map $D(\omega_i, z)$ $(i = 1, 2, 3, 4)$ at $z = -1$ are all algebraic values.*

According to Theorem 2.4 and Remark 3.1 we suspect the following

For the meromorphic differentials

$$\omega(\ell, m, n) = \frac{1}{y^4}x^\ell(x-1)^m(x-z)^n dx$$

on $DegP(z)$ in (3.5), may we have $D(\ell, m, n; -1) \in \overline{\mathbb{Q}}$ only for $(\ell, m, n) = (1, 0, 0)$ and $(2, 0, 0)$?

Let us calculate the period matrix of $\Sigma' : y^5 = x^2(x^2 - 1)$.

Set

$$x = \frac{1}{x_1}, y = -\frac{y_1}{x_1}.$$

So we get an isomorphic curve $\Sigma_1 : y_1^5 = x_1(x_1^2 - 1)$. We have the expression of the basis $\{\omega_1, \omega_2, \omega_3, \omega_4\}$ on Σ_1:

$$\omega_1 = -\frac{dx_1}{y_1^2}, \omega_2 = \frac{dx_1}{y_1^3}, \omega_3 = -\frac{x_1dx_1}{y_1^4}, \omega_4 = -\frac{dx_1}{y_1^4}.$$

Set r, r' be an arc on Σ_1 given by the oriented lines $[0,1], [-1,0]$ with real negative value y_1,respectively. Let $r^{(i)}$ be the arc $\sigma^{i-1}r \quad (i = 1,2,3,4)$. Set $\alpha^{(i)} = r^{(i)} - r^{(i+1)}$ and $\beta^{(i)} = r'^{(i)} - r'^{(i+1)}$.

Set

$$M_1 = \begin{pmatrix} 0 & -1 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & -1 \\ 0 & -1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 & -1 \end{pmatrix}.$$

Then

$$(A_2, A_3, A_5, A_6, B_2, B_3, B_5, B_6) = (\alpha^{(1)}, \dots, \alpha^{(4)}, \beta^{(1)}, \dots, \beta^{(4)})M_1$$

are homology basis of Σ_1 with the intersection matrix

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

These cycles $A_2, A_3, A_5, A_6, B_2, B_3, B_5, B_6$ are the same ones given by K. Koike in [Ko] on the general pentagonal curve by making the limit procedure

$$\lim_{\lambda \to -\infty} y_1^5 = x_1(x_1 - 1)(x_1 + 1)(x_1 + \lambda).$$

Put

$$p_i = \int_{\alpha^{(1)}} \omega_i \quad (i = 1,2,3,4), q_i = \int_{\beta^{(1)}} \omega_i \quad (i = 1,2,3,4),$$

then we have

$$q_1 = -p_1, q_2 = p_2, q_3 = p_3, q_4 = -p_4.$$

Setting $\omega_i' = \omega_i / p_1 \quad (i = 1,2,3,4)$, we have the period matrix of ω_i' for the cycles $(\alpha^{(1)}, \dots, \alpha^{(4)}, \beta^{(1)}, \dots, \beta^{(4)})$.

The period matrix of $\Sigma_1 : y_1^5 = x_1(x_1^2 - 1)$ with respect to the basis $\{\omega_1' \dots, \omega_4'\}$ of $H^0(\Sigma_1, \Omega)$ and the basis

$$\{\alpha^{(1)}, \dots, \alpha^{(4)}, \beta^{(1)}, \dots, \beta^{(4)}\}$$

of $H^1(\Sigma_1, \mathbb{Z})$ is given by

$$\begin{pmatrix} 1 & \zeta_5^3 & \zeta_5^1 & \zeta_5^4 & -1 & -\zeta_5^3 & -\zeta_5^1 & -\zeta_5^4 \\ 1 & \zeta_5^2 & \zeta_5^4 & \zeta_5^1 & 1 & \zeta_5^2 & \zeta_5^4 & \zeta_5^1 \\ 1 & \zeta_5 & \zeta_5^2 & \zeta_5^3 & 1 & \zeta_5 & \zeta_5^2 & \zeta_5^3 \\ 1 & \zeta_5 & \zeta_5^2 & \zeta_5^3 & -1 & -\zeta_5 & -\zeta_5^2 & -\zeta_5^3 \end{pmatrix}.$$

By changing the $\mathbb{Q}$-homology basis to

$$\{\alpha^{(1)} + \beta^{(1)}, \ldots, \alpha^{(4)} + \beta^{(4)}, \alpha^{(1)} - \beta^{(1)}, \ldots, \alpha^{(4)} + \beta^{(4)}\}$$

we know that $\mathrm{Jac}(\Sigma_1) = T(5, -1)$ is isogenous to the direct sum

$$\mathbb{C}^2 / \left(\mathbb{Z} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \mathbb{Z} \begin{pmatrix} \zeta_5^2 \\ \zeta_5 \end{pmatrix} + \mathbb{Z} \begin{pmatrix} \zeta_5^4 \\ \zeta_5^2 \end{pmatrix} + \mathbb{Z} \begin{pmatrix} \zeta_5 \\ \zeta_5^3 \end{pmatrix} \right) +$$
$$\mathbb{C}^2 / \left(\mathbb{Z} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \mathbb{Z} \begin{pmatrix} \zeta_5^3 \\ \zeta_5 \end{pmatrix} + \mathbb{Z} \begin{pmatrix} \zeta_5 \\ \zeta_5^2 \end{pmatrix} + \mathbb{Z} \begin{pmatrix} \zeta_5^4 \\ \zeta_5^3 \end{pmatrix} \right).$$

That means $\mathrm{Jac}(C_1)$ is $\mathbb{Q}$-isogenous to a direct sum of two 2-dimensional abelian varieties of CM type with the CM field $\mathbb{Q}(\zeta_5)$ of type $(3, 4)$ and of type $(4, 2)$, and these types are the same under the isomorphism $\zeta_5 \mapsto \zeta_5^3$.

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