COCYCLE INVARIANTS OF PRETZEL KNOTS AND THEIR TWIST-SPINS

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Abstract. We calculate the cocycle invariants of twist-spins of alternating odd pretzel knots. The calculation leads us to the conclusion that there are non-ribbon 2-knots which admit a non-trivial coloring and whose cocycle invariants take value in $\mathbb{Z}$.

1. Introduction.

A *surface-knot* is a connected, oriented closed surface smoothly embedded in the 4-space $\mathbb{R}^4$. The *triple point number* of a surface-knot F, denoted by $t(F)$, is the minimal number of triple points among all possible projections of F into the 3-space $\mathbb{R}^3$. A surface-knot is called a 2-*knot* if it is an embedding of a 2-sphere. It is known in [11] that a 2-knot F is a ribbon 2-knot if and only if $t(F) = 0$.

The homology and cohomology theory for quandles was developed in [2], that are similar to those of groups. Each 3-cocycle θ of a quandle cohomology defines an invariant of a surface-knot F, called the *cocycle invariant*, denoted by Φ_θ. The invariant Φ_θ takes value in the group ring $\mathbb{Z}[G]$, and in $\mathbb{Z} \subset \mathbb{Z}[G]$ if a surface-knot F admits only trivial colorings or $t(F) = 0$, where G is the coefficient group of the cohomology. In [10], Satoh and Shima proved that if the cocycle invariant $\Phi_\theta(F)$ of a surface-knot F is not an integer, then the triple point number $t(F)$ is greater than three, where θ is a 3-cocycle of the dihedral quandle R_3 with a coefficient group G. A similar result has been obtained for the dihedral quandle R_5 of order 5 by Hatakenaka [5]. For some 2-knots, its cocycle invariants with respect to dihedral quandles were calculated concretely in [1],[2], and [6]. However, it was not known that there is a non-ribbon 2-knot which admits a non-trivial coloring and whose cocycle invariants with respect to the dihedral quandles R_p of order p take value in $\mathbb{Z}$ for any odd prime p. In this paper, we show that there are such 2-knots by calculating the cocycle invariants of twist-spun pretzel knots.

This paper is organized as follows: In Section 2, we review the definition of colorings and shadow colorings by quandles, and decide the (shadow) colorings by dihedral quandles of pretzel knots. In Section 3, we calculate the cocycle invariants of alternating odd pretzel knots, prove the main theorem (Theorem 3.5).

2. Colorings and shadow colorings of pretzel knots.

A *quandle*, X, is a set with a binary operation $(a, b) \mapsto a * b$ satisfying the following conditions: (i) $a * a = a$ for any $a \in X$, (ii) for any $a, b \in X$ there is a unique $c \in X$ such that $a = c * b$, (iii) $(a * b) * c = (a * c) * (b * c)$ for any $a, b, c \in X$. The quandle $R_p = (\mathbb{Z}_p, *)$ defined by $a * b \equiv 2b - a \pmod p$ is called the *dihedral quandle* of order p.

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Let D be a diagram of an oriented knot K, and Σ the set of arcs of D. Given a quandle X, an X-*coloring* for D is a map $C : \Sigma \to X$ which satisfies $C(\gamma) = C(\alpha) * C(\beta)$ at each crossing, where $\alpha, \gamma \in \Sigma$ are under-arcs on the right and left of the over-arc $\beta \in \Sigma$, respectively. If a X-coloring uses only one color we say that it is *trivial*. The colorings by the dihedral quandle R_p are coincident with Fox's p-colorings, and independent of the orientation of a knot. We assume that p is a odd prime in the following.

Let m be a non-negative integer, and $p_1, \ldots, p_m$ non-zero integers. We denote by $P(p_1, \ldots, p_m)$ the pretzel link. of type $(p_1, \cdots, p_m)$ The diagram D_P of $P(p_1, \ldots, p_m)$ is obtained as shown in Figure 1, that is, m is the number of columns, p_i is the number of half-twists on the i-th column. The pretzel link $P(p_1, \ldots, p_m)$ is a knot if and only if (i) $p_1, \ldots, p_m, m$ are odd, or (ii) there is a unique p_i in $\{p_1, \ldots, p_m\}$ such that p_i is even. We say that $P(p_1, \ldots, p_m)$ is odd (or even, resp.) if it is in the case (i) (or (ii), resp.).

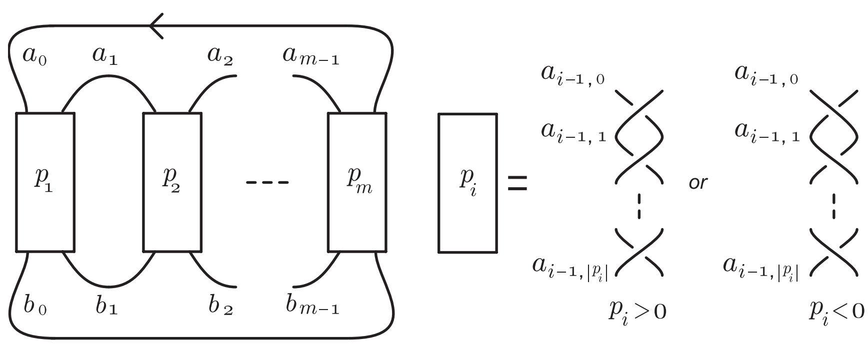

FIGURE 1

We will explicitly find all R_p-colorings of the diagram D_P of the pretzel knot $P(p_1, \ldots, p_m)$. We color the arcs of i-th column by $a_{i-1,0}, a_{i-1,1}, \ldots, a_{i-1,|p_i|} \in R_p$ from the top. See the right of Figure 1. We note that $a_{i,1} = a_{i+1,0}$ if $p_i > 0$, $a_{i,|p_i|-1} = a_{i+1,|p_{i+1}|}$ if $p_i < 0$, and $a_{00} = a_{m0}$, $b_{00} = b_{m0}$. We use the notations a_i, b_i instead of $a_{i,0}, a_{i,|p_i|}$, respectively. The relations between these colors are described by

$$\begin{pmatrix} b_{i-1} \\ b_i \end{pmatrix} = A^{p_i} \begin{pmatrix} a_{i-1} \\ a_i \end{pmatrix}, \text{ where } A = \begin{pmatrix} 0 & 1 \\ -1 & 2 \end{pmatrix}$$

without regard to the sign of p_i $(1 \leqq i \leqq m)$. By induction, we have

$$\begin{pmatrix} b_{i-1} \\ b_i \end{pmatrix} = \begin{pmatrix} -p_i + 1 & p_i \\ -p_i & p_i + 1 \end{pmatrix} \begin{pmatrix} a_{i-1} \\ a_i \end{pmatrix}. \tag{1}$$

It is known that $P(p_1, \ldots, p_m)$ admits a non-trivial R_p-coloring if and only if it holds that $\sum_{i=1}^{m} p_1 p_2 \cdots \widehat{p_i} \cdots p_m = 0 \pmod p$. For example, if there is a unique p_i in $\{p_1, \ldots, p_m\}$ such that p_i is divisible by p, then the colorings of $P(p_1, \ldots, p_m)$ are always trivial. We consider the following two cases with respect to $p_i \pmod p$.

Case 1. Assume that all p_i's are not divisible by p $(1 \leqq i \leqq m)$. Then the relation (1) induce

$$\begin{pmatrix} a_i \\ b_i \end{pmatrix} = \begin{pmatrix} -q_i + 1 & q_i \\ -q_i & q_i + 1 \end{pmatrix} \begin{pmatrix} a_{i-1} \\ b_{i-1} \end{pmatrix},$$

where $q_i = \frac{1}{p_i}$. By induction, we have

$$a_i = Q_i c_0 + a_0, \tag{2}$$

where $Q_i = \sum_{k=1}^{i} q_k$ and $c_0 = b_0 - a_0$. By the definition of the R_p-coloring, the color of each arc of the i-th column is obtained by

$$a_{i-1,j} = a_{i-1} \pm j d_i \quad (0 \leqq j \leqq |p_i|), \tag{3}$$

where $d_i = a_i - a_{i-1} = q_i c_0$, and the symbol '$\pm$' means that the plus '$+$' if $p_i > 0$ the minus '$-$' if $p_i < 0$. Therefore, given the colors $a_0, b_0 \in R_p$, we may decide a R_p-coloring of the diagram D_P of a pretzel knot $P(p_1, \ldots, p_m)$ from the relations (2) and (3).

Case 2. Assume that $p_{i_1}, \ldots, p_{i_n}$ in $\{p_1, \ldots, p_m\}$ are divisible by p for some $n \geqq 2$ ($i_1 < \cdots < i_n$). From the relation (1), we have

$$a_{i_k} = b_{i_k} = a_{i_k+1} = b_{i_k+1} = \cdots = a_{i_{k+1}-1} = b_{i_{k+1}-1}.$$

For i such that $i_k < i < i_{k+1}$, since the top arcs of the i-th column have the same color a_{i_k}, all arcs of it are colored by a_{i_k}. The color of each arc of the j_k-th column is the color (2) obtained by substituting j_k for i. Thus a R_p-coloring of the above diagram D_P is decided by the colors $a_{i_1}, \ldots, a_{i_n}$.

Let X be a quandle. The number of the X-colorings of a knot K is called the *X-coloring number* of K. The X-coloring number is an invariant of K. If a knot K admits the only trivial X-colorings, the X-coloring number of K is equal to the cardinality of the quandle X. From what has been discussed above, we have the following with respect to the R_p-coloring number of the pretzel knot $P(p_1, \ldots, p_m)$.

Proposition 2.1. *Assume that the pretzel knot $P(p_1, \ldots, p_m)$ admit a non-trivial R_p-coloring. Then the R_p-coloring number of $P(p_1, \ldots, p_m)$ is equal to p^2 if all p_i's are not divisible by p, or p^n if $p_{i_1}, \ldots, p_{i_n}$ in $\{p_1, \ldots, p_m\}$ are divisible by p for some $n \geqq 2$.*

Let D be a knot diagram of an oriented knot K. We assume that D is X-colored by a coloring C. A *shadow X-coloring of D extending C* is a map $\widetilde{C} : \widetilde{\Sigma} \to X$, where $\widetilde{\Sigma}$ is the union of Σ and the set of the connected regions separated by the underlying immersed curve of D, satisfying the following conditions: (i) $\widetilde{C}$ restricted to Σ coincides with C, and (ii) if μ and ν are regions separated by an arc α, where μ is on the right of x, then $\widetilde{C}(\nu) = \widetilde{C}(\mu) * \widetilde{C}(\alpha)$ holds. We call the ordered triple $(\widetilde{C}(\mu), \widetilde{C}(\alpha), \widetilde{C}(\beta)) \in X^3$ the *quandle triple* at a crossing point of a diagram, where α is the under-arc on the right of the over-arc β, and μ is the region on the right side of α and β both, which is denote by $\widetilde{C}(x)$.

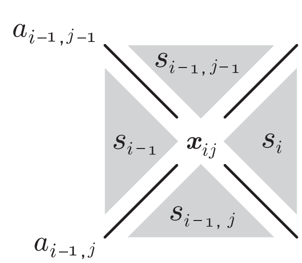

FIGURE 2

Let D_P be the diagram of a pretzel knot $P(p_1, \ldots, p_m)$ defined above. We consider the shadow R_p-colorings of the diagram D_P colored as shown in Figure 1. The shadow R_p-colorings are not depend on the orientation of a diagram (but the quandle triples depend on it). Let x_{ij} be the j-th crossing point from the top of the i-th column ($1 \leqq i \leqq m, 1 \leqq j \leqq |p_i|$). We color the region on the right, left side of x_{ij} by $s_{i-1}, s_i \in R_p$, and upper, under side by $s_{i-1,j-1}, s_{i-1,j} \in R_p$, respectively. See Figure 2. We note that the relations $s_{0,0} = s_{1,0} = \cdots = s_{m-1,0}$, $s_{0,|p_1|} = s_{1,|p_2|} = \cdots = s_{m-1,|p_m|}$, and $s_0 = s_m$ hold. By definition, we have the following relations.

$$\begin{cases} s_{i,0} = 2a_0 - s_0, \\ s_i = 2a_i - s_{i,0} = 2(a_i - a_0) + s_0, \\ s_{ij} = 2a_{ij} - s_i = 2(a_{ij} - a_i + a_0) - s_0. \end{cases} \tag{4}$$

Hence, the shadow R_p-coloring of the R_p-colored diagram D_P of the pretzel knot $P(p_1, \ldots, p_m)$ is decided by the color s_0.

3. Cocycle invariants of pretzel knots and their twist-spins.

We recall diagrams and colorings of 2-knots. For a 2-knot F in $\mathbb{R}^4$, we assume that the projection $p : F \to \mathbb{R}^3$ is a generic map. The singularity set of the projection consists of double points, triple points and branch points. Crossing information is indicated in $p(F)$ as follows: Along every double point curve, two sheets intersect locally, one of which is under the other relative to the projection direction of p. Then the under-sheet is broken by the over-sheet. A *diagram* of F is the image $p(F)$ with such crossing information. Hence, a diagram regarded as a union of disjoint compact, connected surfaces.

Let D be a diagram of a 2-knot F, Σ the set of such connected surfaces in D, and X a quandle. A coloring of D is a map $C : \Sigma \to X$ satisfying $C(\gamma) = C(\alpha) * C(\beta)$ at each double curve, where $\alpha, \beta, \gamma \in \Sigma$ are the three sheets meeting at the double curve such that β is the over-sheet, α, γ are the under-sheets which the normal direction of β points α to γ. See the left of Figure 3.

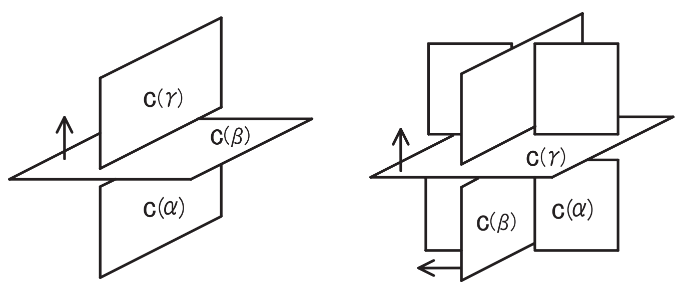

Figure 3

Each triple point t of D is assigned the sign $\epsilon(t) = \pm 1$ induced from the orientation in such a way that $\epsilon(t) = +1$ if and only if the ordered triple of the orientation normals of the top, middle, and bottom sheets, respectively, agree with the orientation of $\mathbb{R}^3$. The colors of the sheets near t are determined by three colors $C(\alpha)$,$C(\beta)$ and $C(\gamma)$, where γ is the top sheet, β is the middle sheet from which the orientation normal of γ points, and α is the bottom sheet from which the orientation normals of β and γ point both. See the right of Figure 3. The ordered triple $(C(\alpha), C(\beta), C(\gamma))$ is called the *color* of t and denoted by $C(t) \in X^3$.

Let X be a quandle, G a abelian group. We may define the cohomology group $H^*(X, G)$ for a quandle X (cf. [2]). A map $\theta : X^3 \to G$ is called a 3-*cocycle* of X if it satisfies

(i) $\theta(a, b, c) = 0$ if $a = b$ or $b = c$,
(ii) for any $a, b, c, d \in X$,

$$\begin{aligned} &\theta(a, c, d) - \theta(a, b, d) + \theta(a, b, c) \\ &= \theta(a * b, c, d) - \theta(a * c, b * c, d) + \theta(a * d, b * d, c * d). \end{aligned}$$

For a 3-cocycle θ, we define the weight $W_\theta(t, C) = \epsilon(t)\theta(C(t)) \in G$, and also define $W_\theta(C) = \sum_t W_\theta(t, C) \in G$. The cocycle invariant $\Phi_\theta(F)$ of a 2-knot F is defined by

$$\Phi_\theta(F) = \sum_C W_\theta(C),$$

which take value in the group ring $\mathbb{Z}[G]$. It is proved in [2] that $\Phi_\theta(F)$ is an invariant of a 2-knot F which does not depend on the choice of a diagram D of F, and if θ and θ' are cohomologous, then $\Phi_\theta(F) = \Phi_{\theta'}(F)$. By definition, the invariant $\Phi_\theta(F)$ is equal to the coloring number in $\mathbb{Z} \subset \mathbb{Z}[G]$ if F is a ribbon 2-knot or admits only trivial colorings.

In the same way, we may define the cocycle invariants for an oriented classical knot K. Let $\widetilde{C}$ be a shadow coloring of a diagram of K, x a crossing. We define $W_\theta(x, \widetilde{C})$ and $W_\theta(\widetilde{C})$ by $W_\theta(x, \widetilde{C}) = \epsilon(x)\theta(\widetilde{C}(x)) \in G$, $W_\theta(\widetilde{C}) = \sum_x W_\theta(x, \widetilde{C}) \in G$, respectively, where $\epsilon(x) = \pm 1$ is the sign of x, $\widetilde{C}(x)$ is the quandle triple at x. Then the state-sum $\sum_{\widetilde{C}} W_\theta(\widetilde{C}) \in \mathbb{Z}[G]$, which takes value in the group ring $\mathbb{Z}[G]$, is independent of the choice of a diagram of a knot K (cf. [3],[8]), denoted by $\Psi_\theta(K)$. If we choose a base point on the diagram D except crossing, then the state-sum $\sum_{\widetilde{C}} W_\theta(\widetilde{C}) \in \mathbb{Z}[G]$ for the restricted shadow X-colorings, where the base point and its adjacent regions receive the same color, is independent on the choice of a base point and a diagram of a knot K (cf. [1]), denoted by $\Psi^*_\theta(K)$.

We consider the case $X = R_p$ and $G = \mathbb{Z}_p$, identify the group ring $\mathbb{Z}[\mathbb{Z}_p]$ with the Laurent polynomial ring $\mathbb{Z}[t^{\pm 1}]/(t^p - 1)$. Then it holds that $\Psi^{(*)}_\theta(K) = \sum_{\widetilde{C}} t^{W_\theta(\widetilde{C})}$. We define the map $\theta_p : R_p^3 \to \mathbb{Z}_p$ by

$$\theta_p(a, b, c) = (a - b)\frac{(2c - b)^p + b^p - 2c^p}{p},$$

where all coefficients of the numerator as a polynomial in a, b, c are divisible by p. It is proved in [7] that $H^3(R_p, \mathbb{Z}_p) \cong \mathbb{Z}_p$ and its generator is given by the map θ_p.

In the following, we assume that the pretzel knot $P(p_1, \ldots, p_m)$ is alternating, odd, and oriented by the orientation indicated in Figure 1, calculate its cocycle invariants associated with the 3-cocycle θ_p.

Lemma 3.1. *For any shadow R_p-coloring $\widetilde{C}$ of the diagram D_P of the alternating odd pretzel knot $P(p_1, \ldots, p_m)$, we have $W_{\theta_p}(\widetilde{C}) = 0$, that is,*

$$\sum_{i=1}^{m} \sum_{j=1}^{|p_i|} W_{\theta_p}(x_{ij}, \widetilde{C}) = 0,$$

where x_{ij} is the j-th crossing from the top of the i-th column of D_P.

Proof. Assume that all p_i's are not divisible by p, positive ($1 \leqq i \leqq m$). The quandle triple $\widetilde{C}(x_{ij})$ and the sign $\epsilon(x_{ij})$ of a triple point x_{ij} of D_P is given by

$$\widetilde{C}(x_{ij}) = \begin{cases} (s_{i-1,j-1}, a_{i-1,j-1}, a_{i-1,j}) & \text{if } j \text{ is even,} \\ (s_{i-1,j}, a_{i-1,j+1}, a_{i-1,j}) & \text{if } j \text{ is odd,} \end{cases}$$
$$\epsilon(x_{ij}) = -1,$$

respectively. The wight $W_{\theta_p}(x_{ij}, \widetilde{C}) \in \mathbb{Z}_p$ is equal to

$$(a_i - jd_i + s_0 - 2a_0)\frac{(a_i + (j-2)d_i)^p + (a_i + jd_i)^p - 2(a_i + (j-1)d_i)^p}{p}$$

with no regard to the parity of j. We use the notation X_{ij} instead of the above numerator modulo p^2. Since $d_i = a_i - a_{i-1} = q_i c_0$, we have $p_i d_i = c_0$. Then it holds that

$$\begin{aligned} \sum_{i=1}^{m}\sum_{j=1}^{p_i} X_{ij} &= \sum_{i=1}^{m} \{(a_i - d_i)^p - (a_i)^p - (a_i + (p_i - 1)d_i)^p + (a_i + p_i d_i)^p\} \\ &= \sum_{i=1}^{m} \{(a_{i-1})^p - (a_i)^p\} - \sum_{i=1}^{m} \{(a_{i-1} + c_0)^p - (a_i + c_0)^p\} \\ &= 0 - 0 = 0 \end{aligned}$$

Furthermore, we have

$$\begin{aligned} \sum_{i=1}^{m}\sum_{j=1}^{p_i} a_i X_{ij} &= \sum_{i=1}^{m} a_i \{(a_{i-1})^p - (a_i)^p\} - \sum_{i=1}^{m} a_i \{(a_{i-1} + p_i d_i)^p - (a_i + p_i d_i)^p\} \\ &= \sum_{i=1}^{m} d_i (a_{i-1})^p - \sum_{i=1}^{m} d_i (a_{i-1} + c_0)^p \end{aligned}$$

and

$$\begin{aligned} \sum_{i=1}^{m}\sum_{j=1}^{p_i} j d_i X_{ij} &= \sum_{i=1}^{m} \{d_i (a_{i-1})^p - d_i (a_{i-1} + p_i d_i)^p\} \\ &\qquad + \sum_{i=1}^{m} p_i d_i \{(a_i + p_i d_i)^p - (a_{i-1} + p_i d_i)^p\} \\ &= \sum_{i=1}^{m} d_i (a_{i-1})^p - \sum_{i=1}^{m} d_i (a_{i-1} + c_0)^p. \end{aligned}$$

Therefore we have $W_{\theta_p}(\widetilde{C}) = 0$.

If $p_{i_1}, \ldots, p_{i_n}$ in $\{p_1, \ldots, p_m\}$ are divisible by p for some $n \geqq 1$ ($i_1 < \cdots < i_n$), it holds that $W_{\theta_p}(x_{ij}, \widetilde{C}) = 0$ for $i \neq i_1, \cdots, i_n$, because all arcs of i-th column are colored by a same color. Hence we may assume that p_i is divisible by p for any i, $m = n$ ($1 \leqq i \leqq m = n$). Since $p_i d_i = 0$ holds we have $W_{\theta_p}(\widetilde{C}) = 0$ immediately.

Assume that all p_i's are negative. Then in the same way we may find the quandle triple and the sign of a crossing x_{ij}, calculate $W_{\theta_p}(\widetilde{C})$, and get the same result, that is, $W_{\theta_p}(\widetilde{C}) = 0$. □

Proposition 3.2. *Let $\widetilde{C}$ be a shadow R_p-coloring of D_P. For any 3-cocycle θ with respect to R_p, it holds that $W_\theta(\widetilde{C}) = 0$.*

Proof. The cohomology class $[\theta_p]$ is a generator of $H^3(R_p, \mathbb{Z}_p)$. Hence, for any 3-cocycle θ with respect to R_p, there is $k \in \mathbb{Z}_p$ such that $[\theta] = k[\theta_p] \in H^3(R_p, \mathbb{Z}_p)$. Then we have $W_\theta(\widetilde{C}) = kW_{\theta_p}(\widetilde{C}) = 0$. □

Proposition 3.3. *If the alternating odd pretzel knot $P(p_1, \dots, p_m)$ admits a non-trivial R_p-coloring, the cocycle invariants $\Psi_\theta(P)$, $\Psi^*_\theta(P)$ of $P(p_1, \dots, p_m)$ are given by*

$$\Psi_\theta(P) = \begin{cases} p^3 & \textit{Case 1}, \\ p^{n+1} & \textit{Case 2}, \end{cases} \qquad \Psi^*_\theta(P) = \begin{cases} p^2 & \textit{Case 1}, \\ p^n & \textit{Case 2}, \end{cases}$$

and otherwise $\Psi_\theta(P) = p^2$, $\Psi^*_\theta(P) = p$, where Case 1, 2 means that $P(p_1, \dots, p_m)$ belong to **Case 1**, **2** *in Section 2, respectively.*

Proof. As discussed in Section 2 the shadow R_p-coloring $\widetilde{C}$ of the diagram D_P is decided by $(a_0, b_0, s_0) \in \mathbb{Z}_p^3$, $(a_{i_1}, \dots, a_{i_n}, s_0) \in \mathbb{Z}_p^{n+1}$ or $(a_0, s_0) \in \mathbb{Z}_p^2$ in **Case 1**, **2** or otherwise, respectively. We may assume that $a_{i_1} = a_0$ without loss of generality. We fix a base point of $P(p_1, \dots, p_m)$ on the top arc colored by a_0. By definition we have $a_0 = s_0$ in the restricted shadow R_p-coloring. The proposition follows from these results and Proposition 3.2 immediately. □

It has been known that $\Psi_{\theta_p} = p\Psi^*_{\theta_p}$ holds for also 2-bridge knots (cf. [6]) and 3-braid knots (cf. [9]), but whether the equality holds for any 2-knot is unknown.

For each non-negative integer r, Zeeman [12] constructed a 2-knot from an oriented classical knot K, which is called the *r-twist-spin* of K and denoted by $\tau^r K$. A twist-spin $\tau^r K$ is a ribbon 2-knot K if and only if $r = 0, 1$ or K is trivial ([4]). In particular, $\tau^1 K$ is a trivial 2-knot for any K.

Proposition 3.4 ([1])**.** (i) *If r is odd, then we have $\Phi_{\theta_p}(\tau^r K) = p$.*
(ii) *If r is even, then we have $\Phi_{\theta_p}(\tau^r K) = \rho^r \Psi^*_{\theta_p}(\tau^r K)$, where $\rho^r : \mathbb{Z}[t^{\pm 1}]/(t^p - 1) \to \mathbb{Z}[t^{\pm 1}]/(t^p - 1)$ is the map induced by $t \to t^r$.*

Theorem 3.5. *If the alternating odd pretzel knot $P(p_1, \dots, p_m)$ admits a non-trivial R_p-coloring and r is even, the cocycle invariant $\Phi_\theta(\tau^r P)$ of the r-twist-spin $\tau^r P$ of $P(p_1, \dots, p_m)$ is given by*

$$\Phi_\theta(\tau^r P) = \begin{cases} p^2 & \textit{Case 1}, \\ p^n & \textit{Case 2}, \end{cases}$$

and otherwise $\Phi_\theta(\tau^r P) = p$.

Proof. If $\theta = \theta_p$, then the theorem follows from Proposition 3.3 and 3.4. By Proposition 3.2, we have $\Phi_\theta(\tau^r P) = \Phi_{\theta_p}(\tau^r P)$ for any 3-cocycle θ with respect to R_p □

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