ISOMORPHISMS OF C*-ALGEBRAS AFTER TENSORING

YUTAKA KATABAMI

Abstract. J. Plastiras exhibited C*-algebras which are not isomorphic but, after tensoring by $M_2$, isomorphic. On proof of non-isomorphism of them, we give two ways which are different from his original one.

0. Introduction

It is well known that the algebra of all complex valued continuous functions on a compact Hausdorff space becomes an abelian C*-algebra with respect to the supremum norm and every abelian C*-algebra is realized as such a C*-algebra by Gelfand’s representation theorem. By this correspondence we can see properties of abelian C*-algebras as those of topological spaces (compact Hausdorff spaces). So we can regard a general C*-algebra as an extended topological object (for example, non-commutative topological space). In the theory of algebraic topology, homology groups and cohomology groups work well as topological invariants. In the theory of C*-algebra, extension theory (resp. K-theory) also works well as homology theory (resp. cohomology theory).

J. Plastiras constructed the example of two C*-algebras such that they are not isomorphic but become isomorphic after tensoring with a matrix algebra. In this paper we look his example from the extension theoretical point of view, and we give the proof of non-existense of isomorphism using K-theory.

1. Preliminaries

Throughout this paper we denote the set of complex numbers, real numbers, integers and nonnegative integers as $\mathbb{C}$, $\mathbb{R}$, $\mathbb{Z}$ and $\mathbb{N}$ respectively. Let $\mathcal{H}$ be a separable infinite dimensional Hilbert space. We denote by $\mathcal{B}(\mathcal{H})$ (resp. $\mathcal{K}(\mathcal{H})$) the set of bounded linear operators (resp. compact operators) on $\mathcal{H}$. $M_n$ stands for the $n \times n$ matrix algebra over $\mathbb{C}$. 

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In this section, we will present some basic facts on Extension theory and K-theory for C*-algebras. Let $A$, $B$ and $C$ be C*-algebras and $\alpha$ (resp. $\beta$) a *-homomorphism from $A$ to $B$ (resp. from $B$ to $C$). We call a short exact sequence $E$ as below an extension of $A$ by $C$:

$$E : 0 \longrightarrow A \xrightarrow{\alpha} B \xrightarrow{\beta} C \longrightarrow 0.$$ 

Then $\alpha$ is injective, $\beta$ is surjective and $\text{Im} \alpha = \text{Ker} \beta$.

When $A$ is a *-subalgebra of $B$ (i.e., if $\xi \in H$ satisfies $\xi a = 0$ for all $a \in A$, then $\xi = 0$), we define the multiplier algebra $M(A)$ for $A$ as follows:

$$M(A) = \{ x \in \mathbb{B}(H) \mid xA \subset A, \ Ax \subset A \}.$$ 

Clearly we have that $A$ becomes a closed two-sided *-ideal of $M(A)$ and the multiplier algebra $M(K(H))$ of $K(H)$ coincides with $\mathbb{B}(H)$. A double centralizer for $A$ is a pair $(L,R)$ of functions $L,R : A \rightarrow A$ satisfying

$$R(xy) = xL(y)$$

for all $x,y \in A$. For an element $x \in A$, $(L_x,R_x)$ becomes a double centralizer of $A$, where

$$L_x : A \ni y \mapsto xy \in A, \quad R_x : A \ni y \mapsto yx \in A.$$ 

It is known that the set of all double centralizers $DC(A)$ for $A$ becomes a C*-algebra and $DC(A)$ is isomorphic to the multiplier algebra $M(A)$ for $A$.

For the above extension (in this case $\alpha$ is injective and $\alpha(A)$ is a closed two-sided *-ideal of $B$), we can uniquely define the *-homomorphism $\sigma$ from $B$ to $M(A)(= DC(A))$ with $\sigma \circ \alpha = \iota$, that is,

$$A \xrightarrow{\alpha} B \xrightarrow{\beta} C \xrightarrow{\iota} M(A),$$

where $\iota(x) = (L_x,R_x) \in DC(A) \cong M(A) (x \in A)$. Indeed $\sigma$ is defined as follows:

$$\sigma(\alpha(x)) = (L(\alpha(x)), R(\alpha(x))) \in DC(A) \cong M(A),$$

where

$$L(\alpha(x)) : A \ni y \mapsto \alpha^{-1}(\alpha(x)\alpha(y)) \in A,$$

$$R(\alpha(x)) : A \ni y \mapsto \alpha^{-1}(\alpha(y)\alpha(x)) \in A.$$
Definition 1.1. (Busby invariant) For an extension 

\[ E : 0 \longrightarrow A \overset{\alpha}{\longrightarrow} B \overset{\beta}{\longrightarrow} C \longrightarrow 0, \]

the Busby invariant for \( E \) is defined as the \(*\)-homomorphism \( \tau_E \) from \( C \) to \( M(A)/A \) given by 

\[ \tau_E(c) = \pi \circ \sigma(b), \]

where \( b \) is a lift of \( c \) through \( \beta \) and \( \pi \) is the quotient map from \( M(A) \) to \( M(A)/A \).

The Busby invariant \( \tau_E \) is the unique \(*\)-homomorphism which makes the following diagram commutative:

\[
\begin{array}{ccc}
0 & \longrightarrow & A \\
\downarrow & & \downarrow \alpha \\
& 0 & \longrightarrow A \longrightarrow B_1 \longrightarrow C \longrightarrow 0 \\
\end{array}
\]

\[
\begin{array}{ccc}
0 & \longrightarrow & A \\
\downarrow & & \downarrow \tau_E \\
& 0 & \longrightarrow A \longrightarrow M(A) \longrightarrow M(A)/A \longrightarrow 0.
\end{array}
\]

Proposition 1.2. Let 

\[ E_1 : 0 \longrightarrow A \longrightarrow B_1 \longrightarrow C \longrightarrow 0, \]

\[ E_2 : 0 \longrightarrow A \longrightarrow B_2 \longrightarrow C \longrightarrow 0 \]

be extensions and \( \tau_1, \tau_2 \) Busby invariants respectively.

(1) (strongly isomorphic) \( \tau_1 = \tau_2 \) if and only if there is an unique \(*\)-isomorphism \( \gamma \) for which the diagram

\[
\begin{array}{ccc}
0 & \longrightarrow & A \\
\downarrow & & \downarrow \gamma \\
& 0 & \longrightarrow A \longrightarrow B_1 \longrightarrow C \longrightarrow 0 \\
\end{array}
\]

is commutative.

(2) (strongly equivalent) There is a unitary \( u \in M(A) \) such that \( \tau_2(c) = \pi(u)\tau_1(c)\pi(u)^* \) if and only if there are a unitary \( v \in M(A) \) and a \(*\)-isomorphism \( \gamma \) such that the diagram

\[
\begin{array}{ccc}
& 0 & \longrightarrow A \\
\downarrow Ad(v) & & \downarrow \gamma \\
0 & \longrightarrow A \longrightarrow B_2 \longrightarrow C \longrightarrow 0
\end{array}
\]

is commutative.
We will define $K_0$ and $K_1$ groups for any C*-algebra $\mathcal{A}$. For the definition of $K_0$-group for a C*-algebra $\mathcal{A}$ (denoted by $K_0(\mathcal{A})$), we will give some properties of projections. Let $\mathcal{M}_n(\mathcal{A})$ be an $n \times n$ matrix algebra with entries of $\mathcal{A}$. For $m,n \in \mathbb{N}$ with $m < n$, an inclusion map $\varphi_{nm}$ from $\mathcal{M}_m(\mathcal{A})$ to $\mathcal{M}_n(\mathcal{A})$ is defined by the following way; for $x \in \mathcal{M}_m(\mathcal{A})$

$$\varphi_{nm}(x) := x \oplus 0_{n-m}$$

where $\oplus$ means the diagonal sum. That is, $x$ is put into left upper part in $\mathcal{M}_n(\mathcal{A})$. Using this $\varphi_{nm}$, we can view $\mathcal{M}_m(\mathcal{A})$ as a subalgebra of $\mathcal{M}_n(\mathcal{A})$. We put $\mathcal{M}_\infty(\mathcal{A}) = \bigcup_{n=1}^{\infty} \mathcal{M}_n(\mathcal{A})$. For projections in $\mathcal{M}_\infty(\mathcal{A})$, we introduce some equivalent relations.

**Definition 1.3.** (Murray-von Neumann equivalence) For projections $p, q \in \mathcal{M}_\infty(\mathcal{A})$, they are Murray-von Neumann equivalent if there exists a partial isometry $v$ in $\mathcal{M}_\infty(\mathcal{A})$ such that $p = v^*v$ and $q = vv^*$, where $^*$ is an involution of $\mathcal{M}_\infty(\mathcal{A})$.

**Definition 1.4.** (Homotopy equivalence) For projections $p, q \in \mathcal{M}_\infty(\mathcal{A})$, they are homotopy equivalent if there exist a positive integer $N$ and a norm continuous path $\{P_t\}_{t \in [0,1]}$ in $\mathcal{M}_N(\mathcal{A})$ such that $P_0 = p$ and $P_1 = q$.

**Proposition 1.5.** For projections $p, q \in \mathcal{M}_\infty(\mathcal{A})$, if $\|p - q\| < 1$, then they are homotopy equivalent.

It is known that the Murray-von Neumann equivalence (algebraic notion) and the homotopy equivalence (topological notion) are the same equivalence for $\mathcal{M}_\infty(\mathcal{A})$.

**Definition 1.6.**

$$V(\mathcal{A}) := \{\text{projections in } \mathcal{M}_\infty(\mathcal{A})\} / \sim$$

where $\sim$ is the Murray-von Neumann equivalent relation.

**Definition 1.7.** For equivalence classes $[p], [q] \in V(\mathcal{A})$, the addition of them is defined by

$$[p] + [q] := [p \oplus q].$$

It can be easily verified that the above operation is well-defined and abelian. So $V(\mathcal{A})$ becomes an abelian semigroup with the unit $[0]$. Now let $V(\mathcal{A}) - V(\mathcal{A})$ be formal differences of $V(\mathcal{A})$ and we define the following equivalence relation.
Definition 1.8. For \([p_1] - [q_1], [p_2] - [q_2] \in V(A) - V(A), [p_1] + [q_1] \approx [p_2] + [q_2]\) if there exist \([r] \in V(A)\) such that 
\[ [p_1] + [q_2] + [r] = [p_2] + [q_1] + [r]. \]

Definition 1.9. \((K_0\text{-group})\) For a unital C*-algebra \(A\), the \(K_0\text{-group}\) for \(A\) is defined by
\[ K_0(A) := \{V(A) - V(A)\} / \approx \]
where \(\approx\) is the above relation.

We observe properties of \(K_0(A)\). Let \(e_0 - f_0, e_1 - f_1\) be elements in \(K_0(A)\).
1. (abelian additivity) \((e_0 - f_0) + (e_1 - f_1) = (e_0 + e_1) - (f_0 + f_1) = (e_1 - f_1) + (e_0 - f_0).\)
2. (the unit) The unit of \(K_0(A)\) is \(e_0 - e_0\) (denoted as 0).
3. (existence of the inverse) The inverse of \(e_0 - f_0\) is \(f_0 - e_0\).

Therefore \(K_0(A)\) becomes an abelian group.

We now define the \(K_1\text{-group}\) for a unital C*-algebras \(A\). We denote by \(U_n(A)\) the set of unitary elements of \(M_n(A)\). This is the topological subgroup with respect to the norm topology. For \(m, n \in \mathbb{N}\), when \(m < n\), an inclusion map \(\phi_{nm}\) from \(U_m(A)\) to \(U_n(A)\) is defined by for \(x \in U_m(A)\)
\[ \phi(x) := x \oplus 1_{n-m}. \]
We put \(U_\infty(A) = \bigcup_{n=1}^\infty U_n(A)\). We denote by \(U_n(A)\) the set of unitary elements homotopic to the unit \(1_n\) of \(M_n(A)\). By the similar argument it is put that \(U_\infty(A) = \bigcup_{n=1}^\infty U_n(A)\).

Definition 1.10. \((K_1\text{-group})\)
\[ K_1(A) := U_\infty(A)/U_\infty(A)_0. \]

The \(K_1\text{-group}\) is an abelian group with the unit \([1]\) under the multiplicative operation
\[ [u][v] := [uv] = [u \oplus v]. \]
For C*-algebras \(\{A_n\}\) and \(*\)-homomorphisms \(\{\varphi_{nm} : A_m \to A_n, (m < n)\}\),
We call \(\{(A_n, \varphi_{nm})\}\) an inductive system of C*-algebras if they satisfy for \(l < m < n\), \(\varphi_{nl} = \varphi_{mn} \circ \varphi_{ml}\). Then we define \(A_0\) and the semi-norm on \(A\) as the following:
\[ A_0 = \{a = (a_n) \in \Pi_{n=1}^\infty A_n | \text{there exists } N_0 \text{ such that } \varphi_{mN_0}(a_{N_0}) = a_m \text{ for } m > N_0\} \]
\[ \|a\|_0 = \lim_{n \to \infty} \|a_n\|. \]
Then the completion \( \mathcal{A} \) of \( \mathcal{A}_0/\{a \in \mathcal{A}_0\|a\|_0 = 0\} \) becomes a C*-algebra, and \( \mathcal{A} \) is called the inductive limit of the system and denoted by \( \lim_{\rightarrow} \mathcal{A}_n \).

**Theorem 1.11.** For an unital C*-algebra \( \mathcal{A}, \mathcal{B} \) and \( i = 0, 1 \).

1. (Stability) \( K_i(\mathcal{A} \otimes \mathbb{K}) = K_i(\mathcal{A}) \).
2. (Distributivity) \( K_i(\mathcal{A} \oplus \mathcal{B}) = K_i(\mathcal{A}) \oplus K_i(\mathcal{B}) \).
3. (Continuity) If \( \mathcal{A} = \lim_{\rightarrow} \mathcal{A}_n \). Then \( K_i(\mathcal{A}) = \lim_{\rightarrow} K_i(\mathcal{A}_n) \).

Combining the basic facts and the following powerful theorem, we can compute \( K \)-groups of typical C*-algebras:

1. \( K_0(\mathbb{C}) = K_0(\mathbb{M}_n) = \mathbb{Z}, \ K_1(\mathbb{C}) = K_1(\mathbb{M}_n) = 0 \).
2. \( K_0(\mathbb{B}(\mathcal{H})) = K_0(\mathbb{B}(\mathcal{H})/\mathbb{K}(\mathcal{H})) = 0, \ K_1(\mathbb{B}(\mathcal{H})) = 0, \ K_1(\mathbb{B}(\mathcal{H})/\mathbb{K}(\mathcal{H})) = \mathbb{Z} \).
3. \( K_0(\mathbb{K}(\mathcal{H})) = \mathbb{Z}, \ K_1(\mathbb{K}(\mathcal{H})) = 0 \).

**Theorem 1.12.** (Six term exact sequence) Let \( \mathcal{J} \) be an closed two-sided \( * \)-ideal in a unital C*-algebra \( \mathcal{A} \). For a short exact sequence

\[
0 \longrightarrow \mathcal{J} \overset{\iota}{\longrightarrow} \mathcal{A} \overset{\pi}{\longrightarrow} \mathcal{A}/\mathcal{J} \longrightarrow 0,
\]

we have the following diagram which is exact at any part:

\[
\begin{array}{ccc}
K_0(\mathcal{J}) & \overset{\iota_*}{\longrightarrow} & K_0(\mathcal{A}) \overset{\pi_*}{\longrightarrow} K_0(\mathcal{A}/\mathcal{J}) \\
\delta_1 \uparrow & & \downarrow \delta_0 \\
K_1(\mathcal{A}/\mathcal{J}) & \overset{\pi_*}{\leftarrow} & K_1(\mathcal{A}) \overset{\iota_*}{\leftarrow} K_1(\mathcal{A})
\end{array}
\]

where \( \iota_* \) and \( \pi_* \) are induced maps from \( \iota \) and \( \pi \) respectively and \( \delta_0 \) is the exponential map and \( \delta_1 \) is the index map.

[the construction of \( \delta_1 \)] For \( x \in K_1(\mathcal{A}/\mathcal{J}) \), we can choose a unitary \( u \) in \( \mathbb{M}_n(\mathcal{A}/\mathcal{J}) \) such that \( x = [u] \). So \( \begin{pmatrix} u & 0 \\ 0 & u^* \end{pmatrix} \) is also a unitary in \( \mathbb{M}_{2n}(\mathcal{A}/\mathcal{J}) \) which is homotopic to \( 1_{2n} \). Choosing a unitary lift \( w \) of \( \begin{pmatrix} u & 0 \\ 0 & u^* \end{pmatrix} \) in \( \mathbb{M}_{2n}(\mathcal{A}) \) and a projection \( P_n = \begin{pmatrix} 1_n & 0_n \\ 0_n & 1_n \end{pmatrix} \) in \( \mathbb{M}_{2n}(\mathcal{A}) \), we can define the index map \( \delta_1 \) from \( K_1(\mathcal{A}/\mathcal{J}) \) to \( K_0(\mathcal{J}) \) as follows:

\[
\delta_1(x) := [wP_n w^*] - [P_n].
\]

[the construction of \( \delta_0 \)] For \( x \in K_0(\mathcal{A}/\mathcal{J}) \), we can choose a projection \( P \) in \( \mathbb{M}_n(\mathcal{A}/\mathcal{J}) \) such that \( x = [P] \) and a self-adjoint lift \( f \) of \( P \) in
$M_n(A)$. Then we have $\exp(2\pi i \sigma) \in U_n(J)$, since

$$\pi(\exp(2\pi i \sigma)) = \exp(2\pi i P) = I_n + 2\pi i P + \frac{(2\pi i P)^2}{2i} + \cdots$$

$$= I_n - P + (I_n + 2\pi i I_n + \frac{(2\pi i P)^2}{2i} I_n + \cdots) P$$

$$= I_n - P + \exp(2\pi i)P = I_n.$$

The exponential map $\delta_0$ from $K_0(A/J)$ to $K_1(J)$ is defined by

$$\delta_0(x) := [\exp(2\pi i \sigma)].$$

2. Original result by J. Plastiras

In this section we describe original result by J. Plastiras. He exhibited two C*-algebras as the following:

$$\mathcal{A} := \{T \oplus T \mid T \in B(H)\} + K(H \oplus H),$$

$$\mathcal{B} := \{0 \oplus T \oplus T \mid 0 \in B(C), T \in B(H)\} + K(C \oplus H \oplus H).$$

Theorem 2.1. In this setting, $M_2 \otimes \mathcal{A}$ is *-isomorphic to $M_2 \otimes \mathcal{B}$.

Proof. By the definition of $\mathcal{A}$ and $\mathcal{B}$, we can see

$$M_2 \otimes \mathcal{A} = \begin{cases}
T_{11} & T_{12} \\
T_{21} & T_{22}
\end{cases} \begin{cases}
T_{11} & T_{12} \\
T_{21} & T_{22}
\end{cases} \mid T_{ij} \in B(H)
+ K(H \oplus H \oplus H \oplus H),$$

$$M_2 \otimes \mathcal{B} = \begin{cases}
0 & T_{12} \\
T_{11} & 0 \\
0 & T_{12}
\end{cases} \begin{cases}
T_{11} & T_{12} \\
T_{21} & T_{22}
\end{cases} \mid T_{ij} \in B(H)
+ K(C \oplus H \oplus H \oplus C \oplus H \oplus H).$$

Let $\{e_i\}$ be a completely orthonormal system for $H$ and $S$ the unilateral shift operator on $H$, i.e., $S e_i = e_{i+1} (n \in \mathbb{N})$. We define a linear operator $U$ from $C \oplus H \oplus H \oplus C \oplus H \oplus H$ to $H \oplus H \oplus H \oplus H$ by

$$U(\lambda_1, \xi, \eta, \lambda_2, \xi', \eta') := (\lambda_1 e_0 + S \xi, \lambda_2 e_0 + S \eta, \xi', \eta').$$
where $\lambda_1, \lambda_2 \in \mathbb{C}$ and $\xi, \xi', \eta, \eta' \in \mathcal{H}$. By the fact

$$
\| (\lambda_1 e_0 + S\xi, \lambda_2 e_0 + S\eta, \xi', \eta') \|^2 \\
= \| (\lambda_1 e_0 + S\xi) \|^2 + \| (\lambda_2 e_0 + S\eta) \|^2 + \| \xi' \|^2 + \| \eta' \|^2 \\
= |\lambda_1|^2 + \| \xi \|^2 + |\lambda_2|^2 + \| \eta \|^2 + \| \xi' \|^2 + \| \eta' \|^2 \\
= \| (\lambda_1, \xi, \eta, \lambda_2, \xi', \eta') \|^2,
$$

we can see $U$ is unitary. Then we have, for $T_{ij} \in \mathbb{B}(\mathcal{H})$ and $K \in \mathbb{K}(\mathbb{C} \oplus \mathcal{H} \oplus \mathcal{H} \oplus \mathbb{C} \oplus \mathcal{H} \oplus \mathcal{H})$,

$$
U(\begin{bmatrix}
0 & T_{11} & 0 & T_{12} \\
T_{11} & 0 & T_{12} \\
0 & T_{21} & 0 & T_{22} \\
T_{21} & 0 & T_{22}
\end{bmatrix}) + K)U^* \\
= U(\begin{bmatrix}
0 & T_{11} & 0 & T_{12} \\
T_{11} & 0 & T_{12} \\
0 & T_{21} & 0 & T_{22} \\
T_{21} & 0 & T_{22}
\end{bmatrix}) [ \begin{array}{c}
(\xi, e_0)
\\
S^*\xi
\\
S^*\eta
\\
(\eta, e_0)
\end{array} ] + UKU^* \\
= U(\begin{bmatrix}
0 & T_{11}S^*\xi + T_{12}\xi' \\
T_{11}S^*\eta + T_{12}\eta' \\
0 & T_{21}S^*\xi + T_{22}\xi' \\
T_{21}S^*\eta + T_{22}\eta'
\end{bmatrix}) + UKU^* \\
= (\begin{bmatrix}
ST_{11}S^* & ST_{12} \\
T_{21}S^* & T_{22}
\end{bmatrix}) + UKU^* \\
= \left( \begin{array}{c}
(\xi) \\
(\eta)
\end{array} \right),
$$

where $(,)$ means the inner product of $\mathcal{H}$. Consequently, it is verified that $U(\mathbb{M}_2 \otimes \mathbb{B})U^* \subset \mathbb{M}_2 \otimes \mathbb{A}$. By the similar computation, we can have $U^*(\mathbb{M}_2 \otimes \mathbb{A})U \subset \mathbb{M}_2 \otimes \mathbb{B}$. So $\mathbb{M}_2 \otimes \mathbb{A}$ is $\ast$-isomorphic to $\mathbb{M}_2 \otimes \mathbb{B}$. $\square$
For comparing $A$ and $B$, we will introduce the notion of the essential commutant algebra and the basic theory of AF-algebras.

**Definition 2.2.** (Essential commutant) For $C \subset B(H)$, the essential commutant for $C$ (denoted by $EC(C)$) is defined by

$$EC(C) := \{ X \in B(H) \mid XY - YX \in K(H) \text{ for all } Y \in C \}$$

**Lemma 2.3.** For a separable Hilbert space $H$, we have

$$EC(B(H)) = C1_H + K(H),$$

where $1_H$ is the identity operator on $H$.

*Proof.* It is trivial that $EC(B(H)) \supset C1_H + K(H)$.

It is sufficient to show that the reverse implication holds. Since $EC(B(H))$ is a closed $^*$-subalgebra of $B(H)$, any element in $EC(B(H))$ is represented by a linear combination of self-adjoint elements. Let $T$ be a self-adjoint element in $EC(B(H))$ and its spectral decomposition

$$T = \int_{-\|T\|}^{\|T\|} \lambda d\varepsilon(\lambda),$$

where $\{\varepsilon(\lambda)\}$ is the right continuous spectral family of projections for $T$.

For $-\|T\| < a < b < \|T\|$, we assume that two projections

$$\int_{-\|T\|}^{a} d\varepsilon(\lambda) \text{ and } \int_{b}^{\|T\|} d\varepsilon(\lambda)$$

are infinitely dimensional. Since $H$ is separable, there exists a partial isometry $V$ such that

$$V^*V = \int_{b}^{\|T\|} d\varepsilon(\lambda), \quad VV^* = \int_{-\|T\|}^{a} d\varepsilon(\lambda).$$

Then we have

$$(VT - TV)V^* = V \int_{b}^{\|T\|} \lambda d\varepsilon(\lambda)V^* - \int_{-\|T\|}^{a} \lambda d\varepsilon(\lambda)$$

$$\geq b \int_{-\|T\|}^{a} \lambda d\varepsilon(\lambda) - \int_{-\|T\|}^{a} \lambda d\varepsilon(\lambda)$$

$$\geq (b - a) \int_{-\|T\|}^{a} \lambda d\varepsilon(\lambda) \notin K(H).$$

This means that $VT - TV \notin K(H)$, i.e., $T \notin EC(B(H))$. 
This fact implies that $\sigma(T)$ has at most one accumulation point. If an accumulation point $c$ exists, then each $\lambda \in \sigma(T) \setminus \{c\}$ is an eigenvalue for $T$ and its eigenprojection is finite dimensional. So we have

$$T - c1_\mathcal{H} \in \mathbb{K}(\mathcal{H}).$$

If an accumulation point does not exist, then $\sigma(T)$ is a finite set of eigenvalues for $T$ and their eigenprojections are finite dimensional except for one point $c$. Also we have

$$T - c1_\mathcal{H} \in \mathbb{K}(\mathcal{H}).$$

Now we will give some facts on approximately finite dimensional C*-algebras (standing for AF-algebras) which are the direct limits of increasing sequences of finite dimensional C*-algebras. For an AF-algebra $A$, $K_1(A) = 0$. This is because finite dimensional C*-algebras are isomorphic to the direct sum of matrices over $\mathbb{C}$, the distributivity and the continuity of $K$-groups, and $K_1(M_n) = 0$.

When we put $K_0(A)_+ := \text{Im}(\iota)$, where $\iota$ is the natural inclusive map from $V(A)$ to $K_0(A)$. By $K_0(A)_+$, $K_0(A)$ becomes the ordered group: for $x, y \in K_0(A)$, $x \leq y$ if $y - x \in K_0(A)_+$

**Definition 2.4.** (Dimension group) The dimension group associated to an AF-algebra $A$ is the ordered group $(K_0(A), K_0(A)_+)$.

**Definition 2.5.** (Scale $\Gamma$) For an unital C*-algebra $A$, the scale of $A$ is defined by

$$\Gamma(A) := \{[P] \mid P \text{ is a projection in } A\}.$$

**Lemma 2.6.** (The theorem of Elliott) For AF-algebras $A$ and $B$, $A$ is $\ast$-isomorphic to $B$ if and only if there is a group isomorphism from $K_0(A)$ to $K_0(B)$ which preserves their scales and their ordered cones.

**Theorem 2.7.** The $A$ is not $\ast$-isomorphic to the $B$.

**Proof.** Since $A(\subset B(\mathcal{H} \oplus \mathcal{H}))$ and $B(\subset B(\mathbb{C} \oplus \mathcal{H} \oplus \mathcal{H}))$ contain compact operators, $A \cong B$ implies $EC(A) \cong EC(B)$. Therefore it is sufficient to prove that $EC(A) \not\cong EC(B)$.

By the above lemma, the essential commutants can be represented as

$$EC(A) = \left\{ \begin{bmatrix} \lambda_{11}1_\mathcal{H} & \lambda_{12}1_\mathcal{H} \\ \lambda_{21}1_\mathcal{H} & \lambda_{22}1_\mathcal{H} \end{bmatrix} \mid \lambda_{ij} \in \mathbb{C} \right\} + \mathbb{K}(\mathcal{H} \oplus \mathcal{H}),$$

$$EC(B) = \left\{ \begin{bmatrix} 0 & 0 & 0 \\ 0 & \mu_{11}1_\mathcal{H} & \mu_{12}1_\mathcal{H} \\ 0 & \mu_{21}1_\mathcal{H} & \mu_{22}1_\mathcal{H} \end{bmatrix} \mid \mu_{ij} \in \mathbb{C} \right\} + \mathbb{K}(\mathbb{C} \oplus \mathcal{H} \oplus \mathcal{H})$$
Let \( \{p_i\}_{i=1}^{\infty} \) (resp. \( \{q_i\}_{i=1}^{\infty} \)) be a family of orthogonal projections of rank 1 on \( \mathcal{H} \oplus \mathcal{H} \) with \( \sum_{i=1}^{\infty} p_i = 1_\mathcal{H} \oplus 0_\mathcal{H} \) (resp. \( \sum_{i=1}^{\infty} q_i = 0_\mathcal{H} \oplus 1_\mathcal{H} \)). We set

\[
A_n = \left\{ \begin{bmatrix} \lambda_{11}1_\mathcal{H} & \lambda_{12}1_\mathcal{H} \\ \lambda_{21}1_\mathcal{H} & \lambda_{22}1_\mathcal{H} \end{bmatrix} + \left( \sum_{i=1}^{n} p_i + q_i \right)x \left( \sum_{i=1}^{n} p_i + q_i \right) \mid \lambda_{ij} \in \mathbb{C}, \ x \in \mathbb{B}(\mathcal{H} \oplus \mathcal{H}) \right\}
\]

\[
\cong \mathbb{M}_2 \oplus \mathbb{M}_{2n}
\]

\[
B_n = \left\{ \begin{bmatrix} 0 & 0 & 0 \\ 0 & \mu_{11}1_\mathcal{H} & \mu_{12}1_\mathcal{H} \\ 0 & \mu_{21}1_\mathcal{H} & \mu_{22}1_\mathcal{H} \end{bmatrix} + \left( r + \sum_{i=1}^{n} p_i + q_i \right)x \left( r + \sum_{i=1}^{n} p_i + q_i \right) \mid \mu_{ij} \in \mathbb{C}, \ x \in \mathbb{B}(\mathbb{C} \oplus \mathcal{H} \oplus \mathcal{H}) \right\}
\]

\[
\cong \mathbb{M}_2 \oplus \mathbb{M}_{2n+1}
\]

where \( r \) is the identity operator of \( \mathbb{B}(\mathbb{C}) \cong \mathbb{C} \). Then we have

\[
EC(\mathfrak{A}) = \bigcup_{n=0}^{\infty} A_n \ \text{and} \ EC(\mathfrak{B}) = \bigcup_{n=0}^{\infty} B_n.
\]

So the essential commutants have the following the Bratteli diagrams which represents the embedded manner of the sequence of increasing finite dimensional C*-algebras.

In these figures numbers are the size of matrices and arrows are represented as the manner of embedding of projections in matrices by the
multiplicity one. By the facts

\[ K_0(A_n) = \mathbb{Z} \oplus \mathbb{Z} \]
\[ K_0(B_n) = \mathbb{Z} \oplus \mathbb{Z}, \]

and the above inductive systems, we can get

\[ K_0(EC(A)) = \lim_{\sim} K_0(A_n) \cong \mathbb{Z} \oplus \mathbb{Z} \]
\[ K_0(EC(B)) = \lim_{\sim} K_0(B_n) \cong \mathbb{Z} \oplus \mathbb{Z} \]

and the following figures of groups, ordered cones and scales:

\[ K_0(EC(A))_+ \quad K_0(EC(B))_+ \]

Then the class of unit in \( K_0(EC(A)) \) is \( \left[ \begin{pmatrix} 1_{\mathcal{H}} & 0 \\ 0 & 1_{\mathcal{H}} \end{pmatrix} \right] \). Therefore we have \( \left[ \begin{pmatrix} 1_{\mathcal{H}} & 0 \\ 0 & 1_{\mathcal{H}} \end{pmatrix} \right] = 2 \left[ \begin{pmatrix} 1_{\mathcal{H}} & 0 \\ 0 & 0 \end{pmatrix} \right] \). In the other hand, the unit class of \( K_0(EC(B)) \) is \( \left[ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1_{\mathcal{H}} & 0 \\ 0 & 0 & 1_{\mathcal{H}} \end{pmatrix} \right] \). So it is immediately realized that there does not exist the element \( x \) in \( K_0(EC(B)) \) such that \( x \oplus x = \left[ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1_{\mathcal{H}} & 0 \\ 0 & 0 & 1_{\mathcal{H}} \end{pmatrix} \right] \) so that we cannot construct the scaled, ordered, and group-isomorphic map. Using Elliott’s result, it follows that \( EC(A) \not\cong EC(B) \).

\[ \square \]

Remark 1. In the above proof of non-isomorphism, the part of AF-algebraic argument is different from the original one.
3. Main result

In this section, we present two ways on the non-isomorphism proof which are different from J. Plastiras. The one is a elementary proof and another is K-theoretical.

At the first, we will present the Fredholm operator, the (Fredholm) Index and their properties.

**Definition 3.1. (Fredholm operator, Index)** We call $T \in \mathcal{B}(\mathcal{H})$ a Fredholm operator if $\pi(T)$ is invertible in $\mathcal{B}(\mathcal{H})/\mathbb{K}(\mathcal{H})$. Then $\text{Im}T$ and $\text{Im}T^*$ are closed and $\ker T$ and $\ker T^*$ are finite-dimensional. The index of $T$ $(\text{Index} T)$ is defined by $\dim \ker T - \dim \ker T^*$.

The followings are well-known:

1. $\text{Index}(T) = \text{Index}(T + K)$ for all $K \in \mathbb{K}(\mathcal{H})$.
2. If $S$ is a unilateral shift operator, $\text{Index}(S) = -1$ and $\text{Index}(S^*) = 1$.

**Lemma 3.2.** If $C^*$-algebras $\mathcal{B}_1$ and $\mathcal{B}_2$ act on $\mathcal{H}$ and contain $\mathbb{K}(\mathcal{H})$ and they are $*$- isomorphic, then the $*$-isomorphism map is given by $\text{Ad}(u)$ where $u$ is a unitary in $\mathcal{B}(\mathcal{H})$.

**Proof.** Let $\varphi$ be an isomorphism between $\mathcal{B}_1$ and $\mathcal{B}_2$. For a completely orthonormal system $\{e_i\}_{i \in \mathbb{H}} \subset \mathcal{H}$, a projection $P_{ij} \in \mathcal{B}(\mathcal{H})$ of rank 1 is defined by $P_{ij} := (\cdot, e_j)e_i$. So $\{P_{ii}\}$ is the family of orthogonal projections of rank 1 in $\mathbb{K}(\mathcal{H})$ such that $\sum P_{ii} = 1_{\mathcal{H}}$. Then it can be found that $\varphi(P_{11}) := Q_{11}$ is a minimal projection where $Q_{11} = (\cdot, f_1)f_1$ with respect to another completely orthonormal system $\{f_i\}_{i \in \mathbb{H}} \subset \mathcal{H}$. Let $v$ be a partial isometry such that $v^*v = P_{11}$ and $vv^* = Q_{11}$. Now all we have to do is that $v$ is extended to the unitary $u$ on $\mathcal{H}$ such that $\varphi(P_{ij}) = uP_{ij}u^*$. The $u$ is defined by $u := \sum \varphi(P_{ki})vP_{ik}$. Then we have, for $\xi \in \mathcal{H},$

$$
\|u\xi\| = \left(\sum \varphi(P_{ki})vP_{ik}(\xi), \sum \varphi(P_{li})vP_{li}(\xi)\right) \\
= \sum_{k,l} (\xi, e_k)(\xi, e_l)(\varphi(P_{ki})f_1, \varphi(P_{li})f_1) \\
= \sum_k (\xi, e_k)^2 = \|\xi\|
$$
\[ uu^* = \sum_{k,l} \phi(P_{kl}) v P_{1k} v^* \phi(P_{1l}) \]
\[ = \sum_k \phi(P_{1k}) v P_{11} v^* \phi(P_{1k}) \]
\[ = \sum_k \phi(P_{1k}) \phi(P_{11}) \phi(P_{1k}) \]
\[ = \sum_k \phi(e_{kk}) = 1, \]

\[ u^* u = \sum_{k,l} P_{kl} v^* \phi(P_{kl}) \phi(P_{1l}) v P_{1l} \]
\[ = \sum_k P_{kl} v^* \phi(P_{11}) v P_{1k} \]
\[ = \sum_k P_{kl} P_{1l} \]
\[ = \sum_k P_{kk} = 1. \]

Since the range of \( u \) is clearly dense, \( u \) is a unitary operator. And also since the following holds;

\[ u P_{ij} = \sum_k \phi(P_{k1}) v P_{1k} P_{ij} \]
\[ = \phi(P_{11}) v P_{ij}, \]
\[ \varphi(P_{ij}) u = \varphi(P_{ij}) \sum_k \phi(P_{kl}) v P_{1k} \]
\[ = \varphi(P_{11}) v P_{ij}. \]

It is shown that an isomorphism \( \varphi \) induces \( Ad(u) \) on \( \mathbb{K}(\mathcal{H}) \) such that \( Ad(u)(a) = uau^* \) for all \( a \in \mathbb{K}(\mathcal{H}) \). It is found that since \( \varphi(ab) = uabu^* = uau^*bu^* = \varphi(a)ubu^* \) for \( b \in B_1 \), we have \( \varphi(b) = ubu^* \). Consequently, \( \varphi = Ad(u) \).

**Theorem 3.3.** Let \( \mathfrak{A} \) and \( \mathfrak{B} \) be as the above. Then we have that \( \mathfrak{A} \) is not \( * \)-isomorphic to \( \mathfrak{B} \).
Proof. It is verified that the following sequences are exact:

\[ E_1 : 0 \longrightarrow \mathbb{K}(\mathcal{H} \oplus \mathcal{H}) \stackrel{\iota}{\longrightarrow} \mathfrak{A} \stackrel{\pi}{\longrightarrow} \mathbb{B}(\mathcal{H} \oplus \mathcal{H})/\mathbb{K}(\mathcal{H} \oplus \mathcal{H}) \longrightarrow 0, \]

\[ E_2 : 0 \longrightarrow \mathbb{K}(\mathcal{C} \oplus \mathcal{H} \oplus \mathcal{H}) \stackrel{\iota}{\longrightarrow} \mathfrak{B} \stackrel{\pi}{\longrightarrow} \mathbb{B}(\mathcal{H} \oplus \mathcal{H})/\mathbb{K}(\mathcal{H} \oplus \mathcal{H}) \longrightarrow 0 \]

where \( \iota \) is the natural inclusion map and \( \pi \) is the surjective map such that \( \pi(T + T + K) = [T] \oplus [T] \), and \( \pi(0 \oplus T + T + L) = [T] \oplus [T] \) on \( E_1 \) and \( E_2 \) respectively. We define a linear operator \( U \) from \( \mathcal{C} \oplus \mathcal{H} \oplus \mathcal{H} \) to \( \mathcal{H} \oplus \mathcal{H} \) by \( U(\lambda, \xi, \eta) := (\lambda e_0 + S\xi, \eta) \). Then the \( U \) is unitary. It is found that \( U\mathfrak{B}U^* \subset \mathbb{B}(\mathcal{H} \oplus \mathcal{H}) \). Since it is checked that

\[ \mathfrak{B} \cong \{STS^* + T \mid T \in \mathbb{B}(\mathcal{H})\} + \mathbb{K}(\mathcal{H} \oplus \mathcal{H}) \subset \mathbb{B}(\mathcal{H} \oplus \mathcal{H}), \]

\( E_2 \) can be slightly modified as

\[ E_2 : 0 \longrightarrow \mathbb{K}(\mathcal{H} \oplus \mathcal{H}) \stackrel{\iota}{\longrightarrow} \mathfrak{B} \stackrel{\pi}{\longrightarrow} \mathbb{B}(\mathcal{H} \oplus \mathcal{H})/\mathbb{K}(\mathcal{H} \oplus \mathcal{H}) \longrightarrow 0 \]

where \( \pi(STS^* + T + L) = [STS^*] \oplus [T] \) is well-defined. By the above lemma, if \( \mathfrak{A} \) is \( \ast \)-isomorphic to \( \mathfrak{B} \), then there exists a unitary \( u \in \mathbb{B}(\mathcal{H} \oplus \mathcal{H}) \) such that

\[ \pi(u) \begin{pmatrix} [T] & 0 \\ 0 & [T] \end{pmatrix} \pi(u)^* = \begin{pmatrix} [STS^*] & 0 \\ 0 & [T] \end{pmatrix} \text{ for all } T \in \mathbb{B}(\mathcal{H}). \]

Let \( u = \begin{pmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{pmatrix} \). Then we have the following relations:

\[ u_{11}T - STS^*u_{11} \in \mathbb{K}(\mathcal{H}), \quad u_{12}T - STS^*u_{12} \in \mathbb{K}(\mathcal{H}), \]
\[ u_{21}T - Tu_{21} \in \mathbb{K}(\mathcal{H}), \quad u_{22}T - Tu_{22} \in \mathbb{K}(\mathcal{H}) \text{ for all } T \in \mathbb{B}(\mathcal{H}). \]

By Lemma 2.3, it is found that \( u = \begin{pmatrix} \lambda_1 S & \lambda_2 S \\ \lambda_3 & \lambda_4 \end{pmatrix} = \begin{pmatrix} S & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \lambda_1 & \lambda_2 \\ \lambda_3 & \lambda_4 \end{pmatrix} \)

where \( \lambda_i \in \mathbb{C} \). The Index of \( u \) is equal to 0 and that of the right hand is equal to \(-1\). Therefore there does not exist a unitary \( u \in \mathbb{B}(\mathcal{H} \oplus \mathcal{H}) \). Consequently, \( \mathfrak{A} \) is not \( \ast \)-isomorphic to \( \mathfrak{B} \).

Theorem 3.4. Let \( \mathfrak{A} \) and \( \mathfrak{B} \) be as the above. Then we have that \( \mathfrak{A} \) is not \( \ast \)-isomorphic to \( \mathfrak{B} \).
Proof. (K-theoretical). We have the following short exact sequences:

\[ E_1 : \ 0 \longrightarrow \mathbb{K} (\mathcal{H} \oplus \mathcal{H}) \longrightarrow \mathfrak{A} \longrightarrow \mathbb{B} (\mathcal{H}) / \mathbb{K} (\mathcal{H}) \longrightarrow 0 \]

\[ E_2 : \ 0 \longrightarrow \mathbb{K} (\mathcal{H} \oplus \mathcal{H}) \longrightarrow \mathfrak{B} \longrightarrow \mathbb{B} (\mathcal{H}) / \mathbb{K} (\mathcal{H}) \longrightarrow 0 \]

where \( \iota \) is the natural inclusion map and \( \pi \) is the surjective map such that \( \pi (T \oplus T + K) = [T] \) and \( \pi (STS^* \oplus T + L) = [T] \). The six term exact sequence is here applied for the first short exact sequence \( E_1 \).

\[
\begin{array}{c}
\delta_1 : K_0 (\mathbb{K} (\mathcal{H} \oplus \mathcal{H})) \longrightarrow K_0 (\mathfrak{A}) \longrightarrow K_0 (\mathbb{B} (\mathcal{H}) / \mathbb{K} (\mathcal{H})) \\
\end{array}
\]

\[
\begin{array}{c}
\delta_1 : K_1 (\mathbb{B} (\mathcal{H}) / \mathbb{K} (\mathcal{H})) \longrightarrow K_1 (\mathfrak{A}) \longrightarrow K_1 (\mathbb{K} (\mathcal{H} \oplus \mathcal{H})) \\
\end{array}
\]

Since it is known that the Fredholm Index correspond to the connected component in \( \mathbb{B} (\mathcal{H}) / \mathbb{K} (\mathcal{H}) \), and then the class \( [S^*] \) is the generator in \( K_1 (\mathbb{B} (\mathcal{H}) / \mathbb{K} (\mathcal{H})) \). So we need to observe where \( [S^*] \) go into \( K_0 (\mathbb{K} (\mathcal{H} \oplus \mathcal{H})) \) through \( \delta_1 \). \( \delta_1 \) is defined by the following:

\[
\delta_1 ([S^*]) := [u^* \oplus u^* (1_{\mathcal{H}} \oplus 1_{\mathcal{H}} \oplus 0 \oplus 0) u \oplus u] - [1_{\mathcal{H}} \oplus 1_{\mathcal{H}} \oplus 0 \oplus 0]
\]

where \( [S^*] \in \mathbb{B} (\mathcal{H}) / \mathbb{K} (\mathcal{H}) \) and \( u \in M_2 (\mathbb{B} (\mathcal{H})) \) is a unitary lift of \( [S \oplus S^*] \in M_2 (\mathbb{B} (\mathcal{H}) / \mathbb{K} (\mathcal{H})) \). So we have \( \delta_1 ([S^*]) = 2[p] \) for a 1-dimensional projection \( p \). From the fact that \( K_0 (\mathbb{B} (\mathcal{H}) / \mathbb{K} (\mathcal{H})) = 0 \), it is easily verified that

\[
K_0 (\mathfrak{A}) \cong K_0 (\mathbb{K} (\mathcal{H} \oplus \mathcal{H})) / \text{Im} \delta_1 = \mathbb{Z} / 2 \mathbb{Z}.
\]

By the similar argument, it will be found that \( K_0 (\mathfrak{B}) = \mathbb{Z} / 2 \mathbb{Z} \) where \( \delta_1 \) is defined by

\[
\delta_1 ([S^*]) := [Su S^* \oplus u^* (S1_{\mathcal{H}} S^* \oplus 1_{\mathcal{H}} \oplus 0 \oplus 0) Su S^* \oplus u] - [S1_{\mathcal{H}} S^* \oplus 1_{\mathcal{H}} \oplus 0 \oplus 0].
\]

Now the unit class \([1] \in K_0 (\mathfrak{A})\) will be compared to \([1] \in K_0 (\mathfrak{B})\).

\[
K_0 (\mathfrak{A}) \ni [1] = [1_{\mathcal{H}} \oplus 1_{\mathcal{H}}] = 2[1_{\mathcal{H}} \oplus 0] = [0] \in \mathbb{Z} / 2 \mathbb{Z},
\]

\[
K_0 (\mathfrak{B}) \ni [1] = [1_{\mathcal{H}} - p \oplus 1_{\mathcal{H}}] = [1_{\mathcal{H}} \oplus 1_{\mathcal{H}}] + [p \oplus 0] = [1] \in \mathbb{Z} / 2 \mathbb{Z}.
\]

This means that the unit class \([1] \in K_0 (\mathfrak{A})\) is different from \([1] \in K_0 (\mathfrak{B})\). Then it can be concluded that \( \mathfrak{A} \) is not \(*\)-isomorphic to \( \mathfrak{B} \). \( \square \)

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References


*Department of Mathematics and Informatics, Graduate School of Science and Technology Chiba University, 1-33, Yayoi-Cho, Inage-Ku, Chiba 263-8522 Japan
E-mail address: 99um0102@g.math.s.chiba-u.ac.jp*