ISOMORPHISMS OF C*-ALGEBRAS AFTER TENSORING

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ABSTRACT. J. Plastiras exhibited C*-algebras which are not isomorphic but, after tensoring by M_2, isomorphic. On proof of non-isomorphism of them, we give two ways which are different from his original one.

0. Introduction

It is well known that the algebra of all complex valued continuous functions on a compact Hausdorff space becomes an abelian C*-algebra with respect to the supremum norm and every abelian C*-algebra is realized as such a C*-algebra by Gelfand's representation theorem. By this correspondence we can see properties of abelian C*-algebras as those of topological spaces (compact Hausdorff spaces). So we can regard a general C*-algebra as an extended topological object (for example, non-commutative topological space). In the theory of algebraic topology, homology groups and cohomology groups work well as topological invariants. In the theory of C*-algebra, extension theory (resp. K-theory) also works well as homology theory (resp. cohomology theory).

J. Plastiras constructed the example of two C*-algebras such that they are not isomorphic but become isomorphic after tensoring with a matrix algebra. In this paper we look his example from the extension theoretical point of view, and we give the proof of non-existense of isomorphism using K-theory.

1. Preliminaries

Throughout this paper we denote the set of complex numbers, real numbers, integers and nonnegative integers as $\mathbb{C}, \mathbb{R}, \mathbb{Z}$ and $\mathbb{N}$ respectively. Let $\mathcal{H}$ be a separable infinite dimensional Hilbert space. We denote by $\mathbb{B}(\mathcal{H})$ (resp. $\mathbb{K}(\mathcal{H})$) the set of bounded linear operators (resp. compact operators) on $\mathcal{H}$. $\mathbb{M}_n$ stands for the $n \times n$ matrix algebra over $\mathbb{C}$.

In this section, we will present some basic facts on Extension theory and K-theory for C*-algebras. Let A, B and C be C*-algebras and α (resp. β) a *-homomorphism from A to B (resp. from B to C). We call a short exact sequence E as below an extension of A by C:

$$E : 0 \longrightarrow A \xrightarrow{\alpha} B \xrightarrow{\beta} C \longrightarrow 0.$$

Then α is injective, β is surjective and $\mathrm{Im}\alpha = \mathrm{Ker}\beta$.

When A is a *-subalgebra of $\mathbb{B}(\mathcal{H})$ and acts non-degenerately on $\mathcal{H}$ (i.e., if $\xi \in \mathcal{H}$ satisfies $\xi a = 0$ for all $a \in A$, then $\xi = 0$), we define the multiplier algebra $M(A)$ for A as follows:

$$M(A) = \{x \in \mathbb{B}(\mathcal{H}) \mid xA \subset A, \quad Ax \subset A\}.$$

Clearly we have that A becomes a closed two-sided *-ideal of $M(A)$ and the multiplier algebra $M(\mathbb{K}(\mathcal{H}))$ of $\mathbb{K}(\mathcal{H})$ coincides with $\mathbb{B}(\mathcal{H})$. A double centralizer for A is a pair (L, R) of functions $L, R : A \longrightarrow A$ satisfying

$$R(x)y = xL(y)$$

for all $x, y \in A$. For an element $x \in A$, (L_x, R_x) becomes a double centralizer of A, where

$$L_x : A \ni y \longmapsto xy \in A, \qquad R_x : A \ni y \longmapsto yx \in A.$$

It is known that the set of all double centralizers $DC(A)$ for A becomes a C*-algebra and $DC(A)$ is isomorphic to the multiplier algebra $M(A)$ for A.

For the above extension (in this case α is injective and $\alpha(A)$ is a closed two-sided *-ideal of B), we can uniquely define the $*$-homomorphism σ from B to $M(A)(= DC(A))$ with $\sigma \circ \alpha = \iota$, that is,

$$\begin{array}{ccc} A & \xrightarrow{\alpha} & B \\ \Big\| & & \Big\downarrow \sigma \\ A & \xrightarrow[\iota]{} & M(A), \end{array}$$

where $\iota(x) = (L_x, R_x) \in DC(A) \cong M(A)$ $(x \in A)$. Indeed σ is defined as follows:

$$\sigma(\alpha(x)) = (L(\alpha(x)), R(\alpha(x))) \in DC(A) \cong M(A),$$

where

$$L(\alpha(x)) : A \ni y \longmapsto \alpha^{-1}(\alpha(x)\alpha(y)) \in A,$$
$$R(\alpha(x)) : A \ni y \longmapsto \alpha^{-1}(\alpha(y)\alpha(x)) \in A.$$

Definition 1.1. *(Busby invariant) For an extension*

$$E : 0 \longrightarrow A \xrightarrow{\alpha} B \xrightarrow{\beta} C \longrightarrow 0 \ ,$$

the Busby invariant for E is defined as the $*$-homomorphism τ_E from C to $M(A)/A$ given by

$$\tau_E(c) = \pi \circ \sigma(b) \ ,$$

where b is a lift of c through β and π is the quotient map from $M(A)$ to $M(A)/A$.

The Busby invariant τ_E is the unique $*$-homomorphism which makes the following diagram commutative :

$$\begin{array}{ccccccccc}
0 & \longrightarrow & A & \xrightarrow{\alpha} & B & \xrightarrow{\beta} & C & \longrightarrow & 0 \\
 & & \| & & \downarrow{\scriptstyle \sigma} & & \downarrow{\scriptstyle \tau_E} & & \\
0 & \longrightarrow & A & \xrightarrow{\iota} & M(A) & \xrightarrow{\pi} & M(A)/A & \longrightarrow & 0.
\end{array}$$

Proposition 1.2. *Let*

$$E_1 : 0 \longrightarrow A \longrightarrow B_1 \longrightarrow C \longrightarrow 0,$$

$$E_2 : 0 \longrightarrow A \longrightarrow B_2 \longrightarrow C \longrightarrow 0$$

be extensions and τ_1, τ_2 Busby invariants respectively.

(1) *(strongly isomorphic) $\tau_1 = \tau_2$ if and only if there is an unique $*$-isomorphism γ for which the diagram*

$$\begin{array}{ccccccccc}
0 & \longrightarrow & A & \longrightarrow & B_1 & \longrightarrow & C & \longrightarrow & 0 \\
 & & \| & & \downarrow{\scriptstyle \gamma} & & \| & & \\
0 & \longrightarrow & A & \longrightarrow & B_2 & \longrightarrow & C & \longrightarrow & 0
\end{array}$$

is commutative.

(2) *(strongly equivalent) There is a unitary $u \in M(A)$ such that $\tau_2(c) = \pi(u)\tau_1(c)\pi(u)^*$ if and only if there are a unitary $v \in M(A)$ and a $*$-isomorphism γ such that the diagram*

$$\begin{array}{ccccccccc}
0 & \longrightarrow & A & \longrightarrow & B_1 & \longrightarrow & C & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle Ad(v)} & & \downarrow{\scriptstyle \gamma} & & \| & & \\
0 & \longrightarrow & A & \longrightarrow & B_2 & \longrightarrow & C & \longrightarrow & 0
\end{array}$$

is commutative.

We will define K_0 and K_1 groups for any C*-algebra $\mathcal{A}$. For the definition of K_0-group for a C*-algebra $\mathcal{A}$ (denoted by $K_0(\mathcal{A})$), we will give some properties of projections. Let $\mathbb{M}_n(\mathcal{A})$ be an $n \times n$ matrix algebra with entries of $\mathcal{A}$. For $m, n \in \mathbb{N}$ with $m < n$, an inclusion map φ_{nm} from $\mathbb{M}_m(\mathcal{A})$ to $\mathbb{M}_n(\mathcal{A})$ is defined by the following way; for $x \in \mathbb{M}_m(\mathcal{A})$

$$\varphi_{nm}(x) := x \oplus 0_{n-m}$$

where $\oplus$ means the diagonal sum. That is, x is put into left upper part in $\mathbb{M}_n(\mathcal{A})$. Using this φ_{nm}, we can view $\mathbb{M}_m(\mathcal{A})$ as a subalgebra of $\mathbb{M}_n(\mathcal{A})$. We put $\mathbb{M}_\infty(\mathcal{A}) = \bigcup_{n=1}^{\infty} \mathbb{M}_n(\mathcal{A})$. For projections in $\mathbb{M}_\infty(\mathcal{A})$, we introduce some equivalent relations.

Definition 1.3. *(Murray-von Neumann equivalence) For projections $p, q \in \mathbb{M}_\infty(\mathcal{A})$, they are Murray-von Neumann equivalent if there exists a partial isometory v in $\mathbb{M}_\infty(\mathcal{A})$ such that $p = v^*v$ and $q = vv^*$, where $*$ is an involution of $\mathbb{M}_\infty(\mathcal{A})$.*

Definition 1.4. *(Homotopy equivalence) For projections $p, q \in \mathbb{M}_\infty(\mathcal{A})$, they are homotopy equivalent if there exist a positive integer N and a norm continuous path $\{P_t\}_{t\in[0,1]}$ in $\mathbb{M}_N(\mathcal{A})$ such that $P_0 = p$ and $P_1 = q$.*

Proposition 1.5. *For projections $p, q \in \mathbb{M}_\infty(\mathcal{A})$, if $\|p - q\| < 1$, then they are homotopy equivalent.*

It is known that the Murray-von Neumann equivalence (algebraic notion) and the homotopy equivalence (topological notion) are the same equivalence for $\mathbb{M}_\infty(\mathcal{A})$.

Definition 1.6.

$$V(\mathcal{A}) := \{projections\ in\ \mathbb{M}_\infty(\mathcal{A})\} / \sim$$

where $\sim$ is the Murray-von Neumann equivalent relation.

Definition 1.7. *For equivalence classes $[p]$, $[q] \in V(\mathcal{A})$, the addition of them is defined by*

$$[p] + [q] := [p \oplus q].$$

It can be easily verified that the above operation is well-defined and abelian. So $V(\mathcal{A})$ becomes an abelian semigroup with the unit $[0]$. Now let $V(\mathcal{A}) - V(\mathcal{A})$ be formal differences of $V(\mathcal{A})$ and we define the following equivalence relation.

Definition 1.8. *For* $[p_1]-[q_1]$, $[p_2]-[q_2] \in V(\mathcal{A})-V(\mathcal{A})$, $[p_1]-[q_1] \approx [p_2]-[q_2]$ *if there exist* $[r] \in V(\mathcal{A})$ *such that* $[p_1]+[q_2]+[r] = [p_2]+[q_1]+[r]$.

Definition 1.9. *(K_0-group) For a unital C*-algebra $\mathcal{A}$, the K_0-group for $\mathcal{A}$ is defined by*

$$K_0(\mathcal{A}) := \{V(\mathcal{A})-V(\mathcal{A})\} / \approx$$

where $\approx$ is the above relation.

We observe properties of $K_0(\mathcal{A})$. Let e_0-f_0, e_1-f_1 be elements in $K_0(\mathcal{A})$.

1.(abelian additivity) $(e_0-f_0)+(e_1-f_1) = (e_0+e_1)-(f_0+f_1)$
$= (e_1-f_1)+(e_0-f_0)$.

2.(the unit) The unit of $K_0(\mathcal{A})$ is e_0-e_0. (denoted as 0)

3.(existence of the inverse) The inverse of e_0-f_0 is f_0-e_0.

Therefore $K_0(\mathcal{A})$ becomes an abelian group.

We now define the K_1-group for a unital C*algebras $\mathcal{A}$, We denote by $\mathcal{U}_n(\mathcal{A})$ the set of unitary elements of $\mathbb{M}_n(\mathcal{A})$. This is the topological subgroup with respect to the norm topology. For $m, n \in \mathbb{N}$, when $m < n$, an inclusion map ϕ_{nm} from $\mathcal{U}_m(\mathcal{A})$ to $\mathcal{U}_n(\mathcal{A})$ is defined by for $x \in \mathcal{U}_m(\mathcal{A})$

$$\phi(x) := x \oplus 1_{n-m}.$$

We put $\mathcal{U}_\infty(\mathcal{A}) = \bigcup_{n=1}^{\infty} \mathcal{U}_n(\mathcal{A})$. We denote by $\mathcal{U}_n(\mathcal{A})_0$ the set of unitary elements homotopic to the unit 1_n of $\mathbb{M}_n(\mathcal{A})$. By the similar argument it is put that $\mathcal{U}_\infty(\mathcal{A})_0 = \bigcup_{n=1}^{\infty} \mathcal{U}_n(\mathcal{A})$.

Definition 1.10. *(K_1-group)*

$$K_1(\mathcal{A}) := \mathcal{U}_\infty(\mathcal{A})/\mathcal{U}_\infty(\mathcal{A})_0.$$

The K_1-group is an abelian group with the unit [1] under the multiplicative operation

$$[u][v] := [uv] = [u \oplus v].$$

For C*-algebras $\{\mathcal{A}_n\}$ and $*$-homomorphisms $\{\varphi_{nm} : \mathcal{A}_m \to \mathcal{A}_n, (m < n)\}$, We call $\{(\mathcal{A}_n, \varphi_{nm})\}$ an inductive system of C*-algebras if they satisfy for $l < m < n$, $\varphi_{nl} = \varphi_{nm} \circ \varphi_{ml}$. Then we define $\mathcal{A}_0$ and the semi-norm on $\mathcal{A}$ as the following:

$$\begin{aligned} \mathcal{A}_0 &= \{a = (a_n) \in \Pi_{n=1}^{\infty} \mathcal{A}_n |\ \textit{there exists } N_0 \\ &\qquad \textit{such that } \ \varphi_{mN_0}(a_{N_0}) = a_m \textit{ for } m > N_0\} \\ \|a\|_0 &= \lim_{n\to\infty} \|a_n\|. \end{aligned}$$

Then the completion $\mathcal{A}$ of $\mathcal{A}_0/\{a \in \mathcal{A}_0 | \|a\|_0 = 0\}$ becomes a C*-algebra, and $\mathcal{A}$ is called the inductive limit of the system and denoted by $\varinjlim \mathcal{A}_n$.

Theorem 1.11. *For an unital C*-algebra $\mathcal{A}$, $\mathcal{B}$ and $i = 0, 1$.*
(1) (Stability) $K_i(\mathcal{A} \otimes \mathbb{K}) = K_i(\mathcal{A})$.
(2) (Distributivity) $K_i(\mathcal{A} \oplus \mathcal{B}) = K_i(\mathcal{A}) \oplus K_i(\mathcal{B})$.
(3) (Continuity) If $\mathcal{A} = \varinjlim \mathcal{A}_n$. Then $K_i(\mathcal{A}) = \varinjlim K_i(\mathcal{A}_n)$.

Combining the basic facts and the following powerful theorem, we can compute K-groups of typical C*-algebras:

(1) $K_0(\mathbb{C}) = K_0(\mathbb{M}_n) = \mathbb{Z},\ K_1(\mathbb{C}) = K_1(\mathbb{M}_n) = 0.$
(2) $K_0(\mathbb{B}(\mathcal{H})) = K_0(\mathbb{B}(\mathcal{H})/\mathbb{K}(\mathcal{H})) = 0,\ K_1(\mathbb{B}(\mathcal{H})) = 0,$
$K_1(\mathbb{B}(\mathcal{H})/\mathbb{K}(\mathcal{H})) = \mathbb{Z}.$
(3) $K_0(\mathbb{K}(\mathcal{H})) = \mathbb{Z},\ K_1(\mathbb{K}(\mathcal{H})) = 0.$

Theorem 1.12. *(Six term exact sequence) Let $\mathcal{J}$ be an closed two-sided $*$-ideal in a unital C*-algebra $\mathcal{A}$. For a short exact sequence*

$$0 \longrightarrow \mathcal{J} \xrightarrow{\ \iota\ } \mathcal{A} \xrightarrow{\ \pi\ } \mathcal{A}/\mathcal{J} \longrightarrow 0\ ,$$

we have the following diagram which is exact at any part:

$$\begin{array}{ccccc} K_0(\mathcal{J}) & \xrightarrow{\ \iota_*\ } & K_0(\mathcal{A}) & \xrightarrow{\ \pi_*\ } & K_0(\mathcal{A}/\mathcal{J}) \\ {\scriptstyle \delta_1}\uparrow & & & & \downarrow{\scriptstyle \delta_0} \\ K_1(\mathcal{A}/\mathcal{J}) & \xleftarrow[\ \pi_*\ ]{} & K_1(\mathcal{A}) & \xleftarrow[\ \iota_*\ ]{} & K_1(\mathcal{A}) \end{array}$$

where ι_* and π_* are induced maps from ι and π respectively and δ_0 is the exponential map and δ_1 is the index map.

[the construction of δ_1] For $x \in K_1(\mathcal{A}/\mathcal{J})$, we can choose a unitary u in $\mathbb{M}_n(\mathcal{A}/\mathcal{J})$ such that $x = [u]$. So $\begin{pmatrix} u & 0 \\ 0 & u^* \end{pmatrix}$ is also a unitary in $\mathbb{M}_{2n}(\mathcal{A}/\mathcal{J})$ which is homotopic to 1_{2n}. Choosing a unitary lift w of $\begin{pmatrix} u & 0 \\ 0 & u^* \end{pmatrix}$ in $\mathbb{M}_{2n}(\mathcal{A})$ and a projection $P_n = \begin{pmatrix} 1_n & 0_n \\ 0_n & 0_n \end{pmatrix} \in \mathbb{M}_{2n}(\mathcal{A})$, we can define the index map δ_1 from $K_1(\mathcal{A}/\mathcal{J})$ to $K_0(\mathcal{J})$ as follows:

$$\delta_1(x) := [wP_nw^*] - [P_n].$$

[the construction of δ_0] For $x \in K_0(\mathcal{A}/\mathcal{J})$, we can choose a projection P in $\mathbb{M}_n(\mathcal{A}/\mathcal{J})$ such that $x = [P]$ and a self-adjoint lift f of P in

$\mathbb{M}_n(\mathcal{A})$. Then we have $\exp(2\pi i f) \in \mathcal{U}_n(\mathcal{J})$, since

$$\begin{aligned}\pi(\exp(2\pi i f)) &= \exp(2\pi i P) = I_n + 2\pi i P + \frac{(2\pi i P)^2}{2i} + \cdots \\ &= I_n - P + (I_n + 2\pi i I_n + \frac{(2\pi i)^2}{2i} I_n + \cdots)P \\ &= I_n - P + \exp(2\pi i)P = I_n.\end{aligned}$$

The exponential map δ_0 from $K_0(\mathcal{A}/\mathcal{J})$ to $K_1(\mathcal{J})$ is defined by

$$\delta_0(x) := [\exp(2\pi i f)].$$

2. Original result by J. Plastiras

In this section we describe original result by J. Plastiras. He exhibited two C*-algebras as the following :

$$\begin{aligned}\mathfrak{A} &:= \{T \oplus T \mid T \in \mathbb{B}(\mathcal{H})\} + \mathbb{K}(\mathcal{H} \oplus \mathcal{H}), \\ \mathfrak{B} &:= \{0 \oplus T \oplus T \mid 0 \in \mathbb{B}(\mathbb{C}), T \in \mathbb{B}(\mathcal{H})\} + \mathbb{K}(\mathbb{C} \oplus \mathcal{H} \oplus \mathcal{H}).\end{aligned}$$

Theorem 2.1. *In this setting,* $\mathbb{M}_2 \otimes \mathfrak{A}$ *is* $*$*-isomorphic to* $\mathbb{M}_2 \otimes \mathfrak{B}$*.*

Proof. By the definition of $\mathfrak{A}$ and $\mathfrak{B}$, we can see

$$\begin{aligned}\mathbb{M}_2 \otimes \mathfrak{A} &= \left\{ \begin{bmatrix} T_{11} & & T_{12} & \\ & T_{11} & & T_{12} \\ T_{21} & & T_{22} & \\ & T_{21} & & T_{22} \end{bmatrix} \mid T_{ij} \in \mathbb{B}(\mathcal{H}) \right\} \\ &+ \mathbb{K}(\mathcal{H} \oplus \mathcal{H} \oplus \mathcal{H} \oplus \mathcal{H}), \\ \mathbb{M}_2 \otimes \mathfrak{B} &= \left\{ \begin{bmatrix} 0 & & & 0 & & \\ & T_{11} & & & T_{12} & \\ & & T_{11} & & & T_{12} \\ 0 & & & 0 & & \\ & T_{21} & & & T_{22} & \\ & & T_{21} & & & T_{22} \end{bmatrix} \mid T_{ij} \in \mathbb{B}(\mathcal{H}) \right\} \\ &+ \mathbb{K}(\mathbb{C} \oplus \mathcal{H} \oplus \mathcal{H} \oplus \mathbb{C} \oplus \mathcal{H} \oplus \mathcal{H}) .\end{aligned}$$

Let $\{e_i\}$ be a completely orthonormal system for $\mathcal{H}$ and S the unilateral shift operator on $\mathcal{H}$, i.e.,$Se_i = e_{i+1} (n \in \mathbb{N})$. We define a linear operator U from $\mathbb{C} \oplus \mathcal{H} \oplus \mathcal{H} \oplus \mathbb{C} \oplus \mathcal{H} \oplus \mathcal{H}$ to $\mathcal{H} \oplus \mathcal{H} \oplus \mathcal{H} \oplus \mathcal{H}$ by

$$U(\lambda_1, \xi, \eta, \lambda_2, \xi', \eta') := (\lambda_1 e_0 + S\xi, \lambda_2 e_0 + S\eta, \xi', \eta'),$$

where $\lambda_1, \lambda_2 \in \mathbb{C}$ and $\xi, \xi', \eta, \eta' \in \mathcal{H}$. By the fact

$$\begin{aligned} &\|(\lambda_1 e_0 + S\xi, \lambda_2 e_0 + S\eta, \xi', \eta')\|^2 \\ &= \|(\lambda_1 e_0 + S\xi)\|^2 + \|(\lambda_2 e_0 + S\eta)\|^2 + \|\xi'\|^2 + \|\eta'\|^2 \\ &= |\lambda_1|^2 + \|\xi\|^2 + |\lambda_2|^2 + \|\eta\|^2 + \|\xi'\|^2 + \|\eta'\|^2 \\ &= \|(\lambda_1, \xi, \eta, \lambda_2, \xi', \eta')\|^2, \end{aligned}$$

we can see U is unitary. Then we have, for $T_{ij} \in \mathbb{B}(\mathcal{H})$ and $K \in \mathbb{K}(\mathbb{C} \oplus \mathcal{H} \oplus \mathcal{H} \oplus \mathbb{C} \oplus \mathcal{H} \oplus \mathcal{H})$,

$$\begin{aligned} &U(\begin{bmatrix} 0 & & & 0 & & \\ & T_{11} & & & T_{12} & \\ & & T_{11} & & & T_{12} \\ 0 & & & 0 & & \\ & T_{21} & & & T_{22} & \\ & & T_{21} & & & T_{22} \end{bmatrix} + K)U^* \begin{bmatrix} \xi \\ \eta \\ \xi' \\ \eta' \end{bmatrix} \\ &= U \begin{bmatrix} 0 & & & 0 & & \\ & T_{11} & & & T_{12} & \\ & & T_{11} & & & T_{12} \\ 0 & & & 0 & & \\ & T_{21} & & & T_{22} & \\ & & T_{21} & & & T_{22} \end{bmatrix} \begin{bmatrix} (\xi, e_0) \\ S^*\xi \\ S^*\eta \\ (\eta, e_0) \\ \xi' \\ \eta' \end{bmatrix} + UKU^* \begin{bmatrix} \xi \\ \eta \\ \xi' \\ \eta' \end{bmatrix} \\ &= U \begin{bmatrix} 0 \\ T_{11}S^*\xi + T_{12}\xi' \\ T_{11}S^*\eta + T_{12}\eta' \\ 0 \\ T_{21}S^*\xi + T_{22}\xi' \\ T_{21}S^*\eta + T_{22}\eta' \end{bmatrix} + UKU' \begin{bmatrix} \xi \\ \eta \\ \xi' \\ \eta' \end{bmatrix} \\ &= \begin{bmatrix} ST_{11}S^*\xi + ST_{12}\xi' \\ ST_{11}S^*\eta + ST_{12}\eta' \\ T_{21}S^*\xi + T_{22}\xi' \\ T_{21}S^*\eta + ST_{22}\eta' \end{bmatrix} + UKU^* \begin{bmatrix} \xi \\ \eta \\ \xi' \\ \eta' \end{bmatrix} \\ &= (\begin{bmatrix} ST_{11}S^* & & ST_{12} & \\ & ST_{11}S^* & & ST_{12} \\ T_{21}S^* & & T_{22} & \\ & T_{21}S^* & & T_{22} \end{bmatrix} + UKU^*) \begin{bmatrix} \xi \\ \eta \\ \xi' \\ \eta' \end{bmatrix}, \end{aligned}$$

where $(\ , \ )$ means the inner product of $\mathcal{H}$. Consequently, it is verified that $U(\mathbb{M}_2 \otimes \mathfrak{B})U^* \subset \mathbb{M}_2 \otimes \mathfrak{A}$. By the similar computation, we can have $U^*(\mathbb{M}_2 \otimes \mathfrak{A})U \subset \mathbb{M}_2 \otimes \mathfrak{B}$. So $\mathbb{M}_2 \otimes \mathfrak{A}$ is $*$-isomorphic to $\mathbb{M}_2 \otimes \mathfrak{B}$. $\square$

For comparing $\mathfrak{A}$ and $\mathfrak{B}$, we will introduce the notion of the essential commutant algebra and the basic theory of AF-algebras.

Definition 2.2. *(Essential commutant) For $\mathfrak{C} \subset \mathbb{B}(\mathcal{H})$, the essential commutant for $\mathfrak{C}$ (denoted by $EC(\mathfrak{C})$) is defined by*

$$EC(\mathfrak{C}) := \{X \in \mathbb{B}(\mathcal{H}) \mid XY - YX \in \mathbb{K}(\mathcal{H}) \text{ for all } Y \in \mathfrak{C}\}$$

Lemma 2.3. *For a separable Hilbert space $\mathcal{H}$, we have*

$$EC(\mathbb{B}(\mathcal{H})) = \mathbb{C}1_{\mathcal{H}} + \mathbb{K}(\mathcal{H}),$$

where $1_{\mathcal{H}}$ is the identity operator on $\mathcal{H}$.

Proof. It is trivial that $\mathrm{EC}(\mathbb{B}(\mathcal{H})) \supset \mathbb{C}1_{\mathcal{H}} + \mathbb{K}(\mathcal{H})$.

It is sufficient to show that the reverse implication holds. Since $\mathrm{EC}(\mathbb{B}(\mathcal{H}))$ is a closed *-subalgebra of $\mathbb{B}(\mathcal{H})$, any element in $\mathrm{EC}(\mathbb{B}(\mathcal{H}))$ is represented by a linear combination of self-adjoint elements. Let T be a self-adjoint element in $\mathrm{EC}(\mathbb{B}(\mathcal{H}))$ and its spectral decomposition

$$T = \int_{-\|T\|}^{\|T\|} \lambda de(\lambda),$$

where $\{e(\lambda)\}$ is the right continuous spectral family of projections for T.

For $-\|T\| < a < b < \|T\|$, we assume that two projections

$$\int_{-\|T\|}^{a} de(\lambda) \text{ and } \int_{b}^{\|T\|} de(\lambda)$$

are infinitely dimensional. Since $\mathcal{H}$ is separable, there exists a partial isometry V such that

$$V^*V = \int_{b}^{\|T\|} de(\lambda), \quad VV^* = \int_{-\|T\|}^{a} de(\lambda).$$

Then we have

$$\begin{aligned}(VT - TV)V^* &= V\int_{b}^{\|T\|} \lambda de(\lambda)V^* - \int_{-\|T\|}^{a} \lambda de(\lambda)\\ &\geq b\int_{-\|T\|}^{a} de(\lambda) - \int_{-\|T\|}^{a} \lambda de(\lambda)\\ &\geq (b-a)\int_{-\|T\|}^{a} de(\lambda) \notin \mathbb{K}(\mathcal{H}).\end{aligned}$$

This means that $VT - TV \notin \mathbb{K}(\mathcal{H})$, i.e., $T \notin EC(\mathbb{B}(\mathcal{H}))$.

This fact implies that $\sigma(T)$ has at most one accumulation point. If an accumulation point c exists, then each $\lambda \in \sigma(T)\backslash\{c\}$ is an eigenvalue for T and its eigenprojection is finite dimensional. So we have

$$T - c1_{\mathcal{H}} \in \mathbb{K}(\mathcal{H}).$$

If an accumulation point does not exist, then $\sigma(T)$ is a finite set of eigenvalues for T and their eigenprojections are finite dimensional except for one point c. Also we have

$$T - c1_{\mathcal{H}} \in \mathbb{K}(\mathcal{H}).$$

□

Now we will give some facts on approximately finite dimensional C*-algebras (standing for AF-algebras) which are the direct limits of increasing sequences of finite dimensional C*-algebras. For an AF-algebra $\mathcal{A}$, $K_1(\mathcal{A}) = 0$. This is because finite dimensional C*-algebras are isomorphic to the direct sum of matrices over $\mathbb{C}$, the distributivity and the continuity of K-groups, and $K_1(\mathbb{M}_n) = 0$.

When we put $K_0(\mathcal{A})_+ := \mathrm{Im}(\iota)$, where ι is the natural inclusive map from $V(\mathcal{A})$ to $K_0(\mathcal{A})$. By $K_0(\mathcal{A})_+$, $K_0(\mathcal{A})$ becomes the ordered group: for $x, y \in K_0(\mathcal{A})$, $x \leq y$ if $y - x \in K_0(\mathcal{A})_+$

Definition 2.4. *(Dimension group) The dimension group associated to an AF-algebra $\mathcal{A}$ is the ordered group $(K_0(\mathcal{A}), K_0(\mathcal{A})_+)$.*

Definition 2.5. *(Scale Γ) For an unital C*-algebra $\mathcal{A}$, the scale of $\mathcal{A}$ is defined by*

$$\Gamma(\mathcal{A}) := \{[P] \mid P \textit{ is a projection in } \mathcal{A}\}\,.$$

Lemma 2.6. *(The theorem of Elliott) For AF-algebras $\mathcal{A}$ and $\mathcal{B}$, $\mathcal{A}$ is $*$-isomorphic to $\mathcal{B}$ if and only if there is a group isomorphism from $K_0(\mathcal{A})$ to $K_0(\mathcal{B})$ which preserves their scales and their ordered cones.*

Theorem 2.7. *The $\mathfrak{A}$ is not $*$-isomorphic to the $\mathfrak{B}$.*

Proof. Since $\mathfrak{A}(\subset \mathbb{B}(\mathcal{H}\oplus\mathcal{H}))$ and $\mathfrak{B}(\subset \mathbb{B}(\mathbb{C}\oplus\mathcal{H}\oplus\mathcal{H}))$ contain compact operators, $\mathfrak{A} \cong \mathfrak{B}$ implies EC($\mathfrak{A}$) $\cong$EC($\mathfrak{B}$). Therefore it is sufficient to prove that EC($\mathfrak{A}$) $\ncong$ EC($\mathfrak{B}$).

By the above lemma, the essential commutants can be represented as

$$EC(\mathfrak{A}) = \left\{ \begin{bmatrix} \lambda_{11}1_{\mathcal{H}} & \lambda_{12}1_{\mathcal{H}} \\ \lambda_{21}1_{\mathcal{H}} & \lambda_{22}1_{\mathcal{H}} \end{bmatrix} \mid \lambda_{ij} \in \mathbb{C} \right\} + \mathbb{K}(\mathcal{H} \oplus \mathcal{H}),$$

$$EC(\mathfrak{B}) = \left\{ \begin{bmatrix} 0 & 0 & 0 \\ 0 & \mu_{11}1_{\mathcal{H}} & \mu_{12}1_{\mathcal{H}} \\ 0 & \mu_{21}1_{\mathcal{H}} & \mu_{22}1_{\mathcal{H}} \end{bmatrix} \mid \mu_{ij} \in \mathbb{C} \right\} + \mathbb{K}(\mathbb{C} \oplus \mathcal{H} \oplus \mathcal{H})$$

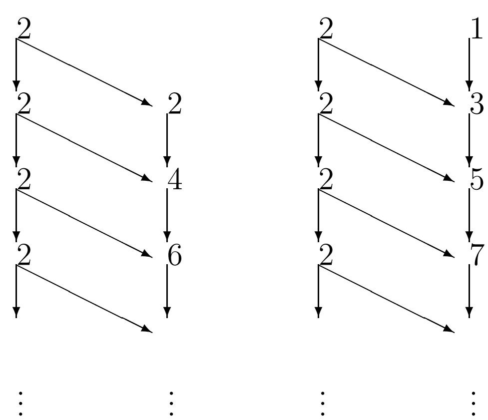
. Let $\{p_i\}_{i=1}^{\infty}$ (resp. $\{q_i\}_{i=1}^{\infty}$) be a family of orthogonal projections of rank 1 on $\mathcal{H} \oplus \mathcal{H}$ with $\sum_{i=1}^{\infty} p_i = 1_{\mathcal{H}} \oplus 0_{\mathcal{H}}$. (resp.$\sum_{i=1}^{\infty} q_i = 0_{\mathcal{H}} \oplus 1_{\mathcal{H}}$). We set

$$\begin{aligned} A_n &= \left\{ \begin{bmatrix} \lambda_{11} 1_{\mathcal{H}} & \lambda_{12} 1_{\mathcal{H}} \\ \lambda_{21} 1_{\mathcal{H}} & \lambda_{22} 1_{\mathcal{H}} \end{bmatrix} + (\sum_{i=1}^{n} p_i + q_i) x (\sum_{i=1}^{n} p_i + q_i) \mid \right. \\ &\qquad \left. \lambda_{ij} \in \mathbb{C},\ x \in \mathbb{B}(\mathcal{H} \oplus \mathcal{H}) \right\} \\ &\cong \mathbb{M}_2 \oplus \mathbb{M}_{2n} \\ B_n &= \left\{ \begin{bmatrix} 0 & 0 & 0 \\ 0 & \mu_{11} 1_{\mathcal{H}} & \mu_{12} 1_{\mathcal{H}} \\ 0 & \mu_{21} 1_{\mathcal{H}} & \mu_{22} 1_{\mathcal{H}} \end{bmatrix} + (r + \sum_{i=1}^{n} p_i + q_i) x (r + \sum_{i=1}^{n} p_i + q_i)) \mid \right. \\ &\qquad \left. \mu_{ij} \in \mathbb{C},\ x \in \mathbb{B}(\mathbb{C} \oplus \mathcal{H} \oplus \mathcal{H}) \right\} \\ &\cong \mathbb{M}_2 \oplus \mathbb{M}_{2n+1}, \end{aligned}$$

where r is the identity operator of $\mathbb{B}(\mathbb{C}) \cong \mathbb{C}$. Then we have

$$EC(\mathfrak{A}) = \overline{\bigcup_{n=0}^{\infty} A_n}^{\|\cdot\|} \qquad and \quad EC(\mathfrak{B}) = \overline{\bigcup_{n=0}^{\infty} B_n}^{\|\cdot\|} .$$

So the essential commutants have the following the Bratteli diagrams which represents the embedded manner of the sequence of increasing finite dimensional C*-algebras.



. EC($\mathfrak{A}$) EC($\mathfrak{B}$)

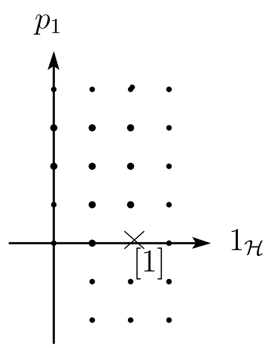
In these figures numbers are the size of matrices and arrows are represented as the manner of embedding of projections in matrices by the multiplicity one. By the facts

$$\begin{aligned} K_0(A_n) &= \mathbb{Z} \oplus \mathbb{Z} \\ K_0(B_n) &= \mathbb{Z} \oplus \mathbb{Z}, \end{aligned}$$

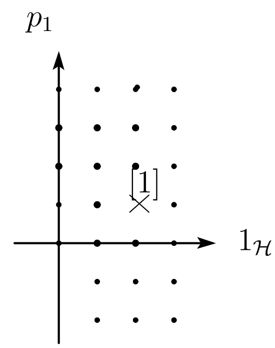
and the above inductive systems, we can get

$$\begin{aligned} K_0(EC(\mathfrak{A})) &= \varinjlim K_0(A_n) \cong \mathbb{Z} \oplus \mathbb{Z} \\ K_0(EC(\mathfrak{B})) &= \varinjlim K_0(B_n) \cong \mathbb{Z} \oplus \mathbb{Z} \end{aligned}$$

and the following figures of groups, ordered cones and scales:



$K_0(EC(\mathfrak{A}))_+$



$K_0(EC(\mathfrak{B}))_+$

Then the class of unit in $K_0(\mathrm{EC}(\mathfrak{A}))$ is $\left[\begin{pmatrix} 1_{\mathcal{H}} & 0 \\ 0 & 1_{\mathcal{H}} \end{pmatrix}\right]$. Therefore we have $\left[\begin{pmatrix} 1_{\mathcal{H}} & 0 \\ 0 & 1_{\mathcal{H}} \end{pmatrix}\right] = 2\left[\begin{pmatrix} 1_{\mathcal{H}} & 0 \\ 0 & 0 \end{pmatrix}\right]$. In the other hand, the unit class of $K_0(\mathrm{EC}(\mathfrak{B}))$ is $\left[\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1_{\mathcal{H}} & 0 \\ 0 & 0 & 1_{\mathcal{H}} \end{pmatrix}\right]$. So it is immediately realized that there dose not exist the element x in $K_0(\mathrm{EC}(\mathfrak{B}))$ such that $x \oplus x = \left[\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1_{\mathcal{H}} & 0 \\ 0 & 0 & 1_{\mathcal{H}} \end{pmatrix}\right]$ so that we cannot construct the scaled, ordered, and group-isomorphic map. Using Elliott's result, it follows that $\mathrm{EC}(\mathfrak{A}) \ncong \mathrm{EC}(\mathfrak{B})$. □

Remark 1. In the above proof of non-isomorphism, the part of AF-algebraic argument is different from the original one.

3. Main result

In this section, we present two ways on the non-isomorphism proof which are different from J. Plastiras. The one is a elementary proof and another is K-theoretical.

At the first, we will present the Fredholm operator, the (Fredholm) Index and their properties.

Definition 3.1. *(Fredholm operator, Index) We call $T \in \mathbb{B}(\mathcal{H})$ a Fredholm operator if $\pi(T)$ is invertible in $\mathbb{B}(\mathcal{H})/\mathbb{K}(\mathcal{H})$. Then* ImT *and* ImT^* *are closed and* kerT *and* kerT^* *are finite-dimensional. The index of T (*IndexT*) is defined by* $\mathrm{dimkerT} - \mathrm{dimkerT}^*$.

The followings are well-known:
(1) $\mathrm{Index(T)} = \mathrm{Index(T+K)}$ for all $K \in \mathbb{K}(\mathcal{H})$.
(2) If S is a unilateral shift operator, $\mathrm{Index(S)} = -1$ and $\mathrm{Index(S^*)} = 1$.

Lemma 3.2. *If C^*-algebras $\mathcal{B}_1$ and $\mathcal{B}_2$ act on $\mathcal{H}$ and contain $\mathbb{K}(\mathcal{H})$ and they are $*$- isomorphic, then the $*$-isomorphism map is given by $Ad(u)$ where u is a unitary in $\mathbb{B}(\mathcal{H})$.*

Proof. Let φ be an isomorphism between $\mathcal{B}_1$ and $\mathcal{B}_2$. For a completely orthonormal system $\{e_i\}_{i\in\mathbb{N}} \subset \mathcal{H}$, a projection $P_{ij} \in \mathbb{B}(\mathcal{H})$ of rank 1 is defined by $P_{ij} := (\ \cdot, e_j)e_i$. So $\{P_{ii}\}$ is the family of orthogonal projections of rank 1 in $\mathbb{K}(\mathcal{H})$ such that $\sum P_{ii} = 1_{\mathcal{H}}$. Then it can be found that $\varphi(P_{11}) := Q_{11}$ is a minimal projection where $Q_{11} = (\ , f_1)f_1$ with respect to another completely orthonormal system $\{f_i\}_{i\in\mathbb{N}} \subset \mathcal{H}$. Let v be a partial isometry such that $v^*v = P_{11}$ and $vv^* = Q_{11}$. Now all we have to do is that v is extended to the unitary u on $\mathcal{H}$ such that $\varphi(P_{ij}) = uP_{ij}u^*$. The u is defined by $u := \sum \varphi(P_{k1})vP_{1k}$. Then we have, for $\xi \in \mathcal{H}$,

$$\begin{aligned}
\|u\xi\| &= (\sum \varphi(P_{k1})vP_{1k}(\xi), \sum \varphi(P_{l1})vP_{1l}(\xi)) \\
&= \sum_{k,l} (\xi, e_k)\overline{(\xi, e_l)}(\varphi(P_{k1})f_1, \varphi(P_{l1})f_1) \\
&= \sum_k (\xi, e_k)^2 = \|\xi\|,
\end{aligned}$$

$$\begin{aligned} uu^* &= \sum_{k,l} \varphi(P_{k1})vP_{1k}P_{l1}v^*\varphi(P_{1l}) \\ &= \sum_{k} \varphi(P_{k1})vP_{11}v^*\varphi(P_{1k}) \\ &= \sum_{k} \varphi(P_{k1})\varphi(P_{11})\varphi(P_{1k}) \\ &= \sum_{k} \varphi(e_{kk}) = 1, \end{aligned}$$

$$\begin{aligned} u^*u &= \sum_{k,l} P_{k1}v^*\varphi(P_{1k})\varphi(P_{l1})vP_{1l} \\ &= \sum_{k} P_{k1}v^*\varphi(P_{11})vP_{1k} \\ &= \sum_{k} P_{k1}P_{11}P_{1k} \\ &= \sum_{k} P_{kk} = 1. \end{aligned}$$

Since the range of u is clearly dense, u is a unitary operator. And also since the following holds;

$$\begin{aligned} uP_{ij} &= \sum_{k} \varphi(P_{k1})vP_{1k}P_{ij} \\ &= \varphi(P_{i1})vP_{1j}, \\ \varphi(P_{ij})u &= \varphi(P_{ij})\sum_{k} \varphi(P_{k1})vP_{1k} \\ &= \varphi(P_{i1})vP_{1j}. \end{aligned}$$

It is shown that an isomorphism φ induces $Ad(u)$ on $\mathbb{K}(\mathcal{H})$ such that $Ad(u)(a) = uau^*$ for all $a \in \mathbb{K}(\mathcal{H})$. It is found that since $\varphi(ab) = uabu^* = uau^*ubu^* = \varphi(a)ubu^*$ for $b \in \mathcal{B}_1$, we have $\varphi(b) = ubu^*$. Consequently, $\varphi = Ad(u)$. □

Theorem 3.3. *Let $\mathfrak{A}$ and $\mathfrak{B}$ be as the above. Then we have that $\mathfrak{A}$ is not $*$-isomorphic to $\mathfrak{B}$.*

Proof. It is verified that the following sequences are exact:

$$E_1 : 0 \longrightarrow \mathbb{K}(\mathcal{H} \oplus \mathcal{H}) \xrightarrow{\iota} \mathfrak{A} \xrightarrow{\pi} \mathbb{B}(\mathcal{H} \oplus \mathcal{H})/\mathbb{K}(\mathcal{H} \oplus \mathcal{H}) \longrightarrow 0,$$

$$E_2 : 0 \longrightarrow \mathbb{K}(\mathbb{C} \oplus \mathcal{H} \oplus \mathcal{H}) \xrightarrow{\iota} \mathfrak{B} \xrightarrow{\pi} \mathbb{B}(\mathcal{H} \oplus \mathcal{H})/\mathbb{K}(\mathcal{H} \oplus \mathcal{H}) \longrightarrow 0$$

where ι is the natural inclusion map and π is the surjective map such that $\pi(T \oplus T + K) = [T] \oplus [T]$, and $\pi(0 \oplus T \oplus T + L) = [T] \oplus [T]$ on E_1 and E_2 respectively. We define a linear operator U from $\mathbb{C} \oplus \mathcal{H} \oplus \mathcal{H}$ to $\mathcal{H} \oplus \mathcal{H}$ by $U(\lambda, \xi, \eta) := (\lambda e_0 + S\xi, \eta)$. Then the U is unitary. It is found that $U\mathfrak{B}U^* \subset \mathbb{B}(\mathcal{H} \oplus \mathcal{H})$. Since it is checked that

$$\mathfrak{B} \cong \{STS^* \oplus T \mid T \in \mathbb{B}(\mathcal{H})\} + \mathbb{K}(\mathcal{H} \oplus \mathcal{H}) \subset \mathbb{B}(\mathcal{H} \oplus \mathcal{H}),$$

E_2 can be slightly modified as

$$E_2 : 0 \longrightarrow \mathbb{K}(\mathcal{H} \oplus \mathcal{H}) \xrightarrow{\iota} \mathfrak{B} \xrightarrow{\pi} \mathbb{B}(\mathcal{H} \oplus \mathcal{H})/\mathbb{K}(\mathcal{H} \oplus \mathcal{H}) \longrightarrow 0$$

where $\pi(STS^* \oplus T + L) = [STS^*] \oplus [T]$ is well-defined. By the above lemma, if $\mathfrak{A}$ is $*$-isomorphic to $\mathfrak{B}$, then there exists a unitary $u \in \mathbb{B}(\mathcal{H} \oplus \mathcal{H})$ such that

$$\pi(u) \begin{pmatrix} [T] & 0 \\ 0 & [T] \end{pmatrix} \pi(u)^* = \begin{pmatrix} [STS^*] & 0 \\ 0 & [T] \end{pmatrix} \ \textit{for all } T \in \mathbb{B}(\mathcal{H}).$$

Let u be $\begin{pmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{pmatrix}$. Then we have the following relations:

$$u_{11}T - STS^*u_{11} \in \mathbb{K}(\mathcal{H}), \quad u_{12}T - STS^*u_{12} \in \mathbb{K}(\mathcal{H}),$$
$$u_{21}T - Tu_{21} \in \mathbb{K}(\mathcal{H}), \ \textit{and} \ \ u_{22}T - Tu_{22} \in \mathbb{K}(\mathcal{H}) \ \textit{for all } T \in \mathbb{B}(\mathcal{H}).$$

By Lemma 2.3, it is found that $u = \begin{pmatrix} \lambda_1 S & \lambda_2 S \\ \lambda_3 & \lambda_4 \end{pmatrix} = \begin{pmatrix} S & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \lambda_1 & \lambda_2 \\ \lambda_3 & \lambda_4 \end{pmatrix}$ where $\lambda_i \in \mathbb{C}$. The Index of u is equal to 0 and that of the right hand is equal to -1. Therefore there does not exist a unitary $u \in \mathbb{B}(\mathcal{H} \oplus \mathcal{H})$. Consequently, $\mathfrak{A}$ is not $*$-isomorphic to $\mathfrak{B}$. □

Theorem 3.4. *Let $\mathfrak{A}$ and $\mathfrak{B}$ be as the above. Then we have that $\mathfrak{A}$ is not $*$-isomorphic to $\mathfrak{B}$.*

Proof. (K-theoretical). We have the following short exact sequences:

$$E_1 : \ 0 \longrightarrow \mathbb{K}(\mathcal{H} \oplus \mathcal{H}) \xrightarrow{\ \iota\ } \mathfrak{A} \xrightarrow{\ \pi\ } \mathbb{B}(\mathcal{H})/\mathbb{K}(\mathcal{H}) \longrightarrow 0$$
$$E_2 : \ 0 \longrightarrow \mathbb{K}(\mathcal{H} \oplus \mathcal{H}) \xrightarrow{\ \iota\ } \mathfrak{B} \xrightarrow{\ \pi\ } \mathbb{B}(\mathcal{H})/\mathbb{K}(\mathcal{H}) \longrightarrow 0$$

where ι is the natural inclusion map and π is the surjective map such that $\pi(T \oplus T + K) = [T]$ and $\pi(STS^* \oplus T + L) = [T]$. The six term exact sequence is here applied for the first short exact sequence E_1.

$$\begin{array}{ccccc} K_0(\mathbb{K}(\mathcal{H} \oplus \mathcal{H})) & \xrightarrow{\ \iota_*\ } & K_0(\mathfrak{A}) & \xrightarrow{\ \pi_*\ } & K_0(\mathbb{B}(\mathcal{H})/\mathbb{K}(\mathcal{H})) \\ \delta_1 \uparrow & & & & \downarrow \delta_0 \\ K_1(\mathbb{B}(\mathcal{H})/\mathbb{K}(\mathcal{H})) & \underset{\pi_*}{\longleftarrow} & K_1(\mathfrak{A}) & \underset{\iota_*}{\longleftarrow} & K_1(\mathbb{K}(\mathcal{H} \oplus \mathcal{H})) \end{array}$$

Since it is known that the Fredholm Index correspond to the connected component in $\mathbb{B}(\mathcal{H})/\mathbb{K}(\mathcal{H})$ and then the class $[S^*]$ is the generator in $K_1(\mathbb{B}(\mathcal{H})/\mathbb{K}(\mathcal{H}))$. So we need to observe where $[S^*]$ go into $K_0(\mathbb{K}(\mathcal{H} \oplus \mathcal{H}))$ through δ_1 . δ_1 is defined by the following:

$$\delta_1([S^*]) := [u^* \oplus u^*(1_{\mathcal{H}} \oplus 1_{\mathcal{H}} \oplus 0 \oplus 0)u \oplus u] - [1_{\mathcal{H}} \oplus 1_{\mathcal{H}} \oplus 0 \oplus 0]$$

where $[S^*] \in \mathbb{B}(\mathcal{H})/\mathbb{K}(\mathcal{H})$ and $u \in \mathbb{M}_2(\mathbb{B}(\mathcal{H}))$ is a unitary lift of $[S \oplus S^*] \in \mathbb{M}_2(\mathbb{B}(\mathcal{H})/\mathbb{K}(\mathcal{H}))$ through $\pi \otimes id_2$. So we have $\delta_1([S^*]) = 2[p]$ for a 1-dimensional projection p. From the fact that $K_0(\mathbb{B}(\mathcal{H})/\mathbb{K}(\mathcal{H})) = 0$, it is easily verified that

$$K_0(\mathfrak{A}) \cong K_0(\mathbb{K}(\mathcal{H} \oplus \mathcal{H}))/\mathrm{Im}\delta_1 = \mathbb{Z}/2\mathbb{Z}.$$

By the similar argument, it will be found that $K_0(\mathfrak{B}) = \mathbb{Z}/2\mathbb{Z}$ where δ_1 is defined by

$$\begin{aligned} \delta_1([S^*]) := [Su^*S^* \oplus u^*(S1_{\mathcal{H}}S^* \oplus 1_{\mathcal{H}} \oplus 0 \oplus 0)SuS^* \oplus u] \\ -[S1_{\mathcal{H}}S^* \oplus 1_{\mathcal{H}} \oplus 0 \oplus 0]. \end{aligned}$$

Now the unit class $[1] \in K_0(\mathfrak{A})$ will be compared to $[1] \in K_0(\mathfrak{B})$.

$$\begin{aligned} &K_0(\mathfrak{A}) \ni [1] = [1_{\mathcal{H}} \oplus 1_{\mathcal{H}}] = 2[1_{\mathcal{H}} \oplus 0] = [0] \in \mathbb{Z}/2\mathbb{Z}, \\ &K_0(\mathfrak{B}) \ni [1] = [1_{\mathcal{H}} - p \oplus 1_{\mathcal{H}}] = [1_{\mathcal{H}} \oplus 1_{\mathcal{H}}] + [p \oplus 0] = [1] \in \mathbb{Z}/2\mathbb{Z}. \end{aligned}$$

This means that the unit class $[1] \in K_0(\mathfrak{A})$ is different from $[1] \in K_0(\mathfrak{B})$. Then it can be concluded that $\mathfrak{A}$ is not $*$-isomorphic to $\mathfrak{B}$. ☐

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