# Two topics related with Fibonacci numbers

## The quintic equation by Galoa theory and the Euler function

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# Outline

1. Motivation
   - Platon's regular Polyhedron
   - The quintic equation on fibonacci can be solved, why?
   - Expantion of $\cos 5\theta$
   - Main Results
2. Numerical calculation
   - Convergence seaquence:

# Outline

# Platon's regular Polyhedron

| name | v | e | f | |
|---|---|---|---|---|
| Tetrahedron | 4 | 6 | 4 | |
| Cube | 8 | 12 | 6 | |
| Octahedron | 6 | 12 | 8 | $\star$ |
| Dodecahedron | 20 | 30 | 12 | |
| Icosahedron | 12 | 30 | 20 | $\star$ |

- If the edges of an Octahedron($\star$) are divided in the golden ratio such that forming an equilateral triangle, then twelve position form an Icosahedron($\star$).(by Mathworld: Wolfram)

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# The quintic equation and the Octahedron

- The regular pentagon is constructed by several methods as the half-angle formula $\tan(\theta/2) = \dfrac{1-\cos(\theta)}{\sin(\theta)}$: $\cos(36^o) = \dfrac{1+\sqrt{5}}{4}$. But a general quintic equation cannot constructed. It is well known by the Galoa theory.

# Fibonacci appears everywhere

In elementary mathematics we encounter many interesting property of problems $\cdots$

- using the `pause` connection between:
  - Octahedron "8"
  - Icosahedron "20"
- Each edge of Octahedron divied by Golden rario:
  - It concludes the quardratic equation.
  - The shape is pentagon.
- using the general `uncover` command:
  - First item.
  - Second item.

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# Outline

# Pascal Triangle with trigonometry

1.
$$\begin{aligned}\cos 3\theta &= \cos^3\theta - 3\cos\theta\,\sin^2\theta\\ \sin 3\theta &= 3\cos^2\theta - \sin^3\theta\end{aligned}$$

2.
$$\begin{aligned}\cos 4\theta &= \cos^4\theta - 6\cos^2\theta\,\sin^2\theta + \sin^4\theta\\ \sin 4\theta &= 4\cos^3\theta\,\sin\theta - 4\cos\theta\,\sin^3\theta\end{aligned}$$

3.
$$\star\begin{cases}\cos 5\theta = \cos^5\theta - 10\cos^3\theta\,\sin^2\theta + 5\cos\theta\,\sin^4\theta\\ \sin 5\theta = 5\cos^4\theta\,\sin\theta - 10\cos^2\theta\,\sin^3\theta + \sin^5\theta\end{cases}$$

# In generaly

$$\begin{cases} \cos n\theta &= \cos^n\theta - \binom{n}{2}\cos^{n-2}\theta\sin^2\theta + \cdots \\ \sin n\theta &= \binom{n}{1}\cos^{n-1}\theta\sin\theta - \binom{n}{3}\cos^{n-3}\theta\sin^3\theta + \cdots \end{cases}$$

In stead of Binomial coefficients:
$\binom{n}{0}, \binom{n}{1}, \binom{n}{2}, \binom{n}{3}, \cdots, \binom{n}{n}$, whether q-binomial
$\begin{bmatrix} n \\ k \end{bmatrix} = \dfrac{(1-q^n)(1-q^{n-1})\cdots(1-q^{n-k+1})}{(1-q^k)(1-q^{k-1})\cdots(1-q)}$ is it possible?

# 5-dim equation

If $c = 2\cos\frac{\pi}{5} = \frac{\sqrt{5}+1}{2} \approx 1.618\cdots$ (Golden ratio), then

$$c^2 - c - 1 = 0.$$

Let $\theta = \frac{\pi}{5}$, that is, $\cos 5\theta = -1$. So

$$-1 = \cos 5\theta = \left(\frac{c}{2}\right)^5 - 10\left(\frac{c}{2}\right)^3\left\{1 - \left(\frac{c}{2}\right)^2\right\} + 5\frac{c}{2}\left\{1 - \left(\frac{c}{2}\right)^2\right\}^2$$

Thus 5-dim equation appears such as

$$c^5 - 5c^3 + 5c + 2 = 0.$$

Note: $x^5 - x + a = 0\,(a \neq 0)$ is not solvable in $n$-th root.

# Outline

# Galoa theory tells us

Galoa theory tells us the necessary and sufficient condition.
The equation is Solvable by Power,

$$c^5 - 5c^3 + 5c + 2 = (c+2)(c^2 - c - 1)^2$$

The pentagon can be drawn by several methods.

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# Numerical convergence

Iteration for the Fibonacci seaquences of fractional:

$$z_{n+1} = \frac{2z_n + 1}{z_n + 1} \quad \Longrightarrow \quad \begin{cases} z_1 = 1 \\ z_2 = \dfrac{3}{2} = \dfrac{F(4)}{F(3)} \\ z_3 = \dfrac{8}{5} = \dfrac{F(6)}{F(5)} \\ z_4 = \dfrac{21}{13} = \dfrac{F(8)}{F(7)} \\ z_5 = \dfrac{55}{34} = \dfrac{F(10)}{F(9)} \\ \dots \end{cases}$$

# Reason why for Iteration

Because of the definiton on Fibonacci,

$$\begin{aligned} z_{n+1} &= \frac{F(2n+2)}{F(2n+1)} \\ &= \frac{F(2n+1)+F(2n)}{F(2n)+F(2n-1)} \\ &= \frac{F(2n)+F(2n-1)+F(2n)}{F(2n)+F(2n-1)} \\ &= \frac{2F(2n)+F(2n-1)}{F(2n)+F(2n-1)} = \frac{2z_n+1}{z_n+1}. \end{aligned}$$

This proves the method.

# Iteration by $\sqrt{5}$

Iteration using Herron method: approximation of $\sqrt{\alpha}$

$$a_{n+1} = \frac{1}{2}\left(a_n + \frac{\alpha}{a_n}\right).$$

So using this method, by letting $\alpha = 5$ and $a_0 = 2$,

$$\Phi_n = \frac{1}{2}(1 + a_n) = \frac{F(3 \cdot 2^n + 1)}{F(3 \cdot 2^n)} \quad \to \quad \frac{1 + \sqrt{5}}{2}$$

# Euler function

From Opera Omnia, the Euler function is defined by

$$\phi(q) = \prod_{k=1}^{\infty} (1 - q^k).$$

Named after Leonhard Euler, it is a prototypical example of a q-series, a modular form, and provides the prototypical example of a relation between combinatorics and complex analysis (Wikipedia).

# Ramanujan's lost notebook

The special values are astonished:

$$\phi(e^{-\pi}) = \frac{e^{\pi/24}\,\Gamma(1/4)}{2^{7/8}\,\pi^{3/4}}$$

$$\phi(e^{-2\pi}) = \frac{e^{\pi/12}\,\Gamma(1/4)}{2\,\pi^{3/4}}$$

(Ramanujan's lost notebook, Part V, p.326) Apostol, Tom M. (1976), Introduction to analytic number theory, Undergraduate Texts in Mathematics, New York-Heidelberg: Springer-Verlag.

# trying till 6th term

Let

$$\phi_n(x) = (1-x)(1-x^2)(1-x^3)\cdots(1-x^n)$$

and then for $n = 2, 3, 4, 5, 6$,

$$\begin{cases} \phi_1(x) = 1 - x, \\ \phi_2(x) = 1 - x - x^2 + x^3, \\ \phi_3(x) = 1 - x - x^2 + x^4 + x^5 - x^6, \\ \phi_4(x) = 1 - x - x^2 + 2\,x^5 - x^8 - x^9 + x^{10}, \\ \phi_5(x) = 1 - x - x^2 + x^5 + x^6 + x^7 - x^8 - \cdots + x^{14} - x^{15} \\ \phi_6(x) = 1 - x - x^2 + x^5 + 2\,x^7 x - x^9 + \cdots + 2\,x^{14} + \cdots + x^{21} \\ \phi_n(x) = ? \end{cases}$$

# Summery table

Each coefficient is $c_k \in \{0, 1, -1\}$:

$$\prod_k (1 - x^k) = 1 + c_1 x + c_2 x^2 + c_3 x^3 + c_4 x^4 + c_5 x^5 + \cdots$$

| | |
|---|---|
| $c_k = 1$ : | $k = 0, 5, 7, 22, 26, 51, 57, 92, 100,$ etc. |
| $c_k = 0$ : | otherwise |
| $c_k = -1$ : | $k = 1, 2, 12, 15, 35, 40, 70, 77,$ etc. |

# Euler's pentagonal number

**Theorem (Euler's pentagonal number thoerem)**

$$\phi(x) = \prod_k (1 - x^k) = 1 + \sum_{r=1}^{\infty} (-1)^r \left( x^{(3r^2-r)/2} + x^{(3r^2+r)/2} \right)$$

Quadratic form:

$$\phi(x)^2 = 1 - 2x - x^2 + x^3 + x^4 + 2x^5 - 2x^6 - 2x^8 - 2x^9 + x^{10} \cdots$$
$$\phi_{10}(x)^3 = 1 - 3x - 5x^3 - 7x^6 + 9x^{10} + 3x^{11} - 6x^{12} - 6x^{13} -$$

**Theorem (Cubic form:(Gauss))**

$$\phi(x)^3 = 1 - 3x - 5x^3 - 7x^6 + 9x^{10} + 11x^{15} - 13x^{21} + \cdots$$

# partition number

$$p(n) = \#\{k \mid n = n_1 + \cdots + n_k, 0 \lneqq n_1 \leq n_2 \leq \cdots \leq n_k\}$$

$$p(1) = 1,\ p(2) = 2; 1+1,\ p(3) = 3; 1+2, 1+1+1,$$
$$p(4) = 5; 1+3, 2+2, 1+1+2, 1+1+1+1,$$

$$\begin{aligned} p(x) &= 1 + x + 2x^2 + 3x^3 + 5x^5 + \cdots \\ &= 1 + \textstyle\sum_{r=1}^{\infty} p(r) x^r \end{aligned}$$

**Theorem (Euler partition number theorem)**

$$\phi(x)\, p(x) = 1$$

# Rough sketch

$$\begin{aligned}\phi(x)^{-1} &= \prod_{n=1}^{\infty}(1+x^n+x^{2n}+x^{3n}+\cdots) \\ &= (1+x+x^2+x^3+\cdots) \\ &\quad \times(1+x^2+x^4+x^6+\cdots) \\ &\quad \times(1+x^3+x^6+x^9+\cdots) \\ &\quad \cdots \\ &\quad \times(1+x^n+x^{2n}+x^{3n}+\cdots) \\ &\quad \cdots\end{aligned}$$

So coefficients of $r$ th:

$$x^{1\cdot k_1}x^{2\cdot k_2}\cdots x^{m\cdot k_m} = x^{1\cdot k_1+2\cdot k_2+\cdots+m\cdot k_m} = x^r,$$

Therefore,

$$r = 1\cdot k_1 + 2\cdot k_2 + \cdots + m\cdot k_m.$$

# Fibonacci case

How does it becomes Fibonacci case?

$$\tilde{F}_n = \tilde{F}_{n-1} + \tilde{F}_{n-2} (n \geq 3)$$

## Definition

Fibonacci :
$\tilde{F}_1 = 1,\ \tilde{F}_2 = 2,\ \tilde{F}_3 = 3,\ \tilde{F}_4 = 5, \tilde{F}_5 = 8, \cdots$
Original Fibonacci (FQ journal):
$F_0 = 0,\ F_1 = 1,\ F_2 = 1,\ F_3 = 2, \cdots$

The following discussion refers to “ Mathematical Omnibus: Thirty Lectures on Classic Mathematics”, D. Fuchs & S. Tabachinikov, Amer Math Soc (2007).

# A few property

A few property are different:

**Theorem**

*Tilde FIbonacci:*

1. $\tilde{F}_1 + \tilde{F}_2 + \tilde{F}_3 + \cdots + \tilde{F}_n = \tilde{F}_{n+2} - 2$
2. $\tilde{F}_1 + \tilde{F}_3 + \tilde{F}_5 + \cdots + \tilde{F}_{2k-1} = \tilde{F}_{2k} - 1$
3. $\tilde{F}_2 + \tilde{F}_4 + \tilde{F}_6 + \cdots + \tilde{F}_{2k} = \tilde{F}_{2k+1} - 1$

**Theorem**

*Original FInobacci:*

1. $F_1 + F_2 + F_3 + \cdots + F_n = F_{n+2} - 1$
2. $F_1 + F_3 + F_5 + \cdots + F_{2k-1} = F_{2k}$
3. $F_2 + F_4 + F_6 + \cdots + F_{2k} = F_{2k+1} - 1$

**Theorem**

*For arbitrary* $n \geq 1$*, it is represented as*

$$n = \tilde{F}_{k_1} + \cdots + \tilde{F}_{k_s}, \quad 1 \leq k_1 < \cdots < k_2$$

*by using Fibonacci numbers.*

Define the sequence $\{g_n\}$ by

$$\prod_{k=1}^{\infty}(1-x^{\tilde{F}_k}) = (1-x)(1-x^2)(1-x^3)(1-x^5)(1-x^8)\cdots$$
$$= 1+g_1x+g_2x^2+g_3x^3+g_4x^4+\cdots.$$

**Theorem**

$$|g_n| \leq 1$$

**Theorem**

*More generally, if $k < l$, then the coefficient for*

$$(1-x^{\tilde{F}_k})(1-x^{\tilde{F}_{k+1}})\cdots(1-x^{\tilde{F}_l})$$

*equals* $\quad g_n = 1, \quad 0, \quad -1, \quad n = \tilde{F}_k, \cdots, \tilde{F}_k + \cdots + \tilde{F}_l.$

# Example No.1

$$\prod_{k=1}^{8}(1-x^{\tilde{F}_k}) = (1-x)(1-x^2)(1-x^3)(1-x^5)(1-x^8)$$
$$= 1+g_1x+g_2x^2+g_3x^3+\cdots+g_{32}x^{32}.$$

Summing up as $32 = 1+2+3+5+8$,

| | |
|---|---|
| $g_n = 1$ | $n = 0, 4, 7, 11, 14, 18, 21, 25, 28, 32$ : total 10 |
| $g_n = 0$ | *otherwise* : total 13 |
| $g_n = -1$ | $n = 1, 2, 8, 12, 13, 19, 20, 24, 30, 31$ : total 10 |

# Example No.2

$$\prod_{k=1}^{89}(1-x^{\tilde{F}_k}) = (1-x)(1-x^2)(1-x^3)\cdots(1-x^{89})$$
$$= 1+g_1x+g_2x^2+g_3x^3+\cdots+g_{231}x^{231}.$$

Summing up as $231 = 1+2+3+5+8+\cdots+89$,

| | |
|---|---|
| $g_n = 1$ | $n = 0, 4, 7, 11, 14, \cdots, 224, 227, 231$ : total 56 |
| $g_n = 0$ | *otherwise* : total 120 |
| $g_n = -1$ | $n = 1, 2, 8, 12, 13, \cdots, 223, 229, 230$ : total 56 |

# Summary

- The first main message of your talk in one or two lines.
- The second main message of your talk in one or two lines.
- Perhaps a third message, but not more than that.

- Outlook
  - Something you haven't solved.
  - Something else you haven't solved.

# For Further Reading I

A. Author.
*Handbook of Everything*.
Some Press, 1990.

S. Someone.
On this and that.
*Journal of This and That*, 2(1):50–100, 2000.

# Acknowledgement

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