Autocontinuity from below of Set Functions and Convergence in Measure

Jun Li, Masami Yasuda, and Ling Zhou

Abstract. In this note, the concepts of strong autocontinuity from below and strong converse autocontinuity from below of set function are introduced. By using four types of autocontinuity from below of monotone measure, the relationship between convergence in measure and pseudo-convergence in measure for sequence of measurable function are discussed.

Keywords: Monotone measure, Autocontinuity from below, Convergence in measure, Pseudo-convergence in measure.

1 Introduction

In non-additive measure theory, there are several different kinds of convergence for sequence of measurable functions, such as almost everywhere convergence, pseudo-almost everywhere convergence, convergence in measure, and convergence pseudo-in measure. The implication relationship between such convergence concepts are closely related to the structural characteristics of set functions. In this direction there are a lot of results ([5, 7, 2, 6, 3, 10, 4, 8, 9, 11, 12, 14, 15]).

Jun Li
School of Science, Communication University of China, Beijing 100024, China
e-mail: lijun@cuc.edu.cn

Masami Yasuda
Department of Mathematics & Informatics, Faculty of Science, Chiba University, Chiba 263-8522, Japan
e-mail: yasuda@math.s.chiba-u.ac.jp

Ling Zhou
College of Applied Sciences, Xinjiang Agricultural University, Xinjiang, Urumqi 830052, China
e-mail: zhoulin5703@163.com

S. Li (Eds.): Nonlinear Maths for Uncertainty and its Appli., AISC 100, pp. 77–83.
springerlink.com

In this note, we further discuss the relationship between convergence in measure and convergence pseudo-in measure for sequence of measurable functions. We shall introduce the concepts of strong autocontinuity from below and strong converse autocontinuity from below of a set function. By using the two types of autocontinuity from below of monotone measures, we investigate the inheriting of convergence in measure and convergence pseudo-in measure for sequence of measurable function under the common addition operation "+" and logic addition operation "$\vee$". The implication relationship between convergence in measure and pseudo-convergence in measure are shown by using autocontinuity from below and converse autocontinuity from below, respectively.

2 Preliminaries

Let X be a non-empty set, $\mathcal{F}$ a σ-algebra of subsets of X, and $(X, \mathcal{F})$ denotes the measurable space.

Definition 1. ([9,15]) Set function $\mu : \mathcal{F} \to [0, +\infty]$ is called a *monotone measure* on $(X, \mathcal{F})$ iff it satisfies the following requirements:

(1) $\mu(\emptyset) = 0$; (vanishing at $\emptyset$)
(2) $A \subset B$ and $A, B \in \mathcal{F} \;\Rightarrow\; \mu(A) \leq \mu(B)$. (monotonicity)

When μ is a monotone measure, the triple $(X, \mathcal{F}, \mu)$ is called a monotone measure space ([9,15]).

In some literature, a set function μ satisfying the conditions (1) and (2) of Definition 1 is called a fuzzy measure or a non-additive measure .

In this paper, all the considered sets are supposed to belong to $\mathcal{F}$ and μ is supposed to be a finite monotone measure, i.e., $\mu(X) < \infty$. All concepts and symbols not defined may be found in [9,15].

Definition 2. ([1]) A set function $\mu : \mathcal{F} \to [0, +\infty)$ is said to have *pseudometric generating property* (for short *p.g.p*), if for any $\{E_n\} \subset \mathcal{F}$ and $\{F_n\} \subset \mathcal{F}$,

$$\mu(E_n) \vee \mu(F_n) \to 0 \implies \mu(E_n \cup F_n) \to 0.$$

Note: The concept of pseudometric generated property goes back to Dobrakov and Farkova in seventies, and this was related to Frechet-Nikodym topology [1,9].

Let $\mathbf{F}$ be the class of all finite real-valued measurable functions on $(X, \mathcal{F}, \mu)$, and let $A \in \mathcal{F}, f \in \mathbf{F}, f_n \in \mathbf{F}$ $(n = 1, 2, \ldots)$ and $\{f_n\}$ denote a sequence of measurable functions. We say that $\{f_n\}$ *converges in measure* μ *to*

f on A, and denote it by $f_n \xrightarrow[A]{\mu} f$, if for any given $\sigma > 0$, $\lim_{n\to+\infty} \mu(\{|f_n - f| \geq \sigma\} \cap A) = 0$; $\{f_n\}$ *converges pseudo-in measure μ to f on A*, and denote it by $f_n \xrightarrow[A]{p,\mu} f$, if for any given $\sigma > 0$, $\lim_{n\to+\infty} \mu(\{|f_n - f| < \sigma\} \cap A) = \mu(A)$; $\{f_n\}$ *converges pseudo-in measure μ to f in A*, and denote it by $f_n \xrightarrow{p.\mu} f$ in A, if $f_n \xrightarrow{p.\mu} f$ on C for all $C \in A \cap \mathcal{F}$.

3 Autocontinuity of Set Function

In [14] Wang introduced the concepts of autocontinuity from below and converse-autocontinuity from below of set function, and discussed the convergence for sequence of measurable functions by using the structure of set functions. Now we shall introduce the concepts of strong autocontinuity from below and strong converse-autocontinuity from below for set functions and show their properties.

Definition 3. ([9,14,15]) Let $(X, \mathcal{F}, \mu)$ be a monotone measure space.

(1) μ is said to be *autocontinuous from below* and denote it by *autoc.*$\uparrow$, if for any $E \in \mathcal{F}, \{F_n\} \subset \mathcal{F}$,

$$\mu(F_n) \to 0 \implies \mu(E - F_n) \to \mu(E);$$

(2) μ is said to be *converse-autocontinuous from below* and denote it by *c.autoc.*$\uparrow$, if for any $A \in \mathcal{F}, \{B_n\} \subset A \cap \mathcal{F}$,

$$\mu(B_n) \to \mu(A) \implies \mu(A - B_n) \to 0.$$

Definition 4. Let $(X, \mathcal{F}, \mu)$ be a monotone measure space.

(1) μ is said to be *strong autocontinuous from below* and denote it by *s.autoc.*$\uparrow$, if

$$\mu(E_n) \vee \mu(F_n) \to 0 \Longrightarrow \mu(A - E_n \cup F_n) \to \mu(A),$$

for any $A \in \mathcal{F}, \{E_n\} \subset \mathcal{F}$ and $\{F_n\} \subset \mathcal{F}$;

(2) μ is said to be *strong converse-autocontinuous from below* and denote it by *s.c.autoc.*$\uparrow$, if

$$\mu(A - E_n) \wedge \mu(A - F_n) \to \mu(A) \Longrightarrow \mu(E_n \cup F_n) \to 0,$$

for any $A \in \mathcal{F}$, $\{E_n\} \subset A \cap \mathcal{F}$ and $\{F_n\} \subset A \cap \mathcal{F}$.

Proposition 1. *If μ is s.autoc.$\uparrow$ (resp. s.c.autoc.$\uparrow$), then it is autoc.$\uparrow$ (resp. c.autoc.$\uparrow$).*

Proposition 2. *If μ is autoc.$\uparrow$ and has p.g.p, then it is s.autoc.$\uparrow$.*

Proposition 3. *If μ is c.autoc.$\uparrow$ and has p.g.p, then it is s.c.autoc.$\uparrow$.*

4 Convergence in Measure

In this section, we study the application relationship between convergence in measure and convergence pseudo-in measure on monotone measure spaces.

The first conclusion of the following theorem due to Wang [15].

Theorem 1. *Let μ be a monotone measure. Then,*

(1) μ is autoc.$\uparrow$ iff $f_n \xrightarrow[A]{p.\mu} f$ whenever $f_n \xrightarrow[A]{\mu} f$, $\forall A \in \mathcal{F}, f, f_n \in \mathbf{F}$;

(2) μ is c.autoc$\uparrow$, iff $f_n \xrightarrow[A]{\mu} f$ whenever $f_n \xrightarrow[A]{p.\mu} f$, $\forall A \in \mathcal{F}, f, f_n \in \mathbf{F}$.

Proof. We only prove (2). Let μ be c.autoco$\uparrow$. If $f_n \xrightarrow[A]{p.\mu} f$, then for any given $\sigma > 0$, we have

$$\lim_{n\to+\infty} \mu(\{|f_n - f| < \sigma\} \cap A) = \mu(A)$$

and therefore, using the converse-autocontinuity from below of μ, we have

$$\begin{aligned}\lim_{n\to+\infty} \mu(\{|f_n - f| \geq \sigma\} \cap A) &= \lim_{n\to+\infty} \mu(A - \{|f_n - f| < \sigma\}) \\ &= 0.\end{aligned}$$

So $f_n \xrightarrow[A]{\mu} f$.

Conversly, for any $A \in \mathcal{F}, \{B_n\} \subset A \cap \mathcal{F}$, and $\mu(B_n) \to \mu(A)$, we define measurable function sequences $\{f_n\}$ by

$$f_n = \chi_{B_n} = \begin{cases} 0 \text{ if } & x \notin B_n \\ 1 \text{ if } & x \in B_n, \end{cases}$$

n = 1, 2,..., and denote $f \equiv 1$. It is easy to see that $f_n \xrightarrow[A]{p.\mu} f$. If it implies $f_n \xrightarrow[A]{\mu} f$, then for $\sigma = \frac{1}{2}$, we have

$$\lim_{n\to+\infty} \mu(\{|f_n - f| \geq \frac{1}{2}\} \cap A) = 0.$$

As

$$\{|f_n - f| \geq \frac{1}{2}\} \cap A = \{1 - \chi_{B_n} \geq \frac{1}{2}\} \cap A = A - B_n.$$

So $\lim_{n\to+\infty} \mu(A - B_n) = 0$. This shows that μ is c.autoc$\uparrow$. $\square$

The following theorems describe the inheriting of convergence in measure and convergence pseudo-in measure for sequence of measurable function under the common addition operation.

Theorem 2. *Let μ be a monotone measure.*

(1) If μ is s.autoc.$\uparrow$, then $f_n \xrightarrow{\mu} f$ and $g_n \xrightarrow{\mu} g$ on A imply

$$\alpha f_n + \beta g_n \xrightarrow[D]{p.\mu} \alpha f + \beta g,$$

for any $D \in A \cap \mathcal{F}, \alpha, \beta \in R^1$.

(2) If μ is s.c.autoc.$\uparrow$, then $f_n \xrightarrow[A]{p.\mu} f$ and $g_n \xrightarrow[A]{p.\mu} g$ imply

$$\alpha f_n + \beta g_n \xrightarrow[A]{\mu} \alpha f + \beta g,$$

for any $A \in \mathcal{F}$, $\alpha, \beta \in R^1$.

Proof. It is similar to the proof of Theorem 1. □

The following Theorem 3 and 4 describe respectively the characteristics of strong autocontinuity from below and strong converse-autocontinuity from below of set functions.

Theorem 3. *The following statements are equivalent:*

(1) μ is s.autoc.$\uparrow$;

(2) $f_n + g_n \xrightarrow[A]{p.\mu} 0$ whenever $f_n \xrightarrow[A]{\mu} 0$ and $g_n \xrightarrow[A]{\mu} 0$, $\forall A \in \mathcal{F}$;

(3) $f_n \vee g_n \xrightarrow[A]{p.\mu} 0$ whenever $f_n \xrightarrow[A]{\mu} 0$ and $g_n \xrightarrow[A]{\mu} 0$, $\forall A \in \mathcal{F}$.

Proof. (1) $\Longrightarrow$ (2). It follows directly from Theorem 2 above.

(2) $\Longrightarrow$ (3). For any $A \in \mathcal{F}$, if $f_n \xrightarrow[A]{\mu} 0$ and $g_n \xrightarrow[A]{\mu} 0$, then $|f_n| \xrightarrow[A]{\mu} 0$ and $|g_n| \xrightarrow[A]{\mu} 0$. By condition (2), we have $|f_n| + |g_n| \xrightarrow{p.\mu} 0$ on A, therefore, for any $\sigma > 0$,

$$\lim_{n \to +\infty} \mu(\{|f_n| + |g_n| < \sigma\} \cap A) = \mu(A).$$

Noting that $|f_n \vee g_n| \leq |f_n| + |g_n|$, we get

$$\{|f_n| + |g_n| < \sigma\} \cap A \subseteq \{|f_n \vee g_n| < \sigma\} \cap A \subseteq A.$$

So

$$\lim_{n \to +\infty} \mu(\{|f_n \vee g_n| < \sigma\} \cap A) = \mu(A).$$

This shows $f_n \vee g_n \xrightarrow{p.\mu} 0$ on A.

(3)$\Longrightarrow$(1). For any $\{E_n\} \subset \mathcal{F}, \{F_n\} \subset \mathcal{F}$ with $\lim_{n\to\infty} \mu(E_n) \vee \mu(F_n) = 0$, we define measurable function sequences $\{f_n\}$ and $\{g_n\}$ by

$$f_n = \chi_{E_n} = \begin{cases} 0 \text{ if } & x \notin E_n \\ 1 \text{ if } & x \in E_n \end{cases}$$

and

$$g_n = \chi_{F_n} = \begin{cases} 0 \text{ if } & x \notin F_n \\ 1 \text{ if } & x \in F_n, \end{cases}$$

n = 1, 2,..., then $f_n \xrightarrow{\mu} 0$ on A and $g_n \xrightarrow{\mu} 0$ on A. Thus, $f_n \vee g_n \xrightarrow{\mu} 0$ on A. Therefore for $\sigma = \frac{1}{2}$, we have

$$\lim_{n\to+\infty} \mu(\{f_n \vee g_n < \frac{1}{2}\} \cap A) = \mu(A).$$

Noting $f_n \vee g_n = \chi_{E_n} \vee \chi_{F_n} = \chi_{E_n \cup F_n}$, and

$$\{\chi_{E_n} \vee \chi_{F_n} < \frac{1}{2}\} \cap A = A - \{\chi_{E_n} \vee \chi_{F_n} \geq \frac{1}{2}\} = A - E_n \cup F_n.$$

So

$$\lim_{n\to+\infty} \mu(A - E_n \cup F_n) = \mu(A).$$

That is, μ is s.autoc.$\uparrow$. □

Theorem 4. *The following statements are equivalent:*

(1) μ *is s.c.autoc.*$\uparrow$*;*

(2) $f_n + g_n \xrightarrow[A]{\mu} 0$ *whenever* $f_n \xrightarrow[A]{p.\mu} 0$ *and* $g_n \xrightarrow[A]{p.\mu} 0$, $\forall A \in \mathcal{F}$*;*

(3) $f_n \vee g_n \xrightarrow[A]{\mu} 0$ *whenever* $f_n \xrightarrow[A]{p.\mu} 0$ *and* $g_n \xrightarrow[A]{p.\mu} 0$, $\forall A \in \mathcal{F}$.

Proof. It is similar to the proof of Theorem 3. □

Acknowledgements. The first author was supported by NSFC Grant No. 70771010. The second author was supported by JSPS Grant No.22540112.

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