Multiple stopping odds problem in Markov-dependent trials

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Problem—1/5

We want to study the odds problem in Markov-dependent trials.

1. For a positive integer N, let $X_1, X_2, \ldots, X_N$ denote 0/1 random variables defined on a probability space $(\Omega, \mathcal{F}, \mathrm{P})$. These 0/1 random variables appears according to non-homogenous Markov chain with the transition probability such that

$$\mathbf{P}_i = \begin{pmatrix} 1-\beta_i & \beta_i \\ \alpha_i & 1-\alpha_i \end{pmatrix}, \tag{1}$$

   where $\beta_i := \mathrm{P}(X_{i+1} = 1 | X_i = 0)$, $\alpha_i := \mathrm{P}(X_{i+1} = 0 | X_i = 1)$ $\beta_0 := \mathrm{P}(X_1 = 0)$ and $\alpha_0 := \mathrm{P}(X_1 = 1) = 1 - \beta_0$. Each α_i and β_i are supposed to be known. We assume $0 < \alpha_i, \beta_i < 1$ for all i.

2. We observe these X_i's sequentially and claim that the ith trial is a success if $X_i = 1$.
3. Objective is to obtain the last success with multiple stopping.
4. What are the optimal stopping rule and the probability of win?

Our new results —2/5

We study a multiple stopping odds problem in Markov-dependent trials.
Hsiau and Yang (2002)

1. Their optimal rule was not of odds form, and they restricted the transition probability to $0 < \alpha_i + \beta_i < 1$.
2. They did not provide the lower bound of probability of win, since it may be not easy by using their form of the optimal stopping rule.

$$\downarrow\downarrow$$

Our new results

1. Even if Markov-dependent trials, the optimal stopping rule can be expressed as of odds form!
2. For multiple stopping case, the optimal multiple stopping rule is given.
3. For single stopping case, the asymptotic lower bound of probability of win is again $1/e$ for any transition probability of Markov chain under some condition!

Optimal single stopping rule—3/5

1. Let

$$p_{ij} := \begin{cases} \mathrm{P}(X_{i+1} = 1 | X_i = 1, X_{i+2} = 0) = (1-\alpha_i)\alpha_{i+1}, & j = i+1, \\ \mathrm{P}(X_{i+1} = 1 | X_{j-1} = 0, X_{j+1} = 0) = \beta_{j-1}\alpha_j, & j > i+1, \end{cases}$$

   and $r_{ij} = p_{ij}/(1-p_{ij})$. This is key setting inspired by the incredible insight of Ferguson (2008) who studied the general dependent sequence of X_i in odds problem.

2. **Theorem 2.1.** Assume that $0 < \alpha_i, \beta_i < 1$ for each $i \in \mathcal{N}$. The optimal single selecting strategy for the non-homogeneous Markov-dependent trials is given by

$$\tau_*^{(1)} = \min\left\{ i \in \mathcal{N} : X_i = 1 \ \& \ \sum_{j=i+1}^{N} r_{ij} < 1 \right\} = \min\left\{ i \geq i_*^{(1)} : X_i = 1 \right\}.$$

   Assume that $X_1 = 1$ a.s., then the probability of win is given by

$$\mathrm{P}_N^{(1)}(\mathrm{win}) = \mathrm{P}_N^{(1)}(\alpha_0, \ldots, \alpha_{N-1}, \beta_0, \ldots, \beta_{N-1}) = R_{i_*^{(1)}-1} V_{i_*^{(1)}-1}^{(1)},$$

   where $R_s = \sum_{j=s+1}^{N} r_{sj}$ and $V_s^{(1)} = \alpha_s \prod_{k=s+1}^{N-1}(1-\beta_k)$.

Optimal multiple stopping rule and typical lower bound of $P_N^{(1)}(win)$—4/5

Theorem 3.1. Suppose that we have at most m $(\in \mathcal{N})$ selection chances. Assume that $0 < \alpha_i, \beta_i < 1$ for each $i \in \mathcal{N}$. The optimal selection strategy $\{\tau_*^{(m)}, \tau_*^{(m-1)}, \cdots, \tau_*^{(1)}\}$ for the non-homogeneous Markov-dependent trials is given by

$$\tau_*^{(\ell)} = \min\{i \geq \max\{\tau_*^{(\ell+1)}, i_*^{(\ell)}\} : X_i = 1\}, \ \ell = 1, 2, \ldots, m,$$

where $\min \emptyset = +\infty$ and $\tau_*^{(m+1)} = 0$. Furthermore, the threshold sequence $\{i_*^{(\ell)}\}_{\ell=1}^m$ is decreasing in m, $1 \leq i_*^{(m)} \leq i_*^{(m-1)} \leq \cdots \leq i_*^{(1)} \leq N$.

Theorem 2.2. Assume that $X_1 = 1$, a.s. If $R_s = \sum_{j=s+1}^N r_{sj}$ with $s = i_*^{(1)} - 1$, then

(i) $\mathrm{P}_N^{(1)}(\mathrm{win}) = R_s V_s^{(1)} > R_s e^{-R_s}$.

(ii) If $R_s = R_{s(N)} \to 1$ as $N \to \infty$, then $\lim_{N\to\infty} \mathrm{P}_N^{(1)}(\mathrm{win}) > 1/e$.

Proof: A typical lower bound of $P_N^{(1)}(win)$—5/5

(i) $V_s^{(1)} = \prod_{k=s+1}^{N-1} q_{sk}/(\prod_{k=s+1}^{\tilde{N}-1}(1-\beta_k))$, where $\tilde{N} = N$ if N is an even integer, and $\tilde{N} = N-1$ if N is an odd integer. Since $1-\beta_k < 1$, we have

$$\mathrm{P}_N^{(1)}(\mathrm{win}) = R_s V_s^{(1)} = \frac{R_s \prod_{k=s+1}^{N-1} q_{sk}}{\prod_{k=s+1}^{\tilde{N}-1}(1-\beta_k)} > R_s \prod_{k=s+1}^{N-1} q_{sk}.$$

From $R_s = \sum_{k=s+1}^{N}(1/q_{sk}-1)$, we have $\sum_{k=s+1}^{N}(1/q_{sj}) = R_s + N - s$. By the inequality for arithmetic mean and geometric mean, we have then

$$\left(\prod_{k=s+1}^{N} \frac{1}{q_{sk}}\right)^{\frac{1}{N-s}} = \left(\frac{1}{\prod_{k=s+1}^{N} q_{sk}}\right)^{\frac{1}{N-s}} \leq \frac{\sum_{k=s+1}^{N} \frac{1}{q_{sk}}}{N-s} = 1 + \frac{R_s}{N-s}$$

and thus $\prod_{k=s+1}^{N} q_{sk} \geq (1 + R_s/(N-s))^{-(N-s)}$. Since $(1+R_s/(N-s))^{-(N-s)} \downarrow e^{-R_s}$ as $N \to \infty$, it follows that

$$\mathrm{P}_N^{(1)}(\mathrm{win}) > R_s \prod_{k=s+1}^{N-1} q_{sk} \geq R\left(1 + \frac{R}{N-s}\right)^{-(N-s)} > R_s e^{-R_s}.$$

(ii) follows immediately form (i).

Thank you for your attention.

Announce! :Workshop SDA2012 at RIMS, Kyoto Univ.

RIMS Workshop "Stochastic Decision Analysis 2012"

1. When: November 19 – 22, 2012
2. Where: Room 420, Research Institute of Mathematical Sciences, Kyoto University, Kyoto 606-8502, Japan
3. Reception: November 20, 18:00-20:00. 5000 JPY. Details to be announced.
4. Organizer: K. Ano (Shibaura Institute of Technology)
5. Registration Fee: Free
6. Submission Deadline: September 30, 2012
7. Submission Form: E-mail submission only.
8. Publication: will be published in "Kokyuroku".

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Message form Thomas Bruss!

Thank you all for your good wishes, because I really feel much much better! You can tell them all!

Thomas Bruss

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