Coupling of two partial differential equations and its application

Hidetoshi TAHARA
(Department of Mathematics, Sophia University, Tokyo, JAPAN)

In this talk, I will present a new approach to the study of nonlinear partial differential equations. Since my research is still in the first stage, as a model study I will discuss only the following partial differential equations:

$$\text{(A)} \qquad \frac{\partial u}{\partial t} = F\Big(t, x, u, \frac{\partial u}{\partial x}\Big)$$

(where $(t,x) \in \mathbb{C}^2$ are variables and $u = u(t,x)$ is the unknown function) and

$$\text{(B)} \qquad \frac{\partial w}{\partial t} = G\Big(t, x, w, \frac{\partial w}{\partial x}\Big)$$

(where $(t,x) \in \mathbb{C}^2$ are variables and $w = w(t,x)$ is the unknown function). For simplcity we suppose that $F(t,x,u_0,u_1)$ (resp. $G(t,x,w_0,w_1)$) is a holomorphic function defined in a neighborhood of the origin of $\mathbb{C}_t \times \mathbb{C}_x \times \mathbb{C}_{u_0} \times \mathbb{C}_{u_1}$ (resp. $\mathbb{C}_t \times \mathbb{C}_x \times \mathbb{C}_{w_0} \times \mathbb{C}_{w_1}$).

My basic question is:

Question. *When can we say that the equations (A) and (B) are equivalent ? or when can we transform (A) into (B) (or (B) into (A)) ?*

One way to treat this question is to consider the coupling of (A) and (B), and solve its coupling equation. The coupling of two partial differential equations (A) and (B) means that we consider the following partial differential equation with infinitely many variables $(t, x, u_0, u_1, \ldots)$

$$(\Phi) \qquad \frac{\partial \phi}{\partial t} + \sum_{m \geq 0} D^m[F](t, x, u_0, \ldots, u_{m+1}) \frac{\partial \phi}{\partial u_m} = G\Big(t, x, \phi, D[\phi]\Big)$$

(where $\phi = \phi(t, x, u_0, u_1, \ldots)$ is the unknown function), or the following partial differential equation with infinitely many variables $(t, x, w_0, w_1, \ldots)$

$$(\Psi) \qquad \frac{\partial \psi}{\partial t} + \sum_{m \geq 0} D^m[G](t, x, w_0, \ldots, w_{m+1}) \frac{\partial \psi}{\partial w_m} = F\Big(t, x, \psi, D[\psi]\Big)$$

(where $\psi = \psi(t, x, w_0, w_1, \ldots)$ is the unknown function). In the equation (Φ) (resp. (Ψ)) the notation D means the following vector field with infinite many variables:

$$D = \frac{\partial}{\partial x} + \sum_{i \geq 0} u_{i+1} \frac{\partial}{\partial u_i} \quad \Big(\text{resp.} \quad D = \frac{\partial}{\partial x} + \sum_{i \geq 0} w_{i+1} \frac{\partial}{\partial w_i} \quad \Big).$$

These two equations (Φ) or (Ψ) is called the coupling equation of (A) and (B).

I will solve this coupling equation and establish the equivalence of (A) and (B) under suitable conditions. Details will be explained in the talk. Way of solving (Φ) is as follows: first we get a formal solution and then we prove the convergence of the formal solution. Since the solution has infinitely many variables, the meaning of the convergence is not so trivial.