Observations on the JWKB treatment of the quadratic barrier

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The JWKB wave function, as a complete asymptotic expansion in powers of $\hbar$, is examined for the inverted-parabola potential with the tool of Borel summation. The usual JWKB connection formulas at the linear turning points, combined with matching in a common Stokes region when $\hbar$ is complex, are sufficient to solve the Schrödinger equation. In the limit of real $\hbar$, the ordinary JWKB connection formula at a linear turning point is not the connection formula of the limit because of a coalescence of Stokes lines and the disappearance of the common Stokes region. The $\sqrt{1+e^{-2\pi(-E)/\hbar}}$ factor, inferred from elementary lowest-order treatments, and the phase factor, which had been extracted from the asymptotic expansion of the parabolic cylinder function, arise in the JWKB method from the Borel sum of the "normalization factors" $e^{\pm i\sum_{n=1}^{\infty} S_R^{(n)}(\infty)\hbar^{2n-1}}$. Real $\hbar$ is a Stokes line for these factors. That the limits of the Borel sums of given JWKB wave functions are different for $\arg\hbar \to \pm 0$ is consistent with the distinctly different, mirror-image graphs of the Stokes lines drawn for opposite signs of $\arg\hbar$.